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ORIGINAL PAPER

On a Navier–Stokes–Ohm problem from plasma physics in multi connected domains

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Abstract
We consider a model from electro-magneto-hydrodynamics describing a plasma in bounded multi connected domains. A nontrivial solution exists for magnetic fields as the equilibrium of this model. Nonlinear stability of the nontrivial solution is proved based on time weighted maximal L_p-regularity.

Keywords Navier–Stokes–Ohm problem · Electro-magneto-hydrodynamics model · Multi connected domains · Stokes operator · Maxwell operator · Maximal L_p-regularity · Well-posedness · Nonlinear stability · Global existence

Mathematics Subject Classification 35B35 · 76E25

1 Introduction

Let Ω be a bounded multi connected domain in $\mathbb{R}^3$ and its boundary $\Sigma = \partial\Omega$ be class C^2. We consider the following electro-magneto-hydrodynamics model which describes a plasma, namely completely ionized gas.

$$
\begin{aligned}
\varrho_1(\partial_t + v_1 \cdot \nabla)v_1 - \mu_1 \Delta v_1 + \nabla \pi_1 &= \mathsf{e} n_1 j_1 + \alpha(v_2 - v_1) \quad \text{in } \Omega,\\
\varrho_2(\partial_t + v_2 \cdot \nabla)v_2 - \mu_2 \Delta v_2 + \nabla \pi_2 &= -z\mathsf{e} n_2 j_2 - \alpha(v_2 - v_1) \quad \text{in } \Omega,\\
\epsilon_0 \partial_t E &= \mathrm{rot}\, H - \mathsf{e}(n_1 v_1 - z n_2 v_2) - \sigma E \quad \text{in } \Omega,\\
\mu_0 \partial_t H &= -\mathrm{rot}\, E, \quad j_k = E + \mu_0 v_k \times H \quad \text{in } \Omega,
\end{aligned}
$$

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$$\begin{aligned} \operatorname{div} v_1 = \operatorname{div} v_2 = \operatorname{div} E = \operatorname{div} H = 0 \quad &\text{in } \Omega, \\ v_1 = v_2 = 0, \quad E \times \nu = 0, \quad H \cdot \nu = 0 \quad &\text{on } \Sigma, \\ v_1(0) = v_{01}, \quad v_2(0) = v_{02}, \quad E(0) = E_0, \quad H(0) = H_0 \quad &\text{in } \Omega. \end{aligned} \tag{1}$$

Here v_1 denotes the velocity of electrons, v_2 that of ions, E and H the electric and the magnetic fields, respectively. The numbers $\varrho_j, n_j, \mu_j > 0$, $j = 1, 2$ denote densities, number densities, viscosities. The numbers $\mathsf{e}, z, \epsilon_0, \mu_0, \alpha, \sigma > 0$ are physical constants which denote elementary charge, charge number, dielectricity and permeability of vacuum, as well as friction and conductivity. In this model we adopt Ohm's law, namely intense of electricity is proportional to the electric field with the electric conductivity constant σ. Therefore we call the problem (1) a Navier–Stokes–Ohm problem. For more background, we refer to Van Kampen–Felderhof [18] and Miyamoto [11].

The energy functional E is given by

$$\mathsf{E} = \int_\Omega \big(\frac{\varrho_1}{2}|v_1|^2 + \frac{\varrho_2}{2}|v_2|^2 + \frac{\epsilon_0}{2}|E|^2 + \frac{\mu_0}{2}|H|^2\big)dx.$$

Energy dissipation reads

$$\partial_t \mathsf{E} = -\int_\Omega \big(\mu_1|\nabla v_1|^2 + \mu_2|\nabla v_2|^2 + \alpha|v_2 - v_1|^2 + \sigma|E|^2\big)dx,$$

since

$$\int_\Omega (\operatorname{rot} E \cdot H - E \cdot \operatorname{rot} H)dx = -\int_\Sigma (E \times \nu) \cdot H d\Sigma = 0$$

by the boundary condition $E \times \nu = 0$.

We identify the equilibria of the system and show that the energy is even a strict Lyapunov functional. We assume that $\partial_t \mathsf{E}(t) = 0$ for $t \in (t_1, t_2)$. Then by the energy identity it holds that

$$\nabla v_1 = \nabla v_2 = E = 0, \quad (t, x) \in (t_1, t_2) \times \Omega,$$

hence by the Poincaré inequality, $v_1 = v_2 = E = 0$ on $(t_1, t_2) \times \Omega$. This implies further that the pressures π_j are constant, and

$$\operatorname{rot} H = 0, \quad \operatorname{div} H = 0 \quad \text{in } \Omega, \quad H \cdot \nu = 0 \quad \text{on } \Sigma. \tag{2}$$

If Ω is a bounded simply connected domain, then by Proposition 1 below there exists a potential φ satisfying

$$\Delta\varphi = 0 \quad \text{in } \Omega, \quad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \Sigma.$$

Therefore, φ is constant in Ω, and $H = \nabla\varphi \equiv 0$. These arguments show that the only equilibrium is the trivial solution $v_1 = v_2 = E = H = 0$, and that the energy E is even a strict Lyapunov functional. On the other hand, if Ω is a bounded multi connected domain, the set of H satisfying the conditions (2) has a nontrivial solution whose dimension is coincide with genus of Ω, in other words the second Betti number of Ω, which was proved by Foias–Temam [3] for $L_2(\Omega)$. Kozono–Yanagisawa [8], [9] and Amrouche–Seoula [1] extended the result for $L_r(\Omega)$. We set the function space

$$X_{har}(\Omega) := \{h \in L_2(\Omega);\ \operatorname{rot} h = 0,\ \operatorname{div} h = 0 \text{ in } \Omega,\ h \cdot \nu = 0 \text{ on } \Sigma\} \tag{3}$$

which was used in Kozono–Yanagisawa [8], [9].

Prüss–Shimizu [13] proved nonlinear stability of the trivial solution which is the equilibrium of (1) in a bounded simply connected domain. In this paper, for the case of a bounded multi connected domain, we prove nonlinear stability of the non-trivial solution which is the equilibrium of (1). Especially we construct a concrete example of the non-trivial solution in $X_{har}(\Omega)$. In order to prove nonlinear stability of the non-trivial equilibrium, we first set the problem (1) abstract formulation by using the linear operator A. Next we see that the kernel of A is the set of non-trivial equilibrium. Then we prove local well-posedness of the problem in time weighted L_p space by maximal regularity of the linear operator as the same way in [13]. Based on the linear stability, non-trivial solution of the problem is exponentially stable.

Navier–Stokes–Ohm problems have been studied by Yoshida and Giga [19], Giga and Yoshida [6] concerning strong local well-posedness, Strhömer [15], [16] considered weak solutions. Giga–Ibrahim–Shen–Yoneda [5] proved existence of global weak solutions. More results are also references given therein.

The Navier–Stokes–Ohm problem (1) consists of a Navier–Stokes system and a Maxwell system which are coupled in a semi-linear way. We regard it as a system of evolution equations. Maximal regularity with time weights enables us to obtain well-posedness for initial values in the scale critical space: $v \in H_2^{1/2}(\Omega)^6$ and $(E_0, H_0) \in L_2(\Omega)^6$, which was found by Fujita–Kato [4] for the Navier–Stokes equations (cf. Corollary 1, below). There are many other such nonlinear, weakly coupled hybrid systems, in other words nonlinear parabolic-hyperbolic systems, in the literature. We think that it would be worthwhile to study such systems also from the abstract point of view in the framework of evolution equations.

This paper is organized as follows. In Sect. 2, we see characters of $X_{har}(\Omega)$ and construct a concrete example of the non-trivial solution in $X_{har}(\Omega)$. We formulate the Navier–Stokes–Ohm problem (1) as an abstract evolution equation in Sect. 3 in the same way as in [13], and state results of a linear problem in Sect. 4. In Sect. 5, we prove local well-posedness of the abstract form of (1). Sect. 6 is devoted to show nonlinear stability of the non-trivial solution.

2 Solutions in $X_{har}(\Omega)$

2.1 Properties of solutions in $X_{har}(\Omega)$

In this subsection, we see properties of solutions in $X_{har}(\Omega)$ defined by (3) based on Temam [17, Appendix 1].

Case 1: Ω is simply connected. In order to make the differences between a simply connected domain and a multiply connected domain, we state well-known results.

Proposition 1 *Let Ω be simply connected. For $h \in X_{har}(\Omega)$, there exists a scalar function φ such that $h = \nabla\varphi$ and satisfy*

$$\Delta\varphi = 0 \quad \text{in } \Omega,$$
$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \Sigma.$$

Theorem 1

$$X_{har}(\Omega) = \{0\}.$$

Proof For every scalar functions ϕ and ψ, by the Green formula we have

$$\int_\Omega (\phi\Delta\psi + (\nabla\phi)(\nabla\psi))\, dV = \int_\Sigma \phi\frac{\partial\psi}{\partial\nu}\, dS.$$

Plugging φ in Proposition 1 in both ϕ and ψ, we obtain

$$\int_{\Omega} |h|^2 \, dV = \int_{\Omega} |\nabla \varphi|^2 \, dV = \int_{\Sigma} \varphi \frac{\partial \varphi}{\partial \nu} \, dS.$$

We know that $h = 0$ because $h \in X_{har}(\Omega)$. This shows that $X_{har}(\Omega) = \{0\}$. □

Case 2: Ω is n-th multiply connected. We make Ω with a finite number of smooth cuts. More precisely, $\Gamma_1, \ldots \Gamma_{n-1}$ are manifolds of dimension 2 and class C^2 such that $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, and the open set $\dot{\Omega} = \Omega \setminus \cap_{i-1}^{n-1} \Gamma_i$ is simply connected. Here Γ_i $(i = 1, 2, \ldots, n-1)$ are not tangent to Σ. We denote Γ_i^+ and Γ_i^- the two sides of Γ_i and ν_i the unit normal on Γ_i oriented from Γ_i^+ towards Γ_i^- and set

$$[\theta]_i = \theta\big|_{\Gamma_i^+} - \theta\big|_{\Gamma_i^-}.$$

Lemma 1 (Appendix 1, Lem.1.1 in [17]) *For $h \in X_{har}(\Omega)$, there exists a scalar function φ such that $h = \nabla\varphi$ and*

$$\begin{aligned}
\Delta \varphi &= 0 \quad in \ \dot{\Omega}, \\
\frac{\partial \varphi}{\partial \nu} &= 0 \quad on \ \Sigma, \\
[\varphi]_i &= \text{const.} \qquad i = 1, 2, \ldots, n-1, \\
\left[\frac{\partial \varphi}{\partial \nu_i}\right]_i &= 0 \qquad i = 1, 2, \ldots, n-1.
\end{aligned}$$

Lemma 2 (Appendix 1, Lem.1.2 in [17]) *There exist functions φ_i $(i = 1, 2, \ldots, n-1)$ unique up to an additional constant such that*

$$\Delta \varphi_i = 0 \quad \text{in } \dot{\Omega}, \tag{4}$$

$$\frac{\partial \varphi_i}{\partial \nu} = 0 \quad \text{on } \Sigma, \tag{5}$$

$$[\varphi_i]_j = 0 \ (i \neq j), \ = 1 \ (i = j), \tag{6}$$

$$\left[\frac{\partial \varphi_i}{\partial \nu_j}\right]_j = 0, \qquad j = 1, 2, \ldots, n-1. \tag{7}$$

Lemma 3 (Appendix 1, Lem.1.3 in [17]) *$X_{har}(\Omega)$ is the space spanned by $\nabla\varphi_1, \ldots, \nabla\varphi_{n-1}$ and its dimension is $n-1$. $h \in X_{har}(\Omega)$ is given by*

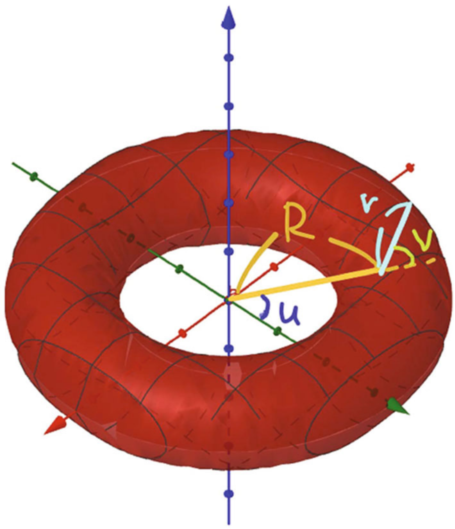
$$h = \nabla\varphi = [\varphi]_1 \nabla\varphi_1 + [\varphi]_2 \nabla\varphi_2 + \cdots + [\varphi]_{n-1} \nabla\varphi_{n-1}.$$

2.2 A concrete vector field of $X_{har}(\Omega)$

In this subsection, we construct a concrete vector field of $X_{har}(\Omega)$ in the case when Ω is a solid torus. Let $0 < r < R_1$. In the xz-plane, we make a circle with radius r with center $(R, 0)$. Turning it around z-axis, we obtain the solid torus Ω (cf. Fig. 1[1]). For the solid torus, the number of smooth cut is equal to 1.

[1] The figure depends on Yu Yoshimi (Graduate School of Human and Environmental Studies, Kyoto University).

Fig. 1 The solid torus



By using parameters $0 \leq u, v < 2\pi$, Ω is represented by

$$\begin{aligned} x &= (R + r\cos v)\cos u, \\ y &= (R + r\cos v)\sin u, \\ z &= r\sin v. \end{aligned}$$

Inversely $r,\ u,\ v$ are expressed by $x,\ y,\ z$

$$\begin{aligned} r^2 &= (\sqrt{x^2+y^2} - R)^2 + z^2, \\ \cos v &= \frac{\sqrt{x^2+y^2} - R}{\sqrt{(\sqrt{x^2+y^2} - R)^2 + z^2}}, \\ \tan u &= \frac{y}{x}. \end{aligned}$$

Toroid $\partial\Omega$ is given by

$$\begin{aligned} x &= (R + R_1\cos v)\cos u, \\ y &= (R + R_1\cos v)\sin u, \\ z &= R_1\sin v \end{aligned}$$

for $0 \leq u, v \leq 2\pi$. The unit outward normal ν to $\partial\Omega$ is given by

$$\nu = \begin{pmatrix} \cos v\cos u \\ \cos v\sin u \\ \sin v \end{pmatrix}.$$

We define a potential φ as

$$\varphi = -\frac{1}{2\pi}\tan^{-1}\frac{y}{x} + 1 = -\frac{1}{2\pi}u + 1. \tag{8}$$

A vector field $h = \nabla\varphi$ is expressed by

$$\left(\frac{y}{2\pi(x^2+y^2)}, \frac{-x}{2\pi(x^2+y^2)}, 0\right) = \left(\frac{\sin u}{2\pi(R+r\cos v)}, \frac{-\cos u}{2\pi(R+r\cos v)}, 0\right). \tag{9}$$

Since the solid torus Ω is biconnected domain, we check that φ in (8) satisfies the conditions in Lemma 2 as $i = 1$. We denote $\Gamma := \Gamma_1$. First we check the condition (4). Laplace operator in the sold torus is the following.

$$\begin{aligned}
\Delta =& \frac{\partial^2}{\partial r^2} + \frac{\cos u\cos v}{R+r\cos v}\frac{\partial}{\partial u}(\sin u\frac{\partial}{\partial r}) - \frac{\sin u\cos v}{R+r\cos v}\frac{\partial}{\partial u}(\cos u\frac{\partial}{\partial r}) \\
&+ \frac{\sin u}{(R+r\cos v)^2}\frac{\partial}{\partial u}(\sin u\frac{\partial}{\partial u}) + \frac{\cos u}{(R+r\cos v)^2}\frac{\partial}{\partial u}(\cos u\frac{\partial}{\partial u}) \\
&+ \frac{\sin u\sin v}{(R+r\cos v)r}\frac{\partial}{\partial u}(\cos u\frac{\partial}{\partial v}) - \frac{\cos u\sin v}{(R+r\cos v)r}\frac{\partial}{\partial u}(\sin u\frac{\partial}{\partial v}) \\
&- \frac{\sin v}{r}\frac{\partial}{\partial v}(\cos v\frac{\partial}{\partial r}) + \frac{\cos v}{r}\frac{\partial}{\partial v}(\sin v\frac{\partial}{\partial r}) \\
&+ \frac{\sin v}{r^2}\frac{\partial}{\partial v}(\sin v\frac{\partial}{\partial v}) + \frac{\cos v}{r^2}\frac{\partial}{\partial v}(\cos v\frac{\partial}{\partial v}).
\end{aligned}$$

The potential $\varphi = \varphi(u)$ in (8) is the function of u, it holds that

$$\begin{aligned}
\Delta\varphi(u) &= \frac{\sin u}{(R+r\cos v)^2}\frac{\partial}{\partial u}(\sin u\frac{\partial}{\partial u})\varphi(u) + \frac{\cos u}{(R+r\cos v)^2}\frac{\partial}{\partial u}(\cos u\frac{\partial}{\partial u})\varphi(u) \\
&= \frac{\sin u\cos u}{(R+r\cos v)^2}\left(-\frac{1}{2\pi}\right) - \frac{\sin u\cos u}{(R+r\cos v)^2}\left(-\frac{1}{2\pi}\right) = 0
\end{aligned}$$

in $\dot{\Omega}$. The condition (5) is verified by

$$\nu\cdot\nabla\varphi = \begin{pmatrix}\cos v\cos u\\ \cos v\sin u\\ \sin v\end{pmatrix}\cdot\begin{pmatrix}\frac{\sin u}{2\pi(R+r\cos v)}\\ -\frac{\cos u}{2\pi(R+r\cos v)}\\ 0\end{pmatrix} = 0.$$

(6) holds as follows

$$[\varphi]_\Gamma = \lim_{u\to+0}\varphi - \lim_{u\to-0}\varphi = \varphi(0) - \varphi(2\pi) = 1.$$

Finally we check the condition (7). Since the normal direction to Γ is angle u, it holds that

$$\left[\frac{\partial\varphi}{\partial u}\right]_\Gamma = \lim_{u\to+0}\frac{\partial\varphi}{\partial u} - \lim_{u\to-0}\frac{\partial\varphi}{\partial u} = -\frac{1}{2\pi} + \frac{1}{2\pi} = 0,$$

which shows that φ satisfies (7).

3 Abstract formulation

Abstract formulation of (1) is essentially the same as the simply connected domain case in [13]. We first recall some well-known results for the Stokes operator as well as for the Maxwell operator. For this we will need the projections of Helmholtz and Weyl. For given $v \in L_q(\Omega)^3$, $1 < q < \infty$, we consider the weak Neumann problem

$$(\nabla\varphi|\nabla\psi)_2 = (v|\nabla\psi)_2, \quad \psi \in H^1_{q'}(\Omega).$$

It is well-known that there is a solution $\varphi \in H^1_q(\Omega)$ which is unique up to a constant. Then we define the Helmholtz projection in $L_q(\Omega)^3$ by means of

$$P_H v := v - \nabla\varphi.$$

This projection is bounded, and it is orthogonal in case $q = 2$. In a similar way we define the Weyl projection. For given $v \in L_q(\Omega)^3$, $1 < q < \infty$, solve the weak Dirichlet problem

$$(\nabla\phi|\nabla\psi)_2 = (v|\nabla\psi)_2, \quad \psi \in {}_0H^1_{q'}(\Omega).$$

Here ${}_0H^1_{q'}(\Omega)$ denotes the closure of the test function $\mathcal{D}(\Omega)$ in $H^1_{q'}(\Omega)$. It is well-known that there is a unique solution $\phi \in {}_0H^1_q(\Omega)$. Then we define the Weyl projection in $L_q(\Omega)^3$ by means of

$$P_W v := v - \nabla\phi.$$

This projection is also bounded and it is orthogonal in case $q = 2$.

Consider the Stokes problem

$$\begin{aligned} \partial_t v - \Delta v + \nabla\pi &= 0 \quad &&\text{in } \Omega,\\ \operatorname{div} v &= 0 &&\text{in } \Omega,\\ v &= 0 &&\text{on } \Sigma,\\ v(0) &= v_0 &&\text{in } \Omega. \end{aligned} \tag{10}$$

To define the Stokes operator, we set $X^S_0 := P_H L_2(\Omega)^3$,

$$X^S_1 = \mathsf{D}(A^S) = \{v \in H^2_q(\Omega)^3 \cap X^S_0 : v = 0 \text{ on } \Sigma\},$$

and $A^S := -P_H\Delta$. It is well-known that $-A^S$ generates a compact analytic C_0-semigroup in X^S_0, which is exponentially stable for bounded domains, and moreover that A^S is positive definite in case $q = 2$.

Consider the Maxwell equations

$$\begin{aligned} \epsilon_0\partial_t E - \operatorname{rot} H &= 0 \quad &&\text{in } \Omega,\\ \mu_0\partial_t H + \operatorname{rot} E &= 0 &&\text{in } \Omega,\\ \operatorname{div} H = \operatorname{div} E &= 0 &&\text{in } \Omega,\\ \nu \times E = \nu \cdot H &= 0 &&\text{on } \Sigma,\\ E(0) = E_0, \quad H(0) &= H_0 &&\text{in } \Omega. \end{aligned}$$

Let $X^M_0 = P_W L_2(\Omega)^3 \times P_H L_2(\Omega)^3$,

$$X^M_1 = \mathsf{D}(A^M) := \{w = (E, H)^T \in H^1_2(\Omega)^6 \cap X^M_0 : \nu \times E = 0 \text{ on } \Sigma\},$$

and define the Maxwell operator by means of

$$A^M := \begin{bmatrix} 0 & -\frac{1}{\epsilon_0}\operatorname{rot} \\ \frac{1}{\mu_0}\operatorname{rot} & 0 \end{bmatrix}.$$

Here $(E, H)^T$ denotes the transposed of (E, H).

Proposition 2 (Prop.2.1 in [13]) *This operator A^M is skew-adjoint in the Hilbert space X^M_0, endowed with the inner product*

$$\langle w|\tilde{w}\rangle := \epsilon_0(w_1|\tilde{w}_1)_2 + \mu_0(w_2|\tilde{w}_2)_2.$$

Hence $-A^M$ *generates a unitary* C_0*-group* $e^{-A^M t}$ *in* X_0^M.

Then we consider the Maxwell equations with conductivity

$$\begin{aligned} \epsilon_0 \partial_t E - \operatorname{rot} H + \sigma E &= 0 \quad \text{in } \Omega, \\ \mu_0 \partial_t H + \operatorname{rot} E &= 0 \quad \text{in } \Omega, \\ \operatorname{div} H = \operatorname{div} E &= 0 \quad \text{in } \Omega, \\ \nu \times E = \nu \cdot H &= 0 \quad \text{on } \Sigma, \\ E(0) = E_0, \quad H(0) &= H_0 \quad \text{in } \Omega. \end{aligned}$$

In X_0^M we define the Maxwell operator with conductivity by means of

$$A^{MC} := \begin{bmatrix} \frac{\sigma}{\epsilon_0} & -\frac{1}{\epsilon_0}\operatorname{rot} \\ \frac{1}{\mu_0}\operatorname{rot} & 0 \end{bmatrix}, \quad \mathsf{D}(A^{MC}) := \mathsf{D}(A^M) = X_1^M.$$

This operator is a bounded accretive perturbation of A^M and therefore is also m-accretive in the Hilbert space X_0^M, hence it is the negative generator of a C_0-semigroup of contractions in X_0^M with compact resolvent. However, A^{MC} is not strongly accretive.

Exponential stability can be proved by means of the Gearhart–Prüss theorem (e.g. Prüss [12]).

Proposition 3 (Prop.2.3 in [13]) C_0*-semigroup* $e^{-A^{MC}t}$ *is exponentially stable on* $R(A^{MC})$. *We have* $X_0^M = N(A^{MC}) \oplus R(A^{MC})$*, and there are constants* $M \geq 1$*,* $\omega \geq 0$ *such that*

$$\left\| e^{-A^{MC}t} - P_0^{MC} \right\|_{\mathcal{B}(X_0^M)} \leq M e^{-\omega t}, \quad t \geq 0,$$

where P_0^{MC} *denotes the projection onto* $N(A^{MC})$ *along* $R(A^{MC})$.

The problem (1) is written the following form.

$$\begin{aligned} \varrho_1 \partial_t v_1 - \mu_1 P_H \Delta v_1 &= \mathsf{e} n_1 P_H E + \alpha(v_2 - v_1) + G_1^{\mathsf{v}} \ \text{ in } \Omega, \\ \varrho_2 \partial_t v_2 - \mu_2 P_H \Delta v_2 &= -z\mathsf{e} n_2 P_H E - \alpha(v_2 - v_1) + G_2^{\mathsf{v}} \ \text{ in } \Omega, \\ \epsilon_0 \partial_t E - \operatorname{rot} H + \sigma E &= -\mathsf{e}(n_1 v_1 - z n_2 v_2), \ \text{ in } \Omega, \\ \mu_0 \partial_t H + \operatorname{rot} E &= 0 \ \text{ in } \Omega, \\ \operatorname{div} v_1 = \operatorname{div} v_2 = \operatorname{div} E = \operatorname{div} H &= 0 \ \text{ in } \Omega, \\ v_1 = v_2 = 0, \quad E \times \nu = 0, \quad H \cdot \nu &= 0, \ \text{ on } \Sigma, \\ v_1(0) = v_{01}, \quad v_2(0) = v_{02}, \quad E(0) = E_0, \quad H(0) &= H_0, \ \text{ in } \Omega, \end{aligned}$$

where

$$\begin{aligned} G_1^{\mathsf{v}} &= P_H[\mathsf{e} n_1 \mu_0 v_1 \times H - \varrho_1 v_1 \cdot \nabla v_1], \\ G_2^{\mathsf{v}} &= -P_H[z\mathsf{e} n_2 \mu_0 v_2 \times H + \varrho_2 v_2 \cdot \nabla v_2]. \end{aligned}$$

For convenience we set $G^{\mathsf{v}} = [G_1^{\mathsf{v}}, G_2^{\mathsf{v}}]^T$, and $G = [G_1^{\mathsf{v}}, G_2^{\mathsf{v}}, 0, 0]^T$. So the structure of the problem is a system of the Stokes equations coupled with the Maxwell system with conduction. The coupling consists of a linear and bounded one, and an unbounded quadratic coupling which fortunately only acts on the velocities.

To formulate this problem abstractly, we define the space $X_0 = X_0^S \times X_0^S \times X_0^M$, whose elements are

$$u = (v, w)^T = (v_1, v_2, w_1, w_2)^T = (v_1, v_2, E, H)^T \in X_0.$$

Then the regularity space is $X_1 = X_1^S \times X_1^S \times X_1^M$, and the principal linear part A_0 is given by

$$A_0 := \mathrm{diag}(\frac{\mu_1}{\varrho_1} A^S, \frac{\mu_2}{\varrho_2} A^S, A^{MC}), \quad \mathsf{D}(A_0) = X_1.$$

Furthermore, the bounded linear perturbation $B_0 + B$ is given by

$$\begin{aligned} Bu &= \Big(\frac{\mathsf{e} n_1}{\varrho_1} P_H E,\ -\frac{z\mathsf{e} n_2}{\varrho_2} P_H E,\ \frac{-\mathsf{e}}{\epsilon_0}(n_1 v_1 - z n_2 v_2),\ 0\Big)^T, \\ B_0 u &= \big(B_0^{\mathsf{v}}, 0, 0\big)^T, \quad B_0^{\mathsf{v}} = \Big(\frac{\alpha}{\varrho_1}(v_2 - v_1),\ -\frac{\alpha}{\varrho_2}(v_2 - v_1)\Big)^T. \end{aligned}$$

Then the problem becomes

$$\dot{u} + A_0 u = B_0 u + Bu + G(u), \quad t > 0, \quad u(0) = u_0, \tag{11}$$

which is a semilinear evolution equation in the base space X_0. In a more detailed form, the abstract problem (11) reads

$$\begin{aligned} \dot{v} + A_0^{\mathsf{v}} v = B_0^{\mathsf{v}} v + B^{\mathsf{v}} w + G^{\mathsf{v}}(v, w), \quad & t > 0, \quad v(0) = v_0, \\ \dot{w} + A_0^{\mathsf{w}} w = B^{\mathsf{w}} v, \quad & t > 0, \quad w(0) = w_0. \end{aligned}$$

4 The linear operator

The base space $X_0 = X_0^S \times X_0^S \times X_0^M$ is endowed with the inner product

$$\begin{aligned} \langle u|\tilde{u}\rangle &= \langle v|\tilde{v}\rangle + \langle w|\tilde{w}\rangle, \\ \langle v|\tilde{v}\rangle &= \varrho_1 (v_1|\tilde{v}_1)_2 + \varrho_2 (v_2|\tilde{v}_2)_2, \quad \langle w|\tilde{w}\rangle = \epsilon_0 (w_1|\tilde{w}_1)_2 + \mu_0 (w_2|\tilde{w}_2)_2. \end{aligned}$$

The fully linearized problem is given by

$$\dot{u} + Au = F, \quad t > 0, \quad u(0) = u_0,$$

where the operator A in X_0 is given by

$$A = \begin{bmatrix} A^{\mathsf{v}} & -B^{\mathsf{v}} \\ -B^{\mathsf{w}} & A^{\mathsf{w}} \end{bmatrix}, \quad \mathsf{D}(A) = X_1. \tag{12}$$

Here we have set

$$A^{\mathsf{v}} = A_0^{\mathsf{v}} - B_0^{\mathsf{v}}, \quad A^{\mathsf{w}} = A_0^{\mathsf{w}} = A^{MC}.$$

Since $B_0 = [B_0^{\mathsf{v}}, 0]^T$ is bounded and $\langle B_0^{\mathsf{v}} v|v\rangle = -\alpha|v_2 - v_1|^2 \le 0$, A^{v} is strongly m-accretive, hence the semigroup $e^{-A^{\mathsf{v}} t}$ is also analytic, exponentially stable and has maximal L_p-regularity in the base space $X_0^{\mathsf{v}} = X_0^S \times X_0^S$.

Further, the perturbations B^{v}, B^{w} are also bounded and $\langle Bu|u\rangle$ is purely imaginary. Therefore A is also m-accretive, but not strongly accretive.

Lemma 4 (Thm. 3.1 in [13]) *The operator A is m-accretive in X_0, hence $-A$ is the generator of the C_0-semigroup e^{-At} of contractions in X_0.*

We also know that the only eigenvalue of A on $i\mathbb{R}$ is possibly 0, all other eigenvalues have strictly positive real parts. By the theorem of Arendt–Batty–Lubich–Phong (cf. [2]) we have the following stability result.

Lemma 5 (Prop. 3.2 in [13]) *It holds that*

$$\lim_{t\to\infty} e^{-At}u_0 = P_A u_0 \quad \text{for each } u_0 \in X_0,$$

where P_A denotes the projection on to the kernel $N(A)$ of A along the range $R(A)$. The semigroup e^{-At} is strongly stable in $R(A)$.

We consider the kernel $N(A)$ of A. If $u = (v, w)^T \in N(A)$, then v and w satisfy

$$\begin{aligned} A_0^{\mathsf{v}} v - B_0^{\mathsf{v}} v - B^{\mathsf{v}} w &= 0, \\ A_0^{\mathsf{w}} w - B^{\mathsf{w}} v &= 0. \end{aligned} \tag{13}$$

Taking the inner product in X_0 with u yields

$$\langle A_0 u|u\rangle - \langle B_0^{\mathsf{v}} v|v\rangle - \langle Bu|u\rangle = 0.$$

Taking real parts, we have

$$0 = \mathrm{Re}\,\langle A_0 u|u\rangle + \alpha|v_2 - v_1|_2^2 \geq \int_\Omega \left(\mu_1|\nabla v_1|^2 + \mu_2|\nabla v_2|^2 + \sigma|w_1|^2\right) dx.$$

This shows that $\nabla v_1 = \nabla v_2 = 0$ and $w_1 = 0$. By the Poincaré inequality, it holds that $v_1 = v_2 = 0$. Plugging in $v_1 = v_2 = w_1 = 0$ in (13) we obtain $\mathrm{rot}\, w_2 = 0$. Now $u \in X_0$, which implies that w_2 also satisfies $\mathrm{div}\, w_2 = 0$ and $w_2 \cdot \nu = 0$. Therefore it holds that

Lemma 6 *The kernel of operator A is*

$$N(A) = \{(0, 0, 0, w_2)^T \in X_0 \mid w_2 \in X_{har}(\Omega)\}. \tag{14}$$

We see the formulation of e^{-At} more precisely. Considering the resolvent problem $\lambda u + Au = F = (f, g)^T$ for λ belongs to $\{\mathrm{Re}\,\lambda \geq 0\} \setminus \{0\}$, we have

$$\lambda v + A^{\mathsf{v}} v = B^{\mathsf{v}} w + f, \quad \lambda w + A^{\mathsf{w}} w = B^{\mathsf{w}} v + g. \tag{15}$$

As $\lambda + A^{\mathsf{w}}$ is invertible for $\{\mathrm{Re}\,\lambda \geq 0\} \setminus \{0\}$, this yields

$$w = (\lambda + A^{\mathsf{w}})^{-1}(g + B^{\mathsf{w}} v).$$

Inserting this into the first equation and setting

$$\widehat{K}(\lambda) = B^{\mathsf{v}}(\lambda + A^{\mathsf{w}})^{-1} B^{\mathsf{w}}(\lambda + A^{\mathsf{v}})^{-1},$$

which reads as

$$(I - \widehat{K}(\lambda))(\lambda + A^{\mathsf{v}})v = f + B^{\mathsf{v}}(\lambda + A^{\mathsf{w}})^{-1} g.$$

This shows that $I - \widehat{K}(\lambda)$ is invertible for $\{\mathrm{Re}\,\lambda \geq 0\} \setminus \{0\}$, and

$$v = (\lambda + A^{\mathsf{v}})^{-1}(I - \widehat{K}(\lambda))^{-1}(f + B^{\mathsf{v}}(\lambda + A^{\mathsf{w}})^{-1} g).$$

Define $K(t) = B^{\mathsf{v}} e^{-A^{\mathsf{w}} t} * B^{\mathsf{w}} e^{-A^{\mathsf{v}} t}$ for $t > 0$. By the operator-valued Paley-Wiener lemma (see e.g. [12]), there is $R \in L_{1,loc}(\mathbb{R}_+; \mathcal{B}(X_0))$ such that

$$(I - \widehat{K}(\lambda))^{-1} = I + \widehat{K}(\lambda)(I - \widehat{K}(\lambda))^{-1} = I + \widehat{R}(\lambda), \quad \{\mathrm{Re}\,\lambda \geq 0\} \setminus \{0\}.$$

Summarizing the above, we have

$$\begin{pmatrix} v \\ w \end{pmatrix} = (\lambda + A)^{-1} \begin{pmatrix} f \\ g \end{pmatrix},$$

where

$$
\begin{aligned}
(\lambda + A)^{-1} &= \begin{pmatrix} (\lambda + A^{\mathsf{v}})^{-1} & 0 \\ 0 & (\lambda + A^{\mathsf{w}})^{-1} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & (\lambda + A^{\mathsf{v}})^{-1} B^{\mathsf{v}} (\lambda + A^{\mathsf{w}})^{-1} \\ (\lambda + A^{\mathsf{w}})^{-1} B^{\mathsf{w}} (\lambda + A^{\mathsf{v}})^{-1} & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} (\lambda + A^{\mathsf{v}})^{-1} & 0 \\ 0 & (\lambda + A^{\mathsf{w}})^{-1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \hat{R}(\lambda) & \hat{R}(\lambda) B^{\mathsf{v}} (\lambda + A^{\mathsf{w}})^{-1} \\ B^{\mathsf{w}} (\lambda + A^{\mathsf{v}})^{-1} \hat{R}(\lambda) & B^{\mathsf{w}} (\lambda + A^{\mathsf{v}})^{-1} (I + \hat{R}(\lambda)) B^{\mathsf{v}} (\lambda + A^{\mathsf{w}})^{-1} \end{pmatrix}.
\end{aligned}
$$

By using the relation between C_0-semigroup and the resolvent via the Laplace transform

$$
\mathcal{L}[e^{-At}](\lambda) = \int_0^\infty e^{-\lambda t} e^{-tA}\, dt = (\lambda + A)^{-1}, \quad \lambda \in \{\operatorname{Re} \lambda \geq 0\} \setminus \{0\},
$$

we obtain the proposition.

Proposition 4 *C^0-semigroup e^{-At} on X_0 has the following expression:*

$$
\begin{aligned}
e^{-At} &= \begin{pmatrix} S^{\mathsf{v}}(t) \\ S^{\mathsf{w}}(t) \end{pmatrix} = \begin{pmatrix} S^{vv}(t) & S^{vw}(t) \\ S^{wv}(t) & S^{ww}(t) \end{pmatrix} \\
&= \begin{pmatrix} e^{-A^{\mathsf{v}} t} * (\delta_0 + R) & e^{-A^{\mathsf{v}} t} * (\delta_0 + R) * B^{\mathsf{v}} e^{-A^{\mathsf{w}} t} \\ e^{-A^{\mathsf{w}} t} * B^{\mathsf{w}} e^{-A^{\mathsf{v}} t} * (\delta_0 + R) & e^{-A^{\mathsf{w}} t} * \left(\delta_0 + B^{\mathsf{w}} e^{-A^{\mathsf{v}} t} * (\delta_0 + R) * B^{\mathsf{v}} e^{-A^{\mathsf{w}} t}\right) \end{pmatrix}
\end{aligned}
$$

for $u(t) = (v(t), w(t))^T \in X_0$ and $t \geq 0$.

Finally in this section, we state the exponential stability result for e^{-At}.

Theorem 2 (Thm 3.5 in [13]) *e^{-At} is exponentially stable on $R(A)$. There exist constants $\omega_1 > 0$ and $M_1 \geq 1$ such that*

$$
\|e^{-At}\|_{\mathcal{B}(R(A))} \leq M_1 e^{-\omega_1 t}, \quad t \geq 0,
$$

which is equivalent to

$$
\|e^{-At} u_0 - P_A u_0\|_{X_0} \leq M_1 e^{-\omega_1 t} \|u_0\|_{X_0}, \quad t \geq 0.
$$

5 Local Well-posedness

In order to obtain local well-posedness, time-weighted maximal L_p-regularity for A^{v} which gives parabolic regularization plays an essential role. For $1 < p < \infty$ and $1/p < \mu \leq 1$ and some $0 < a \leq \infty$, we define

$$
v \in L_{p,\mu}(0, a; X_0^{\mathsf{v}}) \quad \Leftrightarrow \quad t^{1-\mu} v \in L_p(0, a; X_0^{\mathsf{v}}).
$$

$H^1_{p,\mu}$ is defined in the similar way. We introduce the base space

$$
\mathbb{E}_{0,\mu}(a) := \mathbb{E}^{\mathsf{v}}_{0,\mu}(a) \times \mathbb{E}^{\mathsf{w}}_0(a),
$$

$$\mathbb{E}^{\mathsf{v}}_{0,\mu}(a) := L_{p,\mu}(0,a;X^{\mathsf{v}}_0), \quad \mathbb{E}^{\mathsf{w}}_0(a) = C([0,a];X^{\mathsf{w}}_0),$$

where

$$\begin{aligned}
X^{\mathsf{v}}_0 &:= X^S_0 \times X^S_0, \quad X^S_0 = P_H L_2(\Omega)^3,\\
X^{\mathsf{v}}_1 &:= X^S_1 \times X^S_1, \quad X^S_1 = \{v \in H^2_p(\Omega)^3 \cap X^S_0 : \ v = 0 \text{ on } \Sigma\},\\
X^{\mathsf{w}}_0 &:= X^M_0 = P_W L_2(\Omega)^3 \times P_H L_2(\Omega)^3,\\
X^{\mathsf{w}}_1 &:= X^M_1 = \{w = (E,H) \in H^1_2(\Omega)^6 \cap X^M_0 : \ \nu \times E = 0 \text{ on } \Sigma\}.
\end{aligned}$$

We introduce the solution space

$$\begin{aligned}
\mathbb{E}^{\mathsf{v}}_{1,\mu}(a) &:= H^1_{p,\mu}(0,a;X^{\mathsf{v}}_0) \cap L_{p,\mu}(0,a;X^{\mathsf{v}}_1),\\
\mathbb{E}_{1,\mu}(a) &:= \mathbb{E}^{\mathsf{v}}_{1,\mu}(a) \times C([0,a];X^{\mathsf{w}}_0).
\end{aligned}$$

Time trace space of $\mathbb{E}_{1,\mu}(a)$ is given by

$$X_{\gamma,\mu} = \left(B^{2\mu-2/p}_{2,p}(\Omega)^6 \cap X^{\mathsf{v}}_0\right) \times X^{\mathsf{w}}_0.$$

The case $\mu = 1$

$$X_\gamma := X_{\gamma,1}, \quad X_\gamma = \left(B^{2-2/p}_{2,p}(\Omega)^6 \cap X^{\mathsf{v}}_0\right) \times X^{\mathsf{w}}_0.$$

is the natural state space for the problem. Also we set

$$\mathbb{E}_1(a) := \mathbb{E}_{1,1}(a), \quad \mathbb{E}^{\mathsf{v}}_1(a) := \mathbb{E}^{\mathsf{v}}_{1,1}(a).$$

We state the time weighted maximal L_p-regularity result of the Stokes system (10) by Prüss–Simonett [14, Sect. 7] in the context of our problem setting.

Proposition 5 *Let* $1 < p < \infty$ *and* $1/p < \mu \le 1$. *For* $v_0 \in B^{2\mu-2/p}_{2,p}(\Omega)^3 \cap X^S_0$, (10) *admits a unique solution*

$$v \in H^1_{p,\mu}(\mathbb{R}_+;X^S_0) \cap L_{p,\mu}(\mathbb{R}_+;X^S_1), \quad \pi \in L_{p,\mu}(\mathbb{R}_+;H^1_2(\Omega)).$$

The following result is local well-posedness of (11).

Theorem 3 (Thm. 4.1 in [13]) *Let* $4/3 \le p < \infty$ *and* $1/4 + 1/p \le \mu \le 1$. *Then for each initial value*

$$u_0 \in X_{\gamma,\mu} = \left(B^{2\mu-2/p}_{2,p}(\Omega)^6 \cap X^{\mathsf{v}}_0\right) \times X^{\mathsf{w}}_0,$$

there exists $a = a(u_0) > 0$ *and a unique solution* $u \in \mathbb{E}_{1,\mu}(a)$ *of the problem* (11) *satisfies*

$$v \in H^1_p(t_0,a;X^{\mathsf{v}}_0) \cap L_p(t_0,a;X^{\mathsf{v}}_1), \quad w \in C([0,a];X^{\mathsf{w}}_0)$$

for any $t_0 \in (0,a)$. *The solution depends continuously on the data.*

Moreover, the solution exists on a maximal time interval $[0,t_+(u_0))$, *and belongs to*

$$u \in C([0,t_+(u_0));X_{\gamma,\mu}) \cap C((0,t_+(u_0));X_\gamma).$$

If blow up occurs, i.e. if $t_+(u_0) < \infty$, *then* $u([0,t_+))$ *is not relatively compact in* $X_{\gamma,\mu}$. *The solutions generate a local semiflow in the state spaces* $X_{\gamma,\mu}$.

Proof First we see that if the solution $u = (v, w)^T$ belongs to $\mathbb{E}_{1,\mu}(a) = \mathbb{E}^{\mathsf{v}}_{1,\mu}(a) \times \mathbb{E}^{\mathsf{w}}_0(a)$, then it belongs to $\mathbb{E}_1 = \mathbb{E}^{\mathsf{v}}_1 \times \mathbb{E}^{\mathsf{w}}_0$ instantaneously. Indeed, for every $t_0 \in (0, a)$,

$$\begin{aligned}\|v\|_{L_p(t_0,a;X^{\mathsf{v}}_1)} &= \Big(\int_{t_0}^a \|v(t)\|^p_{X^{\mathsf{v}}_1}\,dt\Big)^{\frac{1}{p}}\\ &\le t_0^{\mu-1}\sup_{t_0\le t\le a}\Big(\int_{t_0}^a t^{(1-\mu)p}\|v(t)\|^p_{X^{\mathsf{v}}_1}\,dt\Big)^{\frac{1}{p}} \le t_0^{\mu-1}\|v\|_{L_{p,\mu}(0,a;X^{\mathsf{v}}_1)},\end{aligned}$$

and the same estimate holds for $\partial_t v$.

The proof on this theorem is essentially the same as in [13, Thm. 4.1]. The solution of (11) is expressed by the integral equation for some $a > 0$ determined later

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-\tau)}G(u(\tau))\,d\tau, \quad t \in [0, a], \tag{16}$$

which is decomposed as

$$\begin{aligned} v(t) &= S^{\mathsf{v}}(t)u_0 + \int_0^t S^{\mathsf{vv}}(t-\tau)G^{\mathsf{v}}(u(\tau))\,d\tau,\\ w(t) &= S^{\mathsf{w}}(t)u_0 + \int_0^t S^{\mathsf{wv}}(t-\tau)G^{\mathsf{v}}(u(\tau))\,d\tau.\end{aligned}$$

By Theorem 2, it holds that the first term of right hand side of (16) satisfies $\|e^{-At}u_0\|_{\mathsf{R}(A)} \le M_1\|u_0\|_{X_0}$ for $t \ge 0$. So our task is to estimate the second term of right hand side of (16). We set

$$\begin{aligned} &u_* = (v_*(t), w_*(t))^T, \quad v_*(t) = S^{\mathsf{v}}(t)u_0,\ w_*(t) = S^{\mathsf{w}}(t)u_0,\\ &\tilde u = (\tilde v, \tilde w)^T = (v - v_*, w - w_*)^T.\end{aligned}$$

Now we estimate the nonlinear term which is the second term of the right hand side of (16). We define the space

$$\mathsf{D}([A^{\mathsf{v}}]^\alpha) = \{v \in H^{2\alpha}_2(\Omega)^3 \cap X^S_0;\ v = 0 \text{ on } \Sigma\}.$$

For the convection term $v_j \cdot \nabla v_j$, by the Hölder inequality, the boundedness of P_H, and the Sobolev embedding theorem we have

$$\|P_H(v_j\cdot\nabla v_j)\|_2 \le \|v_j\|_{12}\|\nabla v_j\|_{12/5} \le C\|v\|^2_{H_2^{\frac{5}{4}}} \le C\|v\|^2_{\mathsf{D}([A^{\mathsf{v}}]^{5/8})},$$

which yields

$$\|P_H(v_j\cdot\nabla v_j)\|_{\mathbb{E}^{\mathsf{v}}_{0,\mu}(a)} \le C\|v\|^2_{L_{2p,(1+\mu)/2}(0,a;\mathsf{D}([A^{\mathsf{v}}]^{5/8})}. \tag{17}$$

Let ${}_0\mathbb{E}_{1,\mu}(a)$ denote the space of functions in $\mathbb{E}_{1,\mu}(a)$ with times trace 0 at $t = 0$, and consider a ball $\mathbb{B}_r$ in this space with center origin. We set $\varphi(a) := \|u_*\|_{\mathbb{E}_{1,\mu}(a)}$ and observe that $\varphi(a) \to 0$ as $a \to 0$ uniformly for initial values u_0 belonging to a compact subset of $X_{\gamma,\mu}$. By the assumption $1/4 + 1/p \le \mu$, using the embedding relation

$${}_0H^1_{p,\mu}(0, a; X^{\mathsf{v}}_0) \cap L_{p,\mu}(0, a; X^{\mathsf{v}}_1) \hookrightarrow L_{2p,(1+\mu)/2}(0, a; \mathsf{D}([A^{\mathsf{v}}]^{5/8})$$

(cf. [14, Sect. 4.5.5], [10, Thm. 2.1]), we obtain

$$\|P_H(v_j\cdot\nabla v_j)\|_{\mathbb{E}^{\mathsf{v}}_{0,\mu}(a)} \le C(\varphi(a) + r)^2, \quad \tilde v \in \mathbb{B}_r,$$

and the Lipschitz estimate

$$\|P_H(v_j\cdot\nabla v_j)-P_H(\bar v_j\cdot\nabla\bar v_j)\|_{\mathbb{E}^{\mathsf{v}}_{0,\mu}(a)}\le C(\varphi(a)+r)\|\tilde v-\tilde{\tilde v}\|_{\mathbb{E}^{\mathsf{v}}_{1,\mu}(a)},\quad \tilde v,\ \tilde{\tilde v}\in\mathbb{B}_r$$

in a similar way.

For the nonlinear term $v_j\times w_2$, the Hölder inequality yields

$$\|v_j\times w_2\|_{\mathbb{E}^{\mathsf{v}}_{0,\mu}(a)}\le C\|w\|_{C_b([0,a];X_0^{\mathsf{w}})}\|v_j\|_{L_{p,\mu}(0,a;L_\infty(\Omega))}. \tag{18}$$

If we take β as $\frac{3}{4}<\beta<1$, then the embedding relation $\mathsf{D}([A^{\mathsf{v}}]^\beta)\hookrightarrow L_\infty$ holds. Choosing κ as $\mu<\mu+1-\beta=\kappa\le 1$, we have the embedding

$${}_0H^1_{p,\mu}(0,a;X_0^{\mathsf{v}})\cap L_{p,\mu}(0,a;X_1^{\mathsf{v}})\hookrightarrow L_{p,\kappa}(0,a;\mathsf{D}([A^{\mathsf{v}}]^\beta))$$

(cf. [14, Sect. 4.5.5]) and

$$\begin{aligned}\|v_j\|^p_{L_{p,\mu}(0,a;L_\infty(\Omega))}&\le C\,\|v_j\|^p_{L_{p,\mu}(0,a;)\mathsf{D}([A^{\mathsf{v}}]^\beta)}\\&=C\int_0^a(t^{\kappa-\mu}t^{1-\kappa}\|v_j\|_{\mathsf{D}([A^{\mathsf{v}}]^\beta)})^p\,dt\\&\le C\,a^{p(1-\beta)}\|v_j\|^p_{L_{p,\kappa}(0,a;\mathsf{D}([A^{\mathsf{v}}]^\beta)}.\end{aligned}$$

We set $\psi(a):=\|v_*\|_{L_{p,\mu}(0,a;L_\infty(\Omega))}$. The above embedding implies the estimate

$$\|v_j\times w_2\|_{\mathbb{E}^{\mathsf{v}}_{0,\mu}(a)}\le C\|w\|_{C_b([0,a];X_0^{\mathsf{w}})}(\psi(a)+a^{1-\beta}r),\quad \tilde{\tilde u}\in\mathbb{B}_r,$$

as well as the Lipschitz estimate

$$\begin{aligned}&\|v_j\times w_2-\tilde v_j\times\tilde w_2\|_{\mathbb{E}^{\mathsf{v}}_{0,\mu}(a)}\\&\le C\Big\{\|w_2\|_{C_b([0,a];X_0^{\mathsf{w}})}a^{1-\beta}\|\tilde v_j-\tilde{\tilde v}_j\|_{L_{p,\kappa}(0,a;\mathsf{D}([A^{\mathsf{v}}]^\beta)}\\&\qquad\qquad+(\psi(a)+a^{1-\beta}r)\|\tilde w_2-\tilde{\tilde w}_2\|_{C_b([0,a];X_0^{\mathsf{w}})}\Big\}\\&\le C\left(\psi(a)+2a^{1-\beta}r+a^{1-\beta}\|w_*\|_{C_b([0,a];X_0^{\mathsf{w}})}\right)\|\tilde u-\tilde{\tilde u}\|_{\mathbb{E}_{1,\mu}(a)},\quad \tilde u,\ \tilde{\tilde u}\in\mathbb{B}_r.\end{aligned}$$

Here $\psi(a)\to 0$ as $a\to 0$ locally uniformly in $X_{\gamma,\mu}$. Now we set the map

$$\begin{aligned}(\tilde v,\tilde w)^T&=T(\tilde v,\tilde w)^T\\&=(S^{\mathsf{vv}}*G^{\mathsf{v}}(v_*+\tilde v,w_*+\tilde w),\,S^{\mathsf{wv}}*G^{\mathsf{v}}(v_*+\tilde v,w_*+\tilde w))^T,\end{aligned}$$

where S^{vv} and S^{wv} are defined in Proposition 4. S^{vv} has maximal regularity

$$\|S^{\mathsf{vv}}*f\|_{\mathbb{E}^{\mathsf{v}}_{1,\mu}(a)}\le M_{\mathsf{v}}\|f\|_{\mathbb{E}^{\mathsf{v}}_{0,\mu}(a)} \tag{19}$$

with maximal regularity constant M_{v}. S^{wv} holds the estimate

$$\|S^{\mathsf{wv}}*f\|_{C_b([0,a];X_0^{\mathsf{w}})}\le M_{\mathsf{w}}\|f\|_{L_1(0,a;X_0^{\mathsf{v}})}\le M_{\mathsf{w}}(p'(\mu-1/p))^{-\frac{1}{p'}}a^{\mu-1/p}\|f\|_{\mathbb{E}^{\mathsf{v}}_{0,\mu}(a)},$$

where the constants M_{v}, M_{w} do not depend on a. Choosing first $r>0$ and then $a>0$ small enough, we know that $T:\mathbb{B}_r\to\mathbb{B}_r$ is a self-map and strictly contraction uniformly for initial values $u_0\in\mathcal{K}\subset X_{\gamma,\mu}$ compact. Therefore the contraction mapping principle yields that there is a unique solution $u(t)$ of (11) on $[0,a]$, where a is uniform on $\mathcal{K}$. The characterization of the maximal time of existence is standard argument (cf. [7]). □

If we take $2\mu - 2/p = 1/2$, then it is corresponding result of Fujita-Kato [4], in which obtains well-posedness of the Navier–Stokes equations for initial values in the scale critical space.

Corollary 1 *Let $4/3 \leq p < \infty$. Then for each initial value*

$$u_0 \in \left(B^{1/2}_{2,p}(\Omega)^6 \cap X^{\mathsf{v}}_0\right) \times X^{\mathsf{w}}_0,$$

there exists $a = a(u_0) > 0$ and a unique solution $u \in \mathbb{E}_{1,\mu}(a)$ of the problem (11) satisfies

$$v \in H^1_p(t_0, a; X^{\mathsf{v}}_0) \cap L_p(t_0, a; X^{\mathsf{v}}_1), \quad w \in C([0, a]; X^{\mathsf{w}}_0)$$

for any $t_0 \in (0, a)$. The solution depends continuously on the data.

The solution exists on a maximal time interval $[0, t_+(u_0))$, and belongs to

$$u \in C([0, t_+(u_0)); X_{\gamma,\mu}) \cap C((0, t_+(u_0)); X_\gamma).$$

Remark 1 $B^{1/2}_{2,p}(\Omega)$ has the same scale as $H^{1/2}_2(\Omega)$ and the embedding relations $X^{\mathsf{v}}_0 \hookrightarrow L_2(\Omega)^3$ and $X^{\mathsf{w}}_0 \hookrightarrow L_2(\Omega)^3$ hold. Therefore it shows local well-posedness (also global well-posedness as proved in Theorem 4 below) for initial values in the scale critical space: $v = (v_1, v_2)^T \in H^{1/2}_2(\Omega)^6$ and $(w_1, w_2)^T = (E_0, H_0)^T \in L_2(\Omega)^6$.

Proof Taking $\mu = 1/4 + 1/p$ in Theorem 3, we obtain the result. □

6 Nonlinear stability of equilibria

By Lemma 6, we know that the equilibria of (1) is

$$P_A u_0 = (0, w_E)^T \quad \text{for } u_0 \in X,$$

where $w_E = (0, w_{E2})^T \in X^{\mathsf{w}}_0$, $w_{E2} \in X_{har}(\Omega)$.

The following nonlinear stability of the problem (1) is the main result in this paper.

Theorem 4 *Let $4/3 \leq p < \infty$ and $1/4 + 1/p \leq \mu \leq 1$. The equilibria of the problem (1) are exponentially stable in the state space X_γ.*

1) *For each small $\varepsilon > 0$, there exist constants $t_0 > 0$, $\beta > 0$ and $M \geq 1$ such that the solution u satisfies*

$$\|u(t) - P_A u_0\|_{X_\gamma} = \|v(t)\|_{X^{\mathsf{v}}_\gamma} + \|w(t) - w_E\|_{X^{\mathsf{w}}_0} \leq M e^{-\beta t}, \quad t \geq t_0, \tag{20}$$

provided $\|u_0\|_{X_{\gamma,\mu}} \leq \delta$.

2) *For each small $\varepsilon > 0$, there exists $\delta > 0$ such that the solution u satisfies*

$$\|u(t) - P_A u_0\|_{X_\gamma} \leq \varepsilon, \quad t \geq 0, \tag{21}$$

provided $\|u_0\|_{X_\gamma} \leq \delta$.

Proof 1) Let $u_0 \in X_{\gamma,\mu}$ be small. We note that $P_A u_0 = (0, w_E)^T \in X_\gamma$. By Theorem 3, we know there exists a unique solution $u \in \mathbb{E}_{1,\mu}$ on the maximal interval $[0, t_+(u_0))$, and

$$u \in C([0, t_+(u_0)); X_{\gamma,\mu}) \cap C((0, t_+(u_0)); X_\gamma).$$

More precisely, for $a \in (0, t_+(u_0))$, for every $t_0 \in (0, a)$, the solution u belongs to $\mathbb{E}_{1,\mu}(t_0)$ and this $u = (v, w)^T$ belongs to

$$v \in H^1_p(t_0, a; X^{\mathsf{v}}_0) \cap L_p(t_0, a; X^{\mathsf{v}}_1) =: \mathbb{E}_1(t_0, a), \quad w \in C([0, a]; X^{\mathsf{w}}_0).$$

Therefore, the state space of u is X_γ for $t \geq t_0$. Since the solution of (11) is given by (16), we have

$$u(t) - P_A u_0 = e^{-At}u_0 - P_A u_0 + \int_0^t e^{-A(t-\tau)} G(u(\tau))\, d\tau, \quad t \in [0, a].$$

We set

$$u_*(t) = e^{-At}u_0, \quad \tilde{u}(t) = \int_0^t e^{-A(t-\tau)} G(u(\tau))\, d\tau,$$

and also

$$\begin{aligned} v(t) &= S^{\mathsf{v}}(t)u_0 + S^{\mathsf{vv}} * G^{\mathsf{v}}(t) =: v_*(t) + \tilde{v}(t), \\ w(t) &= (S^{\mathsf{w}}(t)u_0 - w_E) + S^{\mathsf{wv}} * G^{\mathsf{v}}(t) =: (w_*(t) - w_E) + \tilde{w}(t). \end{aligned}$$

By Theorem 2, if we take $0 < \omega < \omega_1$, then the operator $A - \omega$ is also exponentially stable. It holds that

$$\|u_*(t) - P_A u_0\|_{X_\gamma} = \|e^{-At}u_0 - P_A u_0\|_{X_\gamma} \leq M e^{-\omega t} \|u_0\|_{X_\gamma} \quad \text{for } t \geq t_0.$$

By the embedding relations

$$\begin{aligned} {}_0H^1_p(\mathbb{R}_+; X_0^{\mathsf{v}}) \cap L_p(\mathbb{R}_+; X_1^{\mathsf{v}}) &\hookrightarrow L_{2p}(\mathbb{R}_+; \mathsf{D}([A^{\mathsf{v}}]^{5/8}), \\ {}_0H^1_p(\mathbb{R}_+; X_0^{\mathsf{v}}) \cap L_p(\mathbb{R}_+; X_1^{\mathsf{v}}) &\hookrightarrow L_p(\mathbb{R}_+; L_\infty(\Omega)) \end{aligned}$$

(cf. [14, Sect. 4.5.5], [10, Thm. 2.1]), for fixed $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|u_0\|_{X_{\gamma,\mu}} \leq \delta$ implies that

$$\|e^{\omega t} v_*\|_{L_{2p}(t_0,\infty;\mathsf{D}([A^{\mathsf{v}}]^{5/8})}, \ \|e^{\omega t} v_*\|_{L_p(t_0,\infty;L_\infty(\Omega))}, \ \|e^{\omega t} w_*\|_{C_b(t_0,\infty;X_0^{\mathsf{w}})} \leq \varepsilon \tag{22}$$

for all $t_0 \in (0, t_+(u_0))$. For the nonlinear part $\tilde{u}(t) = (\tilde{v}(t), \tilde{w}(t))^T$, we set $\phi(a) = \|e^{\omega t}\tilde{u}\|_{\mathbb{E}_1(t_0,a)}$. From (19) and

$$\|S^{\mathsf{wv}} * f\|_{C([t_0,a];X_0^{\mathsf{w}})} \leq M_{\mathsf{w}} \|f\|_{L_1(t_0,a;X_0^{\mathsf{v}})} \leq M_{\mathsf{w}} \|f\|_{L_p(t_0,a;X_0^{\mathsf{v}})},$$

we obtain

$$\begin{aligned} \phi(a) &\leq M\big(\|e^{\omega t} S^{\mathsf{vv}} * G^{\mathsf{v}}\|_{\mathbb{E}_1^{\mathsf{v}}(t_0,a)} + \|e^{\omega t} S^{\mathsf{wv}} * G^{\mathsf{v}}\|_{C([t_0,a];X_0^{\mathsf{w}})}\big) \\ &\leq M\|e^{\omega t} G^{\mathsf{v}}\|_{L_p(t_0,a;X_0^{\mathsf{v}})}. \end{aligned} \tag{23}$$

(17), (18) and (22) show that

$$\begin{aligned} &\|e^{\omega t} G^{\mathsf{v}}\|_{L_p(t_0,a;X_0^{\mathsf{v}})} \\ &\leq C\big(\|e^{\omega t} v_*\|_{L_{2p}(t_0,a;\mathsf{D}([A^{\mathsf{v}}]^{5/8})} + \|e^{\omega t}\tilde{v}\|_{\mathbb{E}_1^{\mathsf{v}}(t_0,a)}\big)^2 \\ &+ \big(\|e^{\omega t} v_*\|_{L_p(t_0,a;L_\infty(\Omega))} + \|e^{\omega t}\tilde{v}\|_{\mathbb{E}_1^{\mathsf{v}}(t_0,a)}\big) \\ &\quad \times \big(\|e^{\omega t} w_*\|_{C([t_0,a];X_0^{\mathsf{w}})} + \|e^{\omega t}\tilde{w}\|_{C([t_0,a];X_0^{\mathsf{w}})}\big) \\ &\leq C(\epsilon + \phi(a))^2. \end{aligned}$$

Multiplying (23) by MC and combining the above inequality, we have

$$MC\phi(a)(1 - 2MC\varepsilon - MC\phi(a)) \leq (MC)^2\varepsilon^2.$$

Since $x(1-2MC\varepsilon-x) = -(x-\frac{1-2MC\varepsilon}{2})^2 + \frac{(1-2MC\varepsilon)^2}{4}$, if we choose $\varepsilon > 0$ satisfying $(MC)^2\varepsilon^2 \le \frac{(1-2MC\varepsilon)^2}{4}$, namely $\varepsilon \le 1/(4MC)$, then it holds that

$$MC\phi(a) \le \frac{1-2MC\varepsilon}{2}, \quad a \in [t_0, t_+(u_0)),$$

and also

$$\phi(a) \le \frac{2MC}{1-2MC\varepsilon}\varepsilon^2, \quad a \in [t_0, t_+(u_0)).$$

The right-hand side is independent of a, this shows that it is able to extend $t_+(u_0) = \infty$. Therefore we conclude that

$$\sup_{t \ge t_0} \|e^{\omega t}\tilde{u}(t)\|_{X_\gamma} \le 4MC\varepsilon^2$$

and (20) under $\|u_0\|_{X_{\gamma,\mu}} \le \delta$.
2) Let $u_0 \in X_\gamma$ be small. Then the proof in 1) for $t \in [t_0, t_+(u_0))$ is valid for $t \in [0, t_+(u_0))$ and (21) holds. This completes the proof of the theorem. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Amrouche, C., Seoula, H.: L^p-theory for vector potentials and Sobolevs inequalities for vector fields. Math. Models Meth. Appl. Sci. **23**, 37–92 (2013)
2. Arendt, W., Batty, C.J.K., Hieber, M., Neubrander, F.: Vector-valued Laplace transforms and Cauchy problems, Monographs in Mathematics **96**, second edition, Birkhäuser Verlag (2011)
3. Foias, C., Temam, R.: Remarques sur les équations de Navier-Stokes stationaires et les phénomènes successifs de bifurcation. Ann. Scuola. Norm. Sup. Pisa Cl. Sci. **5**, 28–63 (1978)
4. Fujita, H., Kato, T.: On the non-stationary Navier–Stokes system, Rend. Sem. Mat. Univ. Padova **32**, 243–260 (1962)
5. Giga, Y., Ibrahim, S., Shen, S., Yoneda, T.: Global well posedness for a two-fluid model. Differ. Integr. Equ. **31**, 187–214 (2018)
6. Giga, Y., Yoshida, Z.: On the equations of the two-component theory in magnetohydrodynamics. Comm. PDE. **9**, 503–522 (1984)
7. Köhne, M., Prüss, J., Wilke, M.: On quasilinear parabolic evolution equations in weighted L_p-spaces. J. Evol. Equ. **10**, 443–463 (2010)
8. Kozono, H., Yanagisawa, T.: L^r-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains. Indiana Univ. Math. J. **58**, 1853–1920 (2009)

9. Kozono, H., Yanagisawa, T.: L^r-Helmholtz decomposition and its application to the Navier-Stokes equations. Lectures on the analysis of nonlinear partial differential equations. Part 3, 237–290, Morningside Lect. Math., **3**, Int. Press, Somerville, MA, (2013)
10. Kurokiba, M., Ogawa, T.: Singular limit problem for the Keller-Segel system and drift-diffusion in scaling critical spaces. J. Evol. Equ. **20**, 421–457 (2020)
11. Miyamoto, K.: Plasma Physics for Nuclear Fusion. MIT, Cambridge (1981)
12. Prüss, J.: On the spectrum of C^0-semigroups. Trans. Am. Math. Soc. **284**, 847–857 (1984)
13. Prüss, J., Shimizu, S.: On a Navier–Stokes–Ohm problem from plasma physics. J. Evol. Eq. **18**, 351–371 (2018)
14. Prüss, J., Simonett, G.: Moving Interfaces and Quasilinear Parabolic Evolution Problems, Monographs in Mathematics 105. Birkhäuser Verlag (2016)
15. Ströhmer, G.: About an initial-boundary value problem from magneto-hydrodynamics. Math. Z. **209**, 345–362 (1992)
16. Ströhmer, G.: An existence result for partially regular weak solutions of certain abstract evolution equations, with an application to magneto-hydrodynamics. Math. Z. **213**, 373–385 (1993)
17. Temam, R.: Navier–Stokes Equations. Theory and numerical Analysis. American Mathematical Society, Providence (1984)
18. Van Kampen, N.G., Felderhof, B.U.: Theoretical Methods in Plasma Physics. North-Holland, Amsterdam (1967)
19. Yoshida, Z., Giga, Y.: On the Ohm-Navier-Stokes system in magnetohydrodynamics. J. Math. Phys. **24**, 2860–2864 (1983)