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Adjunction and Inversion of Adjunction

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Abstract. We establish adjunction and inversion of adjunction for log canonical centers of arbitrary codimension in full generality.

§1. Introduction

Throughout this paper, we will work over $\mathbb{C}$, the complex number filed. We establish the following adjunction and inversion of adjunction for log canonical centers of arbitrary codimension.

THEOREM 1.1. *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let W be a log canonical center of (X,Δ) and let $\nu\colon Z\to W$ be the normalization of W. Then we have the adjunction formula*

$$\nu^*(K_X+\Delta)=K_Z+B_Z+M_Z$$

with the following properties:

(A) (X,Δ) is log canonical in a neighborhood of W if and only if (Z,B_Z+M_Z) is an NQC generalized log canonical pair, and

(B) (X,Δ) is log canonical in a neighborhood of W and W is a minimal log canonical center of (X,Δ) if and only if (Z,B_Z+M_Z) is an NQC generalized kawamata log terminal pair.

For the definition of NQC generalized log canonical pairs and NQC generalized kawamata log terminal pairs, see [12, Section 2].

In order to formulate adjunction and inversion of adjunction for log canonical centers of arbitrary codimension in full generality, the notion of b-divisors, which was first introduced by Shokurov, is very useful. In fact, the $\mathbb{R}$-divisors B_Z and M_Z in Theorem 1.1 are the traces of certain $\mathbb{R}$-b-divisors $\mathbf{B}$ and $\mathbf{M}$ on Z, respectively. The precise version of Theorem 1.1 is:

THEOREM 1.2. (Adjunction and Inversion of Adjunction) *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let W be a log canonical center of (X,Δ) and let $\nu\colon Z\to W$ be the normalization of W. Then there exist a b-potentially nef $\mathbb{R}$-b-divisor $\mathbf{M}$ and an $\mathbb{R}$-b-divisor $\mathbf{B}$ on Z such that $\mathbf{B}_Z$ is effective with*

$$\nu^*(K_X+\Delta)=\mathbf{K}_Z+\mathbf{M}_Z+\mathbf{B}_Z.$$

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More precisely, there exists a projective birational morphism $p\colon Z' \to Z$ from a smooth quasi-projective variety Z' such that

(i) $\mathbf{M} = \overline{\mathbf{M}_{Z'}}$ *and* $\mathbf{M}_{Z'}$ *is a potentially nef* $\mathbb{R}$*-divisor on* Z',

(ii) $\mathbf{K} + \mathbf{B} = \overline{\mathbf{K}_{Z'} + \mathbf{B}_{Z'}}$,

(iii) $\operatorname{Supp} \mathbf{B}_{Z'}$ *is a simple normal crossing divisor on* Z',

(iv) $\nu \circ p\left(\mathbf{B}_{Z'}^{>1}\right) = W \cap \mathrm{Nlc}(X, \Delta)$ *holds set theoretically, where* $\mathrm{Nlc}(X, \Delta)$ *denotes the non-lc locus of* (X, Δ)*, and*

(v) $\nu \circ p\left(\mathbf{B}_{Z'}^{\geq 1}\right) = W \cap \left(\mathrm{Nlc}(X, \Delta) \cup \bigcup_{W \not\subset W^\dagger} W^\dagger\right)$*, where* $W^\dagger$ *runs over log canonical centers of* (X, Δ) *which do not contain* W*, holds set theoretically.*

Hence, $(Z, \mathbf{B}_Z + \mathbf{M}_Z)$ *is generalized log canonical, that is,* $\mathbf{B}_{Z'}^{>1} = 0$*, if and only if* (X, Δ) *is log canonical in a neighborhood of* W*. Moreover,* $(Z, \mathbf{B}_Z + \mathbf{M}_Z)$ *is generalized kawamata log terminal, that is,* $\mathbf{B}_{Z'}^{\geq 1} = 0$*, if and only if* (X, Δ) *is log canonical in a neighborhood of* W *and* W *is a minimal log canonical center of* (X, Δ)*. We note that* $\mathbf{M}_{Z'}$ *is semi-ample when* $\dim W = 1$*. We also note that if* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier then* $\mathbf{B}$ *and* $\mathbf{M}$ *become* $\mathbb{Q}$*-b-divisors by construction.*

In this paper, the $\mathbb{R}$-b-divisors $\mathbf{B}$ and $\mathbf{M}$ in Theorem 1.2 are defined by using the notion of basic $\mathbb{R}$-slc-trivial fibrations. Here, we explain an alternative definition of $\mathbf{B}$ and $\mathbf{M}$ for the reader's convenience. For the details of Definition 1.3, see [10, Section 5].

Definition 1.3. (see [10, Section 5] and Remark 6.1) Let (X, Δ), W, and $\nu\colon Z \to W$ be as in Theorem 1.2. For any higher birational model $\rho\colon \tilde{Z} \to Z$, we consider all prime divisors T over X such that $a(T, X, \Delta) = -1$ and the center of T on X is W. We take a log resolution $f\colon Y \to X$ of (X, Δ) so that T is a prime divisor on Y and the induced map $f_T\colon T \dashrightarrow \tilde{Z}$ is a morphism. We put $\Delta_T = (\Delta_Y - T)|_T$, where Δ_Y is defined by $K_Y + \Delta_Y = f^*(K_X + \Delta)$. For any prime divisor P on $\tilde{Z}$, we define a real number $\alpha_{P,T}$ by

$$\alpha_{P,T} = \sup\{\lambda \in \mathbb{R} \,|\, (T, \Delta_T + \lambda f_T^* P) \text{ is sub log canonical over the generic point of } P\}.$$

Then the trace $\mathbf{B}_{\tilde{Z}}$ of $\mathbf{B}$ on $\tilde{Z}$ is defined by

$$\mathbf{B}_{\tilde{Z}} = \sum_P (1 - \inf_T \alpha_{P,T}) P$$

where P runs over prime divisors on $\tilde{Z}$ and T runs over prime divisors over X such that $a(T, X, \Delta) = -1$ and the center of T on X is W. When W is a prime divisor on X, T is the strict transform of W on Y. In this case, we can easily check that $\mathbf{B}_{\tilde{Z}} = (f_T)_* \Delta_T$ holds. We consider the $\mathbb{R}$-line bundle $\mathcal{L}$ on X associated to $K_X + \Delta$. We fix an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D_{\tilde{Z}}$ on $\tilde{Z}$ whose associated $\mathbb{R}$-line bundle is $\rho^*\nu^*(\mathcal{L}|_W)$. Then the trace $\mathbf{M}_{\tilde{Z}}$ of $\mathbf{M}$ on $\tilde{Z}$ is defined by

$$\mathbf{M}_{\tilde{Z}} = D_{\tilde{Z}} - K_{\tilde{Z}} - \mathbf{B}_{\tilde{Z}}.$$

We simply write

$$\rho^*\nu^*(K_X + \Delta) = K_{\tilde{Z}} + \mathbf{B}_{\tilde{Z}} + \mathbf{M}_{\tilde{Z}}$$

if there is no danger of confusion (see also Remark 6.1).

As we saw in Definition 1.3, the $\mathbb{R}$-b-divisor $\mathbf{B}$ on Z depends only on the singularities of (X, Δ) near W. Conversely, Theorem 1.2 (ii)–(v) implies that $\mathbf{B}$ remembers properties of the singularities of (X, Δ) near W. If we put $B_Z = \mathbf{B}_Z$ and $M_Z = \mathbf{M}_Z$, then Theorem 1.1 directly follows from Theorem 1.2. Our new formulation of adjunction and inversion of adjunction includes some classical results as special cases. The following corollary is the case of $\dim W = \dim X - 1$ which recovers the classical adjunction and inversion of adjunction.

Corollary 1.4. (Classical Adjunction and Inversion of Adjunction) *In Theorem 1.1, we further assume that* $\dim W = \dim X - 1$*, that is,* W *is a prime divisor on* X*. Then* M_Z *and* B_Z *become zero and Shokurov's different, respectively. Then (A) recovers Kawakita's inversion of adjunction on log canonicity. By (B), we have that* (X, Δ) *is purely log terminal in a neighborhood of* W *if and only if* (Z, B_Z) *is kawamata log terminal.*

We know that we have already had many related results. We only make some remarks on [11] and [3].

Remark 1.5. (Hacon's inversion of adjunction) In [11, Theorem 1], Hacon treated inversion of adjunction on log canonicity for log canonical centers of arbitrary codimension under the extra assumption that Δ is a boundary $\mathbb{Q}$-divisor. However, it is not clear whether $\mathbf{B}(V; X, \Delta)$ in [11] coincides with $\mathbf{B}$ in Theorem 1.2 or not. We do not treat $\mathbf{B}(V; X, \Delta)$ in this paper. In [10, Theorem 5.4], we proved a generalization of [11, Theorem 1]. We note that $\mathbf{B}$ in [10, Theorem 5.4] coincides with $\mathbf{B}$ in Theorem 1.2. Hence Theorem 1.2 can be seen as a complete generalization of [10, Theorem 5.4].

Remark 1.6. (Generalized adjunction and inversion of adjunction by Filipazzi) In [3], Filipazzi established some related results for generalized pairs (see, for example, [3, Theorem 1.6]). Although they are more general than Theorems 1.1 and 1.2 in some sense, they do not include Theorem 1.1.

The main ingredients of Theorem 1.2 are the existence theorem of log canonical modifications established in [10] and the theory of basic slc-trivial fibrations in [6] and [7]. Hence this paper can be seen as a continuation of [7] and [10]. Moreover, the theory of partial resolutions of singularities of pairs in [2] is indispensable. We do not use Kawakita's inversion of adjunction (see [14, Theorem]) nor the Kawamata–Viehweg vanishing theorem. If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then Theorem 1.2 easily follows from [6], [7], and [10]. Unfortunately, however, the framework of basic slc-trivial fibrations discussed in [6] is not sufficient for our purposes in this paper. Hence we establish the following result.

Theorem 1.7. (Corollary 5.2) *Let* $f\colon (X, B) \to Y$ *be a basic* $\mathbb{R}$*-slc-trivial fibration and let* $\mathbf{B}$ *and* $\mathbf{M}$ *be the discriminant and moduli* $\mathbb{R}$*-b-divisors associated to* $f\colon (X, B) \to Y$*, respectively. Then we have the following properties:*

(i) $\mathbf{K} + \mathbf{B}$ *is* $\mathbb{R}$*-b-Cartier, where* $\mathbf{K}$ *is the canonical b-divisor of* Y*, and*

(ii) $\mathbf{M}$ *is b-potentially nef, that is, there exists a proper birational morphism* $\sigma\colon Y' \to Y$ *from a normal variety* Y' *such that* $\mathbf{M}_{Y'}$ *is a potentially nef* $\mathbb{R}$*-divisor on* Y' *and that* $\mathbf{M} = \overline{\mathbf{M}_{Y'}}$ *holds.*

If $f\colon (X,B)\to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration, then Theorem 1.7 is nothing but [6, Theorem 1.2], which is the main result of [6]. More precisely, we establish:

THEOREM 1.8. (see Theorem 5.1) *Let $f\colon (X,B)\to Y$ be a projective surjective morphism from a simple normal crossing pair (X,B) to a smooth quasi-projective variety Y such that every stratum of X is dominant onto Y and $f_*\mathcal{O}_X\simeq\mathcal{O}_Y$ with*

- *$B=B^{\leq 1}$ holds over the generic point of Y,*
- *there exists an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on Y such that $K_Y+B\sim_{\mathbb{R}} f^*D$ holds, and*
- $\operatorname{rank} f_*\mathcal{O}_X(\lceil -(B^{<1})\rceil)=1$.

We assume that there exists a simple normal crossing divisor Σ on Y such that $\operatorname{Supp} D\subset\Sigma$ *and that every stratum of $(X,\operatorname{Supp} B)$ is smooth over $Y\setminus\Sigma$. Let $\mathbf{B}$ and $\mathbf{M}$ be the discriminant and moduli $\mathbb{R}$-b-divisors associated to $f\colon (X,B)\to Y$, respectively. Then*

(i) $\mathbf{K}+\mathbf{B}=\overline{\mathbf{K}_Y+\mathbf{B}_Y}$ holds, where $\mathbf{K}$ is the canonical b-divisor of Y, and

(ii) $\mathbf{M}_Y$ is a potentially nef $\mathbb{R}$-divisor on Y with $\mathbf{M}=\overline{\mathbf{M}_Y}$.

Note that Theorem 1.8 completely generalizes [13, Lemma 2.8]. By Theorem 1.8, we can use the framework of basic slc-trivial fibrations in [6] for $\mathbb{R}$-divisors. We also note that the main part of this paper is devoted to the proof of Theorem 1.8. In the proof of Theorem 1.2, we naturally construct a basic $\mathbb{R}$-slc-trivial fibration $f\colon (V,\Delta_V)\to Z$ by taking a suitable resolution of singularities of the pair (X,Δ). The $\mathbb{R}$-b-divisors $\mathbf{B}$ and $\mathbf{M}$ on Z in Theorem 1.2 are the discriminant and moduli $\mathbb{R}$-b-divisors associated to $f\colon (V,\Delta_V)\to Z$, respectively.

CONJECTURE 1.9. *In Theorem 1.8, $\mathbf{M}_Y$ is semi-ample.*

If Conjecture 1.9 holds true, then $\mathbf{M}$ in Theorem 1.2 is b-semi-ample, that is, $\mathbf{M}_{Z'}$ is semi-ample. Note that Conjecture 1.9 follows from [6, Conjecture 1.4]. When $\dim Y=1$, we can easily check that $\mathbf{M}_Y$ is semi-ample by [9, Corollary 1.4]. Unfortunately, however, it is still widely open. In this paper, we prove Conjecture 1.9 for basic slc-trivial fibrations of relative dimension one under some extra assumption (see Theorem 7.2). Then we establish:

THEOREM 1.10. (see Corollary 7.3) *If W is a codimension two log canonical center of (X,Δ) in Theorem 1.2, then $\mathbf{M}$ is b-semi-ample.*

Theorem 1.10 generalizes Kawamata's result (see [15, Theorem 1]). For the details, see Corollary 7.3.

We briefly look at the organization of this paper. In Section 2, we recall some basic definitions and results. In Section 3, we introduce the notion of basic $\mathbb{R}$-slc-trivial fibrations and recall the main result of [6]. In Section 4, we slightly generalize the main result of [6]. This generalization (see Theorem 4.1) seems to be indispensable in order to treat basic $\mathbb{R}$-slc-trivial fibrations. In Section 5, we establish a fundamental theorem for basic $\mathbb{R}$-slc-trivial fibrations (see Theorems 1.8 and 5.1). In Section 6, we prove the main result, that is, adjunction and inversion of adjunction for log canonical centers of arbitrary codimension, in full generality. More precisely, we first establish Theorem 1.2. Then we see that Theorem 1.1 and Corollary 1.4 easily follow from Theorem 1.2. In Section 7, we treat adjunction and inversion of adjunction for log canonical centers of codimension two.

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§2. Preliminaries

In this paper, we will freely use the standard notation as in [4], [5], [6], and [7]. A *scheme* means a separated scheme of finite type over $\mathbb{C}$. A *variety* means an integral scheme, that is, an irreducible and reduced separated scheme of finite type over $\mathbb{C}$. We note that $\mathbb{Q}$ and $\mathbb{R}$ denote the set of *rational numbers* and *real numbers*, respectively. We also note that $\mathbb{Q}_{>0}$ and $\mathbb{R}_{>0}$ are the set of *positive rational numbers* and *positive real numbers*, respectively. Similarly, $\mathbb{Q}_{\geq 0}$ denotes the set of nonnegative rational numbers.

Here, we collect some basic definitions for the reader's convenience. Let us start with the definition of *potentially nef divisors*.

Definition 2.1. (Potentially nef divisors, see [6, Definition 2.5]) Let X be a normal variety and let D be a divisor on X. If there exist a completion $X^\dagger$ of X, that is, $X^\dagger$ is a complete normal variety and contains X as a dense Zariski open subset, and a nef divisor $D^\dagger$ on $X^\dagger$ such that $D = D^\dagger|_X$, then D is called a *potentially nef* divisor on X. A finite $\mathbb{Q}_{>0}$-linear (resp. $\mathbb{R}_{>0}$-linear) combination of potentially nef divisors is called a *potentially nef* $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor).

We give two important remarks on potentially nef $\mathbb{R}$-divisors.

Remark 2.2. Let D be a nef $\mathbb{R}$-divisor on a smooth projective variety X. Then D is not necessarily a potentially nef $\mathbb{R}$-divisor. This means that D is not always a finite $\mathbb{R}_{>0}$-linear combination of nef Cartier divisors on X.

Remark 2.3. Let X be a normal variety and let D be a potentially nef $\mathbb{R}$-divisor on X. Then $D \cdot C \geq 0$ for every projective curve C on X. In particular, D is π-nef for every proper morphism $\pi\colon X \to S$ to a scheme S.

It is convenient to use *b-divisors* to explain several results. Here we do not repeat the definition of b-divisors. For the details, see [6, Section 2].

Definition 2.4. (Canonical b-divisors) Let X be a normal variety and let ω be a top rational differential form of X. Then (ω) defines a b-divisor $\mathbf{K}$. We call $\mathbf{K}$ the *canonical b-divisor* of X.

Definition 2.5. ($\mathbb{R}$-Cartier closures) The $\mathbb{R}$-*Cartier closure* of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on a normal variety X is the $\mathbb{R}$-b-divisor $\overline{D}$ with trace

$$\overline{D}_Y = f^*D,$$

where $f\colon Y \to X$ is a proper birational morphism from a normal variety Y.

We use the following definition in order to state our results (see Theorem 1.2).

Definition 2.6. ([6, Definition 2.12]) Let X be a normal variety. An $\mathbb{R}$-b-divisor $\mathbf{D}$ of X is *b-potentially nef* (resp. *b-semi-ample*) if there exists a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$, that is, $\mathbf{D}$ is the $\mathbb{R}$-Cartier closure of $\mathbf{D}_{X'}$, and that $\mathbf{D}_{X'}$ is potentially nef (resp. semi-ample). An $\mathbb{R}$-b-divisor $\mathbf{D}$ of X is *$\mathbb{R}$-b-Cartier* if there is a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$. Obviously, $\mathbf{D}$ is said to be *$\mathbb{Q}$-b-Cartier* when $\mathbf{D}_{X'}$ is $\mathbb{Q}$-Cartier and $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

For the reader's convenience, let us recall the definition of singularities of pairs. The following definition is standard and is well known.

Definition 2.7. (Singularities of pairs) Let X be a variety and let E be a prime divisor on Y for some birational morphism $f\colon Y \to X$ from a normal variety Y. Then E is called a divisor *over* X. A *normal pair* (X, Δ) consists of a normal variety X and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let (X, Δ) be a normal pair and let $f\colon Y \to X$ be a projective birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

with

$$f_*\left(\sum_E a(E, X, \Delta)E\right) = -\Delta,$$

where E runs over prime divisors on Y. We call $a(E, X, \Delta)$ the *discrepancy* of E with respect to (X, Δ). Note that we can define the discrepancy $a(E, X, \Delta)$ for any prime divisor E over X by taking a suitable resolution of singularities of X. If $a(E, X, \Delta) \geq -1$ (resp. > -1) for every prime divisor E over X, then (X, Δ) is called *sub log canonical* (resp. *sub kawamata log terminal*). We further assume that Δ is effective. Then (X, Δ) is called *log canonical* and *kawamata log terminal* if it is sub log canonical and sub kawamata log terminal, respectively. When Δ is effective and $a(E, X, \Delta) > -1$ holds for every exceptional divisor E over X, we say that (X, Δ) is *purely log terminal*.

Let (X, Δ) be a log canonical pair. If there exists a projective birational morphism $f\colon Y \to X$ from a smooth variety Y such that both $\mathrm{Exc}(f)$, the exceptional locus of f, and $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}\Delta$ are simple normal crossing divisors on Y and that $a(E, X, \Delta) > -1$ holds for every f-exceptional divisor E on Y, then (X, Δ) is called *divisorial log terminal* (*dlt*, for short). It is well known that if (X, Δ) is purely log terminal then it is divisorial log terminal.

In this paper, the notion of non-lc loci and log canonical centers is indispensable.

Definition 2.8. (Non-lc loci and log canonical centers) Let (X, Δ) be a normal pair. If there exist a projective birational morphism $f\colon Y \to X$ from a normal variety Y and a prime divisor E on Y such that (X, Δ) is sub log canonical in a neighborhood of the generic point of $f(E)$ and that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* of (X, Δ).

From now on, we further assume that Δ is effective. The *non-lc locus* of (X, Δ), denoted by $\mathrm{Nlc}(X, \Delta)$, is the smallest closed subset Z of X such that the complement $(X \setminus Z, \Delta|_{X \setminus Z})$ is log canonical. We can define a natural scheme structure on $\mathrm{Nlc}(X, \Delta)$ by the non-lc ideal sheaf $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ of (X, Δ). For the definition of $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$, see [4, Section 7].

We omit the precise definition of *NQC generalized log canonical pairs* and *NQC generalized kawamata log terminal pairs* here since we need it only in Theorem 1.1 and the statement of Theorem 1.2 is sharper than that of Theorem 1.1. For the basic definitions and properties of *generalized polarized pairs*, we recommend the reader to see [12, Section 2]. Note that the notion of generalized pairs plays a crucial role in the recent study of higher-dimensional algebraic varieties.

DEFINITION 2.9. Let X be an equidimensional reduced scheme. Note that X is not necessarily regular in codimension one. Let D be an $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-divisor), that is, D is a finite formal sum $\sum_i d_i D_i$, where D_i is an irreducible reduced closed subscheme of X of pure codimension one and $d_i \in \mathbb{R}$ (resp. $d_i \in \mathbb{Q}$) for every i such that $D_i \neq D_j$ for $i \neq j$. We put

$$D^{<1} = \sum_{d_i<1} d_i D_i, \quad D^{=1} = \sum_{d_i=1} D_i, \quad D^{>1} = \sum_{d_i>1} d_i D_i, \quad \text{and} \quad \lceil D \rceil = \sum_i \lceil d_i \rceil D_i,$$

where $\lceil d_i \rceil$ is the integer defined by $d_i \leq \lceil d_i \rceil < d_i + 1$. We note that $\lfloor D \rfloor = -\lceil -D \rceil$ and $\{D\} = D - \lfloor D \rfloor$. Similarly, we put

$$D^{\geq 1} = \sum_{d_i \geq 1} d_i D_i.$$

Let D be an $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-divisor) as above. We call D a *subboundary* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) if $D = D^{\leq 1}$ holds. When D is effective and $D = D^{\leq 1}$ holds, we call D a *boundary* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor).

We further assume that $f\colon X \to Y$ is a surjective morphism onto a variety Y such that every irreducible component of X is dominant onto Y. Then we put

$$D^v = \sum_{f(D_i) \subsetneq Y} d_i D_i \quad \text{and} \quad D^h = \sum_{f(D_i) = Y} d_i D_i.$$

We call D^v (resp. D^h) the *vertical part* (resp. *horizontal part*) of D with respect to $f\colon X \to Y$.

§3. On basic slc-trivial fibrations

Roughly speaking, a basic slc-trivial fibration is a canonical bundle formula for simple normal crossing pairs. It was first introduced in [6] based on [8]. Let us start with the definition of simple normal crossing pairs.

DEFINITION 3.1. (Simple normal crossing pairs) The pair (X, B) consists of an equidimensional reduced scheme X and an $\mathbb{R}$-divisor B on X. We say that the pair (X, B) is *simple normal crossing* at a point $x \in X$ if X has a Zariski open neighborhood U of x that can be embedded in a smooth variety M, where M has a regular system of parameters $(x_1, \ldots, x_p, y_1, \ldots, y_r)$ at $x = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$B|_U = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, B) is a *simple normal crossing pair* if it is simple normal crossing at every point of X.

Let (X, B) be a simple normal crossing pair and let $\nu\colon X^\nu \to X$ be the normalization. We define B^ν by $K_{X^\nu} + B^\nu = \nu^*(K_X + B)$, that is, B^ν is the sum of the inverse images of B and the singular locus of X. Then a *stratum* of (X, B) is an irreducible component of X or the ν-image of some log canonical center of (X^ν, B^ν).

Let (X, B) be a simple normal crossing pair and let $X = \bigcup_{i\in I} X_i$ be the irreducible decomposition of X. Then a *stratum* of X means an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$. It is easy to see that W is a stratum of X if and only if W is a stratum of $(X, 0)$.

We introduce the notion of basic slc-trivial fibrations. In [6], we only treat basic $\mathbb{Q}$-slc-trivial fibrations.

Definition 3.2. (Basic slc-trivial fibrations, see [6, Definition 4.1]) A *pre-basic* $\mathbb{Q}$*-slc-trivial* (*resp.* $\mathbb{R}$*-slc-trivial*) *fibration* $f\colon (X, B) \to Y$ consists of a projective surjective morphism $f\colon X \to Y$ and a simple normal crossing pair (X, B) satisfying the following properties:

(1) Y is a normal variety,

(2) every stratum of X is dominant onto Y and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$,

(3) B is a $\mathbb{Q}$-divisor (resp. an $\mathbb{R}$-divisor) such that $B = B^{\leq 1}$ holds over the generic point of Y, and

(4) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) D on Y such that $K_X + B \sim_{\mathbb{Q}} f^*D$ (resp. $K_X + B \sim_{\mathbb{R}} f^*D$), that is, $K_X + B$ is $\mathbb{Q}$-linearly (resp. $\mathbb{R}$-linearly) equivalent to f^*D.

If a pre-basic $\mathbb{Q}$-slc-trivial (resp. $\mathbb{R}$-slc-trivial) fibration $f\colon (X, B) \to Y$ also satisfies

(5) $\operatorname{rank} f_*\mathcal{O}_X(\lceil -(B^{<1})\rceil) = 1$,

then it is called a *basic* $\mathbb{Q}$*-slc-trivial* (resp. $\mathbb{R}$*-slc-trivial*) *fibration.*

If there is no danger of confusion, we sometimes use (*pre-*)*basic slc-trivial fibrations* to denote (*pre-*)*basic* $\mathbb{Q}$*-slc-trivial fibrations* or (*pre-*)*basic* $\mathbb{R}$*-slc-trivial fibrations.*

Remark 3.3. (see Remark 4.5) The condition $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ in (2) in Definition 3.2 does not play an important role. Moreover, we have to treat the case where $\mathcal{O}_Y \subsetneq f_*\mathcal{O}_X$ in this paper. The reader can find that we do not need the condition $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ in many places in [6]. Hence it may be better to remove the condition $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ from the definition of pre-basic slc-trivial fibrations (see [6, Definition 4.1] and Definition 3.2). However, we keep it here not to cause unnecessary confusion.

Note that the condition $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ always holds for basic slc-trivial fibrations even when we remove it from the definition of pre-basic slc-trivial fibrations. We will see it more precisely. It is sufficient to see that if every stratum of X is dominant onto Y with

$\operatorname{rank} f_*\mathcal{O}_X(\lceil -(B^{<1})\rceil) = 1$ then the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ must be an isomorphism. We note that there are natural inclusions

$$\mathcal{O}_Y \hookrightarrow f_*\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_X(\lceil -(B^{<1})\rceil)$$

since $\lceil -(B^{<1})\rceil$ is effective. Hence $\mathcal{O}_Y \hookrightarrow f_*\mathcal{O}_X$ is an isomorphism over some nonempty Zariski open subset of Y and $\operatorname{rank} f_*\mathcal{O}_X = 1$ holds. We consider the Stein factorization

$$f\colon X \longrightarrow Z := \operatorname{Spec}_Y f_*\mathcal{O}_X \xrightarrow{\alpha} Y$$

of $f\colon X \to Y$. Since every irreducible component of X is dominant onto Y, Z is a variety. Moreover, $\alpha\colon Z \to Y$ is birational since $\operatorname{rank} f_*\mathcal{O}_X = 1$. By Zariski's main theorem, $\alpha\colon Z \to Y$ is an isomorphism. Hence the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

In order to define discriminant $\mathbb{R}$-b-divisors and moduli $\mathbb{R}$-b-divisors for basic slc-trivial fibrations, we need the notion of *induced (pre-)basic slc-trivial fibrations*.

Definition 3.4. (Induced (pre-)basic slc-trivial fibrations, [6, 4.3]) Let $f\colon (X,B) \to Y$ be a (pre-)basic slc-trivial fibration and let $\sigma\colon Y' \to Y$ be a generically finite surjective morphism from a normal variety Y'. Then we have an *induced (pre-)basic slc-trivial fibration* $f'\colon (X', B_{X'}) \to Y'$, where $B_{X'}$ is defined by $\mu^*(K_X+B) = K_{X'} + B_{X'}$, with the following commutative diagram:

$$\begin{tikzcd} (X', B_{X'}) \arrow[r, "\mu"] \arrow[d, "f'"'] & (X,B) \arrow[d, "f"] \\ Y' \arrow[r, "\sigma"'] & Y, \end{tikzcd}$$

where X' coincides with $X \times_Y Y'$ over a nonempty Zariski open subset of Y'. More precisely, $(X', B_{X'})$ is a simple normal crossing pair with a morphism $X' \to X \times_Y Y'$ that is an isomorphism over a nonempty Zariski open subset of Y' such that X' is projective over Y' and that every stratum of X' is dominant onto Y'.

Now we are ready to define *discriminant $\mathbb{R}$-b-divisors* and *moduli $\mathbb{R}$-b-divisors* for basic slc-trivial fibrations.

Definition 3.5. (Discriminant and moduli $\mathbb{R}$-b-divisors, [6, 4.5]) Let $f\colon (X,B) \to Y$ be a (pre-)basic slc-trivial fibration as in Definition 3.2. Let P be a prime divisor on Y. By shrinking Y around the generic point of P, we assume that P is Cartier. We set

$$b_P = \max\left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} (X^\nu, B^\nu + t\nu^* f^* P) \text{ is sub log canonical} \\ \text{over the generic point of } P \end{array} \right\},$$

where $\nu\colon X^\nu \to X$ is the normalization and $K_{X^\nu} + B^\nu = \nu^*(K_X+B)$, that is, B^ν is the sum of the inverse images of B and the singular locus of X, and set

$$B_Y = \sum_P (1-b_P)P,$$

where P runs over prime divisors on Y. Then it is easy to see that B_Y is a well-defined $\mathbb{R}$-divisor on Y and is called the *discriminant $\mathbb{R}$-divisor* of $f\colon (X,B) \to Y$. We set

$$M_Y = D - K_Y - B_Y$$

and call M_Y the *moduli* $\mathbb{R}$*-divisor* of $f\colon (X,B)\to Y$. By definition, we have

$$K_X+B\sim_{\mathbb{R}} f^*(K_Y+B_Y+M_Y).$$

Let $\sigma\colon Y'\to Y$ be a proper birational morphism from a normal variety Y' and let $f'\colon (X',B_{X'})\to Y'$ be an induced (pre-)basic slc-trivial fibration by $\sigma\colon Y'\to Y$. We can define $B_{Y'}$, $K_{Y'}$ and $M_{Y'}$ such that $\sigma^*D=K_{Y'}+B_{Y'}+M_{Y'}$, $\sigma_*B_{Y'}=B_Y$, $\sigma_*K_{Y'}=K_Y$ and $\sigma_*M_{Y'}=M_Y$. We note that $B_{Y'}$ is independent of the choice of $(X',B_{X'})$, that is, $B_{Y'}$ is well defined. Hence there exist a unique $\mathbb{R}$-b-divisor $\mathbf{B}$ such that $\mathbf{B}_{Y'}=B_{Y'}$ for every $\sigma\colon Y'\to Y$ and a unique $\mathbb{R}$-b-divisor $\mathbf{M}$ such that $\mathbf{M}_{Y'}=M_{Y'}$ for every $\sigma\colon Y'\to Y$. Note that $\mathbf{B}$ is called the *discriminant* $\mathbb{R}$*-b-divisor* and that $\mathbf{M}$ is called the *moduli* $\mathbb{R}$*-b-divisor* associated to $f\colon (X,B)\to Y$. We sometimes simply say that $\mathbf{M}$ is the *moduli part* of $f\colon (X,B)\to Y$.

Let $f\colon X\to Y$ be a proper surjective morphism from an equidimensional normal scheme X onto a normal variety Y such that every irreducible component of X is dominant onto Y. Let B be an $\mathbb{R}$-divisor on X such that K_X+B is $\mathbb{R}$-Cartier. Assume that (X,B) is sub log canonical over the generic point of Y. Let $\sigma\colon Y'\to Y$ be a generically finite surjective morphism from a normal variety Y'. Then we have the following commutative diagram:

$$\begin{tikzcd}
(X',B_{X'}) \arrow[r, "\mu"] \arrow[d, "f'"'] & (X,B) \arrow[d, "f"] \\
Y' \arrow[r, "\sigma"'] & Y,
\end{tikzcd}$$

where X' is the normalization of the main components of $X\times_Y Y'$ and $B_{X'}$ is defined by $K_{X'}+B_{X'}=\mu^*(K_X+B)$. Then we can define the discriminant $\mathbb{R}$-divisor B_Y on Y and the discriminant $\mathbb{R}$-b-divisor $\mathbf{B}$ as in Definition 3.5. Let $f\colon (X,B)\to Y$ be a (pre-)basic slc-trivial fibration and let $\nu\colon X^\nu\to X$ be the normalization with $K_{X^\nu}+B^\nu=\nu^*(K_X+B)$. Then the discriminant $\mathbb{R}$-b-divisor $\mathbf{B}$ associated to $f\colon (X,B)\to Y$ defined in Definition 3.5 obviously coincides with that of $f\circ\nu\colon (X^\nu,B^\nu)\to Y$ by definition.

Let us see the main result of [6].

THEOREM 3.6. ([6, Theorem 1.2]) *Let $f\colon (X,B)\to Y$ be a basic $\mathbb{Q}$-slc-trivial fibration and let $\mathbf{B}$ and $\mathbf{M}$ be the discriminant and moduli $\mathbb{Q}$-b-divisors associated to $f\colon (X,B)\to Y$, respectively. Then we have the following properties:*

(i) $\mathbf{K}+\mathbf{B}$ is $\mathbb{Q}$-b-Cartier, where $\mathbf{K}$ is the canonical b-divisor of Y, and

(ii) $\mathbf{M}$ is b-potentially nef, that is, there exists a proper birational morphism $\sigma\colon Y'\to Y$ from a normal variety Y' such that $\mathbf{M}_{Y'}$ is a potentially nef $\mathbb{Q}$-divisor on Y' and that $\mathbf{M}=\overline{\mathbf{M}_{Y'}}$.

The following result was established in [9].

THEOREM 3.7. ([9, Corollary 1.4]) *In Theorem 3.6, if Y is a curve, then $\mathbf{M}_Y$ is semi-ample.*

We close this section with important remarks on [6].

Remark 3.8. In (d) in [6, Section 6], we assume that $\operatorname{Supp} M_Y \subset \operatorname{Supp} \Sigma_Y$. However, this conditions is unnecessary. This is because if P is not an irreducible component of $\operatorname{Supp} \Sigma_Y$ then we can always take a prime divisor Q on V such that $\operatorname{mult}_Q(-B_V + h^*B_Y) = 0$, $h(Q) = P$, and $\operatorname{mult}_Q h^*P = 1$ (see [6, Proposition 6.3 (iv)]).

Remark 3.9. In [6, 6.1], we assume that $\operatorname{Supp}(B - f^*(B_Y + M_Y))$ is a simple normal crossing divisor on X. However, we do not need this assumption. All we need in [6, 6.1] is the fact that the support of $\{\Delta\}$ is a simple normal crossing divisor on X. We note that

$$\operatorname{Supp}\{\Delta\} \subset \operatorname{Supp}(B - f^*(B_Y + M_Y))$$

always holds since $\Delta = K_{X/Y} + B - f^*(B_Y + M_Y)$.

§4. Fundamental theorem for basic $\mathbb{Q}$-slc-trivial fibrations

In this section, we will slightly generalize the main theorem of [6] (see Theorem 3.6). The following theorem is the main result of this section.

Theorem 4.1. (see [6, Theorem 1.2]) *Let $f\colon (X, B) \to Y$ be a basic $\mathbb{Q}$-slc-trivial fibration such that Y is a smooth quasi-projective variety. We write $K_X + B \sim_{\mathbb{Q}} f^*D$. Assume that there exists a simple normal crossing divisor Σ on Y such that $\operatorname{Supp} D \subset \Sigma$ and that every stratum of $(X, \operatorname{Supp} B)$ is smooth over $Y \setminus \Sigma$. Then*

(i) $\mathbf{K} + \mathbf{B} = \overline{\mathbf{K}_Y + \mathbf{B}_Y}$ *holds, and*

(ii) $\mathbf{M}_Y$ *is a potentially nef $\mathbb{Q}$-divisor on Y with* $\mathbf{M} = \overline{\mathbf{M}_Y}$.

In Section 5, Theorem 4.1 will be generalized for basic $\mathbb{R}$-slc-trivial fibrations (see Theorems 1.8 and 5.1). We note that Theorem 4.1 is indispensable for the proof of Theorem 5.1 in Section 5. For the proof of Theorem 4.1, we prepare a lemma on simultaneous partial resolutions of singularities of pairs. Let us recall the main result of [2].

Theorem 4.2. ([2, Theorem 1.4]) *Let X be a reduced scheme, and let D be a $\mathbb{Q}$-divisor on X. Let U be the largest open subset of X such that $(U, D|_U)$ is a simple normal crossing pair. Then there is a morphism $f\colon \tilde{X} \to X$, which is a composition of blow-ups, such that*

- *the exceptional locus $\operatorname{Exc}(f)$ is of pure codimension one,*
- *putting $\tilde{D} = f_*^{-1}D + \operatorname{Exc}(f)$ then $(\tilde{X}, \tilde{D})$ is a simple normal crossing pair, and*
- *f is an isomorphism over U.*

Remark 4.3. (Functoriality, see [2, Remark 1.5 (3)]) By [2, Remark 1.5 (3)], for every reduced scheme X and a $\mathbb{Q}$-divisor D on X we may take $f_X\colon \tilde{X} \to X$ of Theorem 4.2 satisfying the following funtoriality. Suppose that we are given an étale or a smooth morphism $\phi\colon X \to Y$ of reduced schemes and $\mathbb{Q}$-divisors D_X and D_Y on X and Y respectively such that

- $\phi^*D_Y = D_X$, and
- the number of irreducible components of X (resp. $\operatorname{Supp} D_X$) at a point $x \in X$ coincides with that of Y (resp. $\operatorname{Supp} D_Y$) at $\phi(x) \in Y$ for every $x \in X$.

Then, the morphisms $f_X\colon \tilde{X}\to X$ and $f_Y\colon \tilde{Y}\to Y$ as in Theorem 4.2 form the diagram of the fiber product

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\phi}"] \arrow[d, "f_X"'] \arrow[dr, phantom, "\square"] & \tilde{Y} \arrow[d, "f_Y"] \\ X \arrow[r, "\phi"'] & Y, \end{tikzcd}$$

that is, $\tilde{X} = X \times_Y \tilde{Y}$.

The following lemma is a key lemma for the proof of Theorem 4.1.

Lemma 4.4. *Let (X, B) be a simple normal crossing pair such that B is a $\mathbb{Q}$-divisor. Let $f\colon X\to Y$ be a surjective morphism onto a smooth variety Y such that every stratum of $(X, \operatorname{Supp} B)$ is smooth over Y. We put $\Delta = K_X + B$ and assume that $b\Delta \sim 0$ for some positive integer b. We consider a b-fold cyclic cover*

$$\pi\colon \widetilde{X} = \operatorname{Spec}_X \bigoplus_{i=0}^{b-1} \mathcal{O}_X(\lfloor i\Delta\rfloor) \longrightarrow X$$

associated to $b\Delta \sim 0$. We put $K_{\widetilde{X}} + B_{\widetilde{X}} = \pi^*(K_X + B)$. Let $\widetilde{U}$ be the largest Zariski open subset of $\widetilde{X}$ such that $(\widetilde{U}, B_{\widetilde{X}}|_{\widetilde{U}})$ is a simple normal crossing pair. Then there exists a morphism $d\colon V\to \widetilde{X}$ given by a composite of blow-ups such that

(i) d is an isomorphism over $\widetilde{U}$,

(ii) (V, B_V) is a simple normal crossing pair, where $K_V + B_V = d^*(K_{\widetilde{X}} + B_{\widetilde{X}})$, and

(iii) every stratum of $(V, \operatorname{Supp} B_V)$ is smooth over Y.

Proof. Let us quickly recall the b-fold cyclic cover $\pi\colon \widetilde{X}\to X$. We fix a rational function ϕ on X such that $b\Delta = \operatorname{div}(\phi)$. As usual, we can define an $\mathcal{O}_X$-algebra structure of $\bigoplus_{i=0}^{b-1}\mathcal{O}_X(\lfloor i\Delta\rfloor)$ by $b\Delta = \operatorname{div}(\phi)$. We note that

$$\mathcal{O}_X(\lfloor i\Delta\rfloor) \times \mathcal{O}_X(\lfloor j\Delta\rfloor) \to \mathcal{O}_X(\lfloor (i+j)\Delta\rfloor)$$

is well defined for $0\le i, j\le b-1$ by $\lfloor i\Delta\rfloor + \lfloor j\Delta\rfloor \le \lfloor (i+j)\Delta\rfloor$ and that

$$\mathcal{O}_X(\lfloor (i+j)\Delta\rfloor) \simeq \mathcal{O}_X(\lfloor (i+j-b)\Delta\rfloor)$$

for $i+j\ge b$ defined by the multiplication with ϕ^{-1}. We put

$$\pi\colon \widetilde{X} = \operatorname{Spec}_X \bigoplus_{i=0}^{b-1} \mathcal{O}_X(\lfloor i\Delta\rfloor)$$

and call it a b-fold cyclic cover associated to $b\Delta\sim 0$. By construction, $\pi\colon \widetilde{X}\to X$ is étale outside $\operatorname{Supp}\{\Delta\}$. We note that $\widetilde{X}$ is normal over a neighborhood of the generic point of every irreducible component of $\operatorname{Supp}\{\Delta\}$. We also note that $(\widetilde{X}, B_{\widetilde{X}})$ is simple normal crossing in codimension one. Throughout this proof, we will freely use the following commutative diagram.

$$\begin{tikzcd}
(X,B) \arrow[d, "f"'] & (\widetilde{X}, B_{\widetilde{X}}) \arrow[l, "\pi"'] \arrow[dl, "\widetilde{f}"'] & (V, B_V) \arrow[l, "d"'] \arrow[dll, "h"] \\
Y & &
\end{tikzcd}$$

Step 1. Let U and Z be affine open neighborhood of $x \in X$ and $y = f(x) \in Y$, respectively. Without loss of generality, we may assume that U is a simple normal crossing divisor on a smooth affine variety W since (X, B) is a simple normal crossing pair. By shrinking W, U, and Z suitably, we get the following commutative diagram

$$\begin{tikzcd}
U \arrow[r, hook, "\iota"] \arrow[dr, "f|_U"'] & W \arrow[d, "p"] \\
& Z
\end{tikzcd}$$

where ι is the natural closed embedding $U \hookrightarrow W$. From now on, we will repeatedly shrink W, U, and Z suitably without mentioning it explicitly. Since every stratum of X is smooth over Y, we may assume that p is a smooth morphism between smooth affine varieties.

Step 2. Since $p \colon W \to Z$ is a smooth morphism, there exists a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "g"] \arrow[dr, "p"'] & Z \times \mathbb{C}^n \arrow[d, "p_1"] \\
& Z
\end{tikzcd}$$

where g is étale and p_1 is the first projection (see, for example, [1, Chapter VII, Definition (1.1) and Theorem (1.8)]). By choosing a coordinate system $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$ suitably and shrinking U and W if necessary, we may further assume that U is defined by a monomial

$$x_1 \cdots x_p = 0$$

on W, where $x_i = g^* z_i$ for $1 \leq i \leq p$, and

$$B|_U = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U \quad \text{with} \quad \alpha_i \in \mathbb{Q}$$

holds, where $y_i = g^* z_{p+i}$ for $1 \leq i \leq r$. Here, we used the hypothesis that every stratum of $(X, \operatorname{Supp} B)$ is smooth over Y.

Step 3. We put $L = (z_1 \cdots z_p = 0)$ in $\mathbb{C}^n$. Then we have the following commutative diagram.

$$\begin{tikzcd}
U \arrow[r, "g|_U"] \arrow[dr, "f|_U"'] & Z \times L \arrow[d, "p_1"] \\
& Z
\end{tikzcd}$$

Note that $g|_U$ is étale because it is the base change of g by $L \hookrightarrow \mathbb{C}^n$. We put

$$D = \sum_{i=1}^{r} \alpha_i(z_{p+i} = 0)$$

on $\mathbb{C}^n$. Let $p_2\colon Z \times \mathbb{C}^n \to \mathbb{C}^n$ be the second projection. Then $B|_U = g^* p_2^* D|_U$ holds.

STEP 4. Without loss of generality, we may assume that $K_Z \sim 0$ by shrinking Z suitably. Then $K_U \sim 0$ holds. Hence, by using the second projection $p_2\colon Z \times \mathbb{C}^n \to \mathbb{C}^n$, we have

$$0 \sim b\Delta = b(K_U + B|_U) \sim bg|_U^* \big(p_2^* D|_{Z\times L}\big).$$

Since $g|_U$ is étale, we see that all the coefficients of $bp_2^* D|_{Z\times L}$ are integers. Since p_2 is the second projection and $D+L$ is a simple normal crossing divisor on $\mathbb{C}^n$, all the coefficients of bD are integers. Therefore, we have $bD \sim 0$. We fix a rational function σ on $\mathbb{C}^n$ such that $bD = \operatorname{div}(\sigma)$. We consider the b-fold cyclic cover $\alpha\colon M \to \mathbb{C}^n$ associated to $bD = \operatorname{div}(\sigma)$. We put $N = \alpha^{-1}L$. We define B_N by $K_N + B_N = (\alpha|_N)^*(K_L + D|_L)$ and put $B_{Z\times N} = p_2^* B_N$, where $p_2\colon Z \times N \to N$ is the second projection. Then we get the following commutative diagram:

$$\begin{tikzcd}
U' \arrow[r, "g'"] \arrow[d] & Z \times N \arrow[r, "p_2"] \arrow[d, "id_Z \times (\alpha|_N)"] & N \arrow[r, hook] \arrow[d, "\alpha|_N"] & M \arrow[d, "\alpha"] \\
U \arrow[r, "g|_U"] \arrow[dr, "\widetilde{f}"'] & Z \times L \arrow[r, "p_2"'] \arrow[d, "p_1"] & L \arrow[r, hook] & \mathbb{C}^n \\
 & Z & &
\end{tikzcd}$$

where $g'\colon U' \to Z \times N$ is the base change of $g|_U\colon U \to Z \times L$ by $id_Z \times (\alpha|_N)$. We put $B_{U'} = g'^* B_{Z\times N}$. Then $K_{U'} + B_{U'}$ is equal to the pullback of $K_U + B|_U$ to U'.

STEP 5. Since $\alpha\colon M \to \mathbb{C}^n$ is the b-fold cyclic cover associated to $bD = \operatorname{div}(\sigma)$, we see that

$$M = \operatorname{Spec}_{\mathbb{C}^n} \bigoplus_{i=0}^{b-1} \mathcal{O}_{\mathbb{C}^n}(\lfloor iD \rfloor).$$

Since $p_2 \circ g\colon W \to Z \times \mathbb{C}^n \to \mathbb{C}^n$ is the composition of an étale morphism and the second projection, we have $g^* p_2^* \lfloor iD \rfloor|_U = \lfloor ig^* p_2^* D \rfloor|_U = \lfloor iB \rfloor|_U = \lfloor iB|_U \rfloor$, where the last equality follows from that $(U, B|_U)$ is a simple normal crossing pair. Let σ_U be a rational function on U which is the pullback of σ. Then $bB|_U = \operatorname{div}(\sigma_U)$ because we have $bD = \operatorname{div}(\sigma)$. By the construction of $U' \to U$, we see that

$$U' = \operatorname{Spec}_U \bigoplus_{i=0}^{b-1} \mathcal{O}_U(\lfloor iB|_U \rfloor)$$

and $U' \to U$ is the b-fold cyclic cover associated to $bB|_U = \operatorname{div}(\sigma_U)$.

We recall that $\Delta = K_X + B$ and $\widetilde{X} \to X$ is the b-fold cyclic cover associated to $b\Delta = \operatorname{div}(\phi)$. We put ϕ_U as the restriction of ϕ to U. Then, the morphism $\pi^{-1}(U) \to U$ is the b-fold cyclic cover associated to $b\Delta|_U = \operatorname{div}(\phi_U)$. Now $\Delta|_U - B|_U$ is a Cartier divisor on U and $b(\Delta|_U - B|_U) = \operatorname{div}(\phi_U \cdot \sigma_U^{-1})$. With this relation, we construct a b-fold cyclic cover

$\tau\colon \overline{U} \to U$. Then τ is étale, $\tau^*(\Delta|_U - B|_U)$ is Cartier and $\tau^*(\Delta|_U - B|_U) \sim 0$. So there exists a rational function ξ on $\overline{U}$ such that $\xi^b = \tau^*(\phi_U \cdot \sigma_U^{-1})$, equivalently, $\tau^*(\Delta|_U - B|_U) = \mathrm{div}(\xi)$. From this, the b-fold cyclic cover $U_1^\dagger \to \overline{U}$ associated to $b\tau^*\Delta|_U = \mathrm{div}(\tau^*\phi_U)$ is isomorphic to the b-fold cyclic cover $U_2^\dagger \to \overline{U}$ associated to $b\tau^*B|_U = \mathrm{div}(\tau^*\sigma_U) = \mathrm{div}(\tau^*\phi_U \cdot \xi^{-b})$. Since $\tau\colon \overline{U} \to U$ is étale, the construction of $U_2^\dagger$ shows that $U_2^\dagger \to \overline{U}$ is the base change of $U' \to U$ by $\overline{U} \to U$. Similarly, we see that $U_1^\dagger \to \overline{U}$ is the base change of $\pi^{-1}(U) \to U$ by $\overline{U} \to U$.

$$\begin{tikzcd} U_2^\dagger \arrow[r, "a_2"] \arrow[d] \arrow[dr, phantom, "\Box"] & U' \arrow[d] \\ \overline{U} \arrow[r, "\tau"'] & U \end{tikzcd} \qquad \begin{tikzcd} U_1^\dagger \arrow[r, "a_1"] \arrow[d] \arrow[dr, phantom, "\Box"] & \pi^{-1}(U) \arrow[d] \\ \overline{U} \arrow[r, "\tau"'] & U \end{tikzcd}$$

We put $a_1\colon U_1^\dagger \to \pi^{-1}(U)$ and $a_2\colon U_2^\dagger \to U'$. By construction, a_1 and a_2 are étale. We see that the composition $U_1^\dagger \to \pi^{-1}(U) \to U$ is isomorphic to the composition $U_2^\dagger \to U' \to U$ by construction. By this isomorphism, we obtain that $a_1^*(B_{\widetilde{X}}|_{\pi^{-1}(U)})$ is isomorphic to $a_2^* B_{U'}$.

In this way, there exist étale morphisms $a\colon U^\dagger \to \pi^{-1}(U)$ and $a'\colon U^\dagger \to U'$ over Z such that $U_1^\dagger \simeq U^\dagger \simeq U_2^\dagger$ with the following commutative diagram:

$$\begin{tikzcd} & U^\dagger \arrow[dl, "a"'] \arrow[dr, "a'"] & \\ \pi^{-1}(U) \arrow[d, "\pi"'] & & U' \arrow[d] \\ U \arrow[rr, equal] & & U \end{tikzcd}$$

such that $a^*(B_{\widetilde{X}}|_{\pi^{-1}(U)}) = a'^* B_{U'}$.

Step 6. We apply [2, Theorem 1.4] (see Theorem 4.2) to the pair $(\widetilde{X}, B_{\widetilde{X}})$. Then we obtain a morphism $d\colon V \to \widetilde{X}$ given by a composite of blow-ups satisfying (i) and (ii). Hence, all we have to do is to check that $d\colon V \to \widetilde{X}$ satisfies (iii).

Step 7. Recall that $B_{Z\times N} = p_2^* B_N$, where $p_2\colon Z \times N \to N$, and $B_{U'} = g'^* B_{Z\times N}$. Recall also the relation $a^*(B_{\widetilde{X}}|_{\pi^{-1}(U)}) = a'^* B_{U'}$. We apply [2, Theorem 1.4] (see Theorem 4.2) to N and B_N, and we obtain a morphism $\beta\colon N' \to N$ given by a composite of blow-ups. We apply [2, Theorem 1.4] again to $Z \times N$ and $B_{Z\times N}$. Then we get a morphism $id_Z \times \beta\colon Z \times N' \to Z \times N$ by the functoriality of [2, Theorem 1.4] (see Remark 4.3). We put $\widehat{V} = d^{-1}(\pi^{-1}(U)) \subset V$, and we apply [2, Theorem 1.4] to the pair of $U^\dagger$ and $a^*(B_{\widetilde{X}}|_{\pi^{-1}(U)})$, and the pair of U' and $B_{U'}$. Then we obtain morphisms $V^\dagger \to U^\dagger$ and $V' \to U'$.

We check that we may apply the functoriality (see Remark 4.3) to the morphisms

$$g'\colon U' \to Z \times N,\ a'\colon U^\dagger \to U', \text{ and } a\colon U^\dagger \to \pi^{-1}(U)$$

(see the diagram in the next paragraph) and divisors

$$B_{Z\times N} \text{ on } Z \times N,\ B_{U'} \text{ on } U', \text{ and } B_{\widetilde{X}}|_{\pi^{-1}(U)} \text{ on } \pi^{-1}(U) \text{ and their pullbacks.}$$

We only check the second condition of Remark 4.3 for schemes because the case of divisors can be proved by the same way. By construction, g' is the base change of $g|_U\colon U \to Z \times L$ by the morphism $Z \times N \to Z \times L$. Because $Z \times L$ is a simple normal crossing divisor on $Z \times \mathbb{C}^n$ and $g|_U$ is étale, By arguing locally, we see that $g|_U$ satisfies the second condition of Remark 4.3. Then so does g' since g' is constructed by the base change of $g|_U$. Similarly, a' (resp. a) is constructed with the base change of $\tau\colon \overline{U} \to U$ by $U' \to U$ (resp. $\pi^{-1}(U) \to U$), and U is a simple normal crossing divisor on W. Thus, the same argument as above implies that a' and a satisfy the second condition of Remark 4.3. Thus, we may apply the functoriality (see Remark 4.3) to the above morphisms and divisors.

Applying the functoriality (see Remark 4.3), we have the following diagram:

$$\begin{tikzcd}
\widehat{V} \arrow[d, "d|_{\widehat{V}}"'] \arrow[dr, phantom, "\square"] & V^{\dagger} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \times N' \arrow[d] \\
\pi^{-1}(U) & U^{\dagger} \arrow[l, "a"] \arrow[r, "a'"'] & U' \arrow[r, "g'"'] & Z \times N,
\end{tikzcd}$$

where each square is the fiber product. By construction, all the upper horizontal morphisms are étale. Let $B_{V^{\dagger}}$ (resp. $B_{\widehat{V}}$) be the sum of the birational transform of $a'^{*}B_{U'}$ (resp. $B_{\widetilde{X}}|_{\pi^{-1}(U)}$) and the exceptional locus of $V^{\dagger} \to U^{\dagger}$ (resp. $\widehat{V} \to \pi^{-1}(U)$). Then, every stratum of $(V^{\dagger}, \operatorname{Supp} B_{V^{\dagger}})$ is smooth over Z. Since $V^{\dagger} \to Z$ is smooth and $V^{\dagger} \to \widehat{V}$ is étale, we see that $\widehat{V} \to Z$ is smooth. By a similar argument, we see that every stratum of $(\widehat{V}, \operatorname{Supp} B_{\widehat{V}})$ is smooth over Z. This implies that $d\colon V \to \widetilde{X}$ satisfies (iii).

We finish the proof of Lemma 4.4.

Before we start the proof of Theorem 4.1, we make an important remark on [6].

Remark 4.5. (see Remark 3.3) In Theorem 4.1, we can write

$$K_X + B + \frac{1}{b}\operatorname{div}(\varphi) = f^{*}D$$

for some positive integer b and a rational function $\varphi \in \Gamma(X, \mathcal{K}_X^{*})$, where $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$ and $\mathcal{K}_X^{*}$ denotes the sheaf of invertible elements in $\mathcal{K}_X$, such that $b(K_X + B - f^{*}D) \sim 0$. In general, b is larger than $b(F, B_F)$ in [6, Section 6]. We take a b-fold cyclic cover $\pi\colon \widetilde{X} \to X$ associated to $b\Delta \sim 0$, where $\Delta = K_X + B - f^{*}D$, as in [6, Section 6]. Then the general fiber of $h\colon V \to Y$ is not necessarily connected in [6, Section 6]. Moreover, V is not necessarily connected. This means that [6, Proposition 6.3 (ii)] does not hold true since the natural map $\mathcal{O}_Y \to h_{*}\mathcal{O}_V$ is not always an isomorphism. Fortunately, the condition $h_{*}\mathcal{O}_V \simeq \mathcal{O}_Y$ is not necessary for the proof of the other properties of [6, Proposition 6.3]. We note that the condition $h_{*}\mathcal{O}_V \simeq \mathcal{O}_Y$ is unnecessary in [6, Lemma 7.3 and Theorem 8.1]. Hence it may be better to remove the condition $f_{*}\mathcal{O}_X \simeq \mathcal{O}_Y$ from (2) in Definition 3.2.

Let b be the smallest positive integer such that $b(K_X + B - f^{*}D) \sim 0$. Then we can write

$$K_X + B + \frac{1}{b}\operatorname{div}(\varphi) = f^{*}D.$$

As usual, we consider the b-fold cyclic cover $\pi\colon \widetilde{X} \to X$ associated to $b\Delta = \operatorname{div}(\varphi^{-1})$, where $\Delta = K_X + B - f^*D$. Let $b^\sharp$ be any positive integer with $b^\sharp \geq 2$. We put $\varphi^\sharp = \varphi^{b^\sharp}$. Then we get

$$K_X + B + \frac{1}{bb^\sharp}\operatorname{div}(\varphi^\sharp) = f^*D.$$

Let $\pi^\sharp\colon X^\sharp \to X$ be the $bb^\sharp$-fold cyclic cover associated to $bb^\sharp\Delta = \operatorname{div}\left((\varphi^\sharp)^{-1}\right)$. We take the H-invariant part of $\pi^\sharp\colon X^\sharp \to X$, where H is the subgroup of the Galois group $\operatorname{Gal}(X^\sharp/X) \simeq \mathbb{Z}/bb^\sharp\mathbb{Z}$ of $\pi^\sharp\colon X^\sharp \to X$ corresponding to $b\mathbb{Z}/bb^\sharp\mathbb{Z}$. Then we can recover $\pi\colon \widetilde{X} \to X$. Note that $\pi^\sharp\colon X^\sharp \to X$ decomposes into $b^\sharp$ components and that each component is isomorphic to $\pi\colon \widetilde{X} \to X$.

Let us prove Theorem 4.1.

Proof of Theorem 4.1. Here, we only explain how to modify the proof of [6, Theorem 1.2] by using Lemma 4.4.

By taking a completion as in [6, Lemma 4.12], we may further assume that Y is projective. By Lemma 4.4, we can construct a commutative diagram (6.4) in [6, Section 6] satisfying (a)–(g) such that $\Sigma_Y = \Sigma$ holds without taking birational modifications of Y. Here, we do not require the condition $\operatorname{Supp} M_Y \subset \operatorname{Supp} \Sigma_Y$ in (d) in [6, Section 6] (see Remark 3.8). We also do not require the condition that the general fiber of $h\colon V \to Y$ is connected (see Remark 4.5). The covering arguments and [6, Proposition 6.3] work without any modifications. We note that Y is a smooth projective variety. In what follows, we apply the proof of [6, Theorem 8.1]. Let $\gamma\colon Y' \to Y$ be a projective birational morphism from a normal variety Y'. By replacing Y' with a higher model if necessary, we may assume that Y' is smooth and that $\gamma^{-1}\Sigma_Y$ is a simple normal crossing divisor on Y'. With [6, Lemma 7.3], we construct $\tau\colon \overline{Y} \to Y$ a unipotent reduction of the local monodromies around Σ_Y. Then the induced fibration over $\overline{Y}$ satisfies [6, Proposition 6.3 (iv), (v)]. As in the proof of [6, Theorem 8.1], we get a diagram:

$$\begin{tikzcd} \overline{Y} \arrow[d, "\tau"'] & \overline{Y}' \arrow[l, "\gamma'"'] \arrow[d, "\tau'"] \\ Y & Y' \arrow[l, "\gamma"] \end{tikzcd}$$

such that τ' is finite and the induced fibration over $\overline{Y}'$ satisfies [6, Proposition 6.3 (iv), (v)]. By [6, Theorem 3.1], we see that $\mathbf{M}_{\overline{Y}}$ is a nef Cartier divisor and $\gamma'^*\mathbf{M}_{\overline{Y}} = \mathbf{M}_{\overline{Y}'}$. Moreover, we have $\tau^*\mathbf{M}_Y = \mathbf{M}_{\overline{Y}}$ and $\tau'^*\mathbf{M}_{Y'} = \mathbf{M}_{\overline{Y}'}$ because τ and τ' are both finite (see [6, Lemma 4.10]). Thus, we have that $\mathbf{M}_Y$ is a nef $\mathbb{Q}$-divisor and $\gamma^*\mathbf{M}_Y = \mathbf{M}_{Y'}$. This is Theorem 4.1 (ii). Theorem 4.1 (i) immediately follows from Theorem 4.1 (ii). So we are done.

§5. Fundamental theorem for basic $\mathbb{R}$-slc-trivial fibrations

In this section, we will establish the following fundamental theorem for basic $\mathbb{R}$-slc-trivial fibrations.

Theorem 5.1. (see Theorem 1.8) *Let $f\colon (X, B) \to Y$ be a basic $\mathbb{R}$-slc-trivial fibration such that Y is a smooth quasi-projective variety. We write $K_X + B \sim_{\mathbb{R}} f^*D$. Assume that there exists a simple normal crossing divisor Σ on Y such that* $\operatorname{Supp} D \subset \Sigma$ *and that every stratum of* $(X, \operatorname{Supp} B)$ *is smooth over $Y \setminus \Sigma$. Then*

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(i) $\mathbf{K}+\mathbf{B}=\overline{\mathbf{K}_Y+\mathbf{B}_Y}$ *holds, and*

(ii) $\mathbf{M}_Y$ *is a potentially nef* $\mathbb{R}$*-divisor on* Y *with* $\mathbf{M}=\overline{\mathbf{M}_Y}$.

By Theorem 5.1, which is obviously a generalization of Theorem 4.1, we can use the theory of basic slc-trivial fibrations in [6] and [7] for $\mathbb{R}$-divisors. The following formulation may be useful. Hence we state it explicitly here for the reader's convenience. We note that if $f\colon (X,B)\to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration then Corollary 5.2 is nothing but [6, Theorem 1.2].

Corollary 5.2. ([6, Theorem 1.2]) *Let* $f\colon (X,B)\to Y$ *be a basic* $\mathbb{R}$*-slc-trivial fibration and let* $\mathbf{B}$ *and* $\mathbf{M}$ *be the discriminant and moduli* $\mathbb{R}$*-b-divisors associated to* $f\colon (X,B)\to Y$*, respectively. Then we have the following properties:*

(i) $\mathbf{K}+\mathbf{B}$ *is* $\mathbb{R}$*-b-Cartier, where* $\mathbf{K}$ *is the canonical b-divisor of* Y*, and*

(ii) $\mathbf{M}$ *is b-potentially nef, that is, there exists a proper birational morphism* $\sigma\colon Y'\to Y$ *from a normal variety* Y' *such that* $\mathbf{M}_{Y'}$ *is a potentially nef* $\mathbb{R}$*-divisor on* Y' *and that* $\mathbf{M}=\overline{\mathbf{M}_{Y'}}$ *holds.*

Remark 5.3. (see [9, Corollary 1.4]) In Theorem 5.1 and Corollary 5.2, we can easily see that $\mathbf{M}_Y$ is semi-ample when Y is a curve by Theorem 3.7 and Lemma 5.4 below.

Let us start with an easy lemma.

Lemma 5.4. *Let* $f\colon (X,B)\to Y$ *be a basic* $\mathbb{R}$*-slc-trivial fibration with* $K_X+B\sim_{\mathbb{R}} f^*D$*. Then there exist a* $\mathbb{Q}$*-divisor* B_i *on* X*, a* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor* D_i *on* Y*, and a positive real number* r_i *for* $1\le i\le k$ *such that*

(1) $\sum_{i=1}^k r_i=1$ *with* $\sum_{i=1}^k r_iB_i=B$ *and* $\sum_{i=1}^k r_iD_i=D$,

(2) $\operatorname{Supp} B=\operatorname{Supp} B_i$, $\lfloor B^{>1}\rfloor=\lfloor B_i^{>1}\rfloor$*, and* $\lceil -(B^{<1})\rceil=\lceil -(B_i^{<1})\rceil$ *hold for every* i,

(3) if $\operatorname{coeff}_S(B)\in\mathbb{Q}$ *for a prime divisor* S *on* X*, then* $\operatorname{coeff}_S(B)=\operatorname{coeff}_S(B_i)$ *holds for every* i,

(4) $\operatorname{Supp} D=\operatorname{Supp} D_i$ *holds for every* i,

(5) if $\operatorname{coeff}_T(D)\in\mathbb{Q}$ *for a prime divisor* T *on* Y*, then* $\operatorname{coeff}_T(D)=\operatorname{coeff}_T(D_i)$ *holds for every* i*, and*

(6) $K_X+B_i\sim_{\mathbb{Q}} f^*D_i$ *holds for every* i.

In particular, $f\colon (X,B_i)\to Y$ *is a basic* $\mathbb{Q}$*-slc-trivial fibration with* $K_X+B_i\sim_{\mathbb{Q}} f^*D_i$ *for every* i*. Moreover, if* $t_1,\dots,t_k$ *are real numbers such that* $0\le t_i\le 1$ *for every* i *with* $\sum_{i=1}^k t_i=1$*, then* $f\colon \left(X,\sum_{i=1}^k t_iB_i\right)\to Y$ *is a basic* $\mathbb{R}$*-slc-trivial fibration with* $K_X+\sum_{i=1}^k t_iB_i\sim_{\mathbb{R}} f^*\left(\sum_{i=1}^k t_iD_i\right)$.

Proof. The proof of [6, Lemma 11.1] works with some suitable minor modifications. Therefore, we can take B_i, D_i, and r_i for $1 \leq i \leq k$ satisfying (1)–(6). By (2), $B_i = B_i^{\leq 1}$ holds over the generic point of Y for every i. By (2) again, $\operatorname{rank} f_*\mathcal{O}_X(\lceil -(B_i^{<1})\rceil) = \operatorname{rank} f_*\mathcal{O}_X(\lceil -(B^{<1})\rceil) = 1$. Hence $f\colon (X, B_i) \to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration with $K_X + B_i \sim_{\mathbb{Q}} f^*D_i$ for every i. We put $\widetilde{B} = \sum_{i=1}^k t_iB_i$. Then $\widetilde{B} = \widetilde{B}^{\leq 1}$ holds over the generic point of Y by (2). By (2) again, we see that $\lceil -(\widetilde{B}^{<1})\rceil = \lceil -(B^{<1})\rceil$ holds. Therefore, $f\colon (X, \widetilde{B}) \to Y$ is a basic $\mathbb{R}$-slc-trivial fibration.

We also need the following lemma.

Lemma 5.5. *Let $f\colon (X, B) \to Y$ be a basic $\mathbb{R}$-slc-trivial fibration. Let $\mathbf{B}$ denote the discriminant $\mathbb{R}$-b-divisor associated to $f\colon (X, B) \to Y$. Suppose that there are $\mathbb{Q}$-divisors $B_1, \ldots, B_k$ on X and real numbers $r_1, \ldots, r_k$ such that $\sum_{i=1}^k r_i = 1$ and $\sum_{i=1}^k r_iB_i = B$. We put*

$$\mathcal{P} = \left\{ \sum_{i=1}^k t_iB_i \,\middle|\, 0 \leq t_i \leq 1 \text{ for every } i \text{ with } \sum_{i=1}^k t_i = 1 \right\}.$$

Assume that $f\colon (X, \Delta) \to Y$ has the structure of a basic $\mathbb{R}$-slc-trivial fibration for every $\Delta \in \mathcal{P}$. For $\Delta \in \mathcal{P}$, $\mathbf{B}^{\Delta}$ denotes the discriminant $\mathbb{R}$-b-divisor of the basic $\mathbb{R}$-slc-trivial fibration $f\colon (X, \Delta) \to Y$. Then, we can find $\Delta_1, \ldots, \Delta_l \in \mathcal{P}$ which are $\mathbb{Q}_{\geq 0}$-linear combinations of $B_1, \ldots, B_k$ and positive real numbers $s_1, \ldots, s_l$ such that

- *$\sum_{j=1}^l s_j = 1$ and $\sum_{j=1}^l s_j\Delta_j = B$, and*
- *$\mathbf{B}_Y = \sum_{j=1}^l s_j\mathbf{B}_Y^{\Delta_j}$.*

Here, $\mathbf{B}_Y$ (resp. $\mathbf{B}_Y^{\Delta_j}$) is the trace of the discriminant $\mathbb{R}$-b-divisor $\mathbf{B}$ (resp. $\mathbf{B}^{\Delta_j}$) on Y.

Proof. Since $\mathbf{B}$ is an $\mathbb{R}$-b-divisor, it is sufficient to prove the lemma for a resolution of $Y' \to Y$ and the induced basic slc-trivial fibrations $f'\colon (X', B_{X'}) \to Y$ and $f'\colon (X', (B_i)_{X'}) \to Y'$. Moreover, by Definition 3.5 and taking the normalization of X, we may assume that X is a disjoint union of smooth varieties. Therefore, by replacing X, Y, B, and B_i, we may assume that Y is smooth and there are simple normal crossing divisors Σ_X on X and Σ_Y on Y such that

- $\operatorname{Supp} B \subset \Sigma_X$ and $\operatorname{Supp} B_i \subset \Sigma_X$ for every i,
- $\Sigma_X^v \subset f^{-1}\Sigma_Y \subset \Sigma_X$, where Σ_X^v is the vertical part of Σ_X,
- f is smooth over $Y \setminus \Sigma_Y$, and
- Σ_X is relatively simple normal crossing over $Y \setminus \Sigma_Y$.

Then it is clear that $\operatorname{Supp} \mathbf{B}_Y \subset \Sigma_Y$ and $\operatorname{Supp} \mathbf{B}_Y^{\Delta} \subset \Sigma_Y$ for all $\Delta \in \mathcal{P}$. We consider a rational convex polytope

$$\mathcal{C} = \left\{ \boldsymbol{v} = (v_1, \ldots, v_k) \in [0,1]^k \,\middle|\, \sum_j v_j = 1 \right\} \subset [0,1]^k.$$

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Then we may identify $\mathcal{C}$ with $\mathcal{P}$ by putting $\Delta_{\boldsymbol{v}} = \sum_i v_i B_i \in \mathcal{P}$ for $\boldsymbol{v} = (v_1, \ldots, v_k) \in \mathcal{C}$. We define $\boldsymbol{v}_0 \in \mathcal{C}$ to be the point such that $\Delta_{\boldsymbol{v}_0} = B$.

Fix a prime divisor Q on Y which is a component of Σ_Y. We shrink Y near the generic point of Q so that all components of f^*Q dominate Q. We can write $f^*Q = \sum_i m_{P_i} P_i$, where P_i are components of Σ_X such that $f(P_i) = Q$, and $m_{P_i} = \mathrm{coeff}_{P_i}(f^*Q)$. We fix a component $P_{(B,Q)}$ of f^*Q such that

$$\frac{1 - \mathrm{coeff}_{P_{(B,Q)}}(B)}{m_{P_{(B,Q)}}} = \min_{P_i} \left\{ \frac{1 - \mathrm{coeff}_{P_i}(B)}{m_{P_i}} \right\}.$$

Note that $\frac{1-\mathrm{coeff}_{P_{(B,Q)}}(B)}{m_{P_{(B,Q)}}}$ is the log canonical threshold of (X, B) with respect to f^*Q over the generic point of Q because $(X, B + \mu f^*Q)$ is sub log canonical over the generic point of Q if and only if $\mathrm{coeff}_{P_i}(B) + \mu m_{P_i} \leq 1$ for all P_i. For every component P_i of f^*Q, we can define a function

$$H^{(P_i)}(\boldsymbol{v}) := \frac{1 - \mathrm{coeff}_{P_{(B,Q)}}(\Delta_{\boldsymbol{v}})}{m_{P_{(B,Q)}}} - \frac{1 - \mathrm{coeff}_{P_i}(\Delta_{\boldsymbol{v}})}{m_{P_i}}$$

and the half space

$$H^{(P_i)}_{\leq 0} := \{\boldsymbol{v} \in \mathcal{C} \mid H^{(P_i)}(\boldsymbol{v}) \leq 0\}.$$

It is easy to check that $H^{(P_i)}$ are rational affine functions and the half spaces $H^{(P_i)}_{\leq 0}$ contain $\boldsymbol{v}_0$ since $\boldsymbol{v}_0$ is the point such that $\Delta_{\boldsymbol{v}_0} = B$. Therefore, the set

$$\mathcal{C}_Q := \mathcal{C} \cap \left(\bigcap_{P_i} H^{(P_i)}_{\leq 0} \right)$$

is a rational polytope in $\mathcal{C}$ containing $\boldsymbol{v}_0$, where P_i runs over components of f^*Q. We put

$$\begin{aligned} t(\Delta_{\boldsymbol{v}}, Q) :=& 1 - \mathrm{coeff}_Q \left(\mathbf{B}_Y^{\Delta_{\boldsymbol{v}}} \right) \\ =& \sup\{\mu \in \mathbb{R} \,|\, (X, \Delta_{\boldsymbol{v}} + \mu f^*Q) \text{ is sub log canonical over the generic point of } Q\}. \end{aligned}$$

Then, by the definitions of $H^{(P_i)}_{\leq 0}$, every $\boldsymbol{v} \in \mathcal{C}_Q$ satisfies

$$t(\Delta_{\boldsymbol{v}}, Q) = \min_{P_i} \left\{ \frac{1 - \mathrm{coeff}_{P_i}(\Delta_{\boldsymbol{v}})}{m_{P_i}} \right\} = \frac{1 - \mathrm{coeff}_{P_{(B,Q)}}(\Delta_{\boldsymbol{v}})}{m_{P_{(B,Q)}}}. \tag{1}$$

Here, to prove the first equality we used the fact that $(X, \Delta_{\boldsymbol{v}} + \mu f^*Q)$ is sub log canonical over the generic point of Q if and only if $\mathrm{coeff}_{P_i}(\Delta_{\boldsymbol{v}}) + \mu m_{P_i} \leq 1$ for all P_i.

Finally, we define

$$\mathcal{C}' := \bigcap_Q \mathcal{C}_Q,$$

where Q runs over all irreducible components of Σ_Y. It is easy to see that $\mathcal{C}'$ is a rational polytope in $\mathcal{C}$ and $\mathcal{C}'$ contains $\boldsymbol{v}_0$. Thus, we can find rational points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l$ and positive real numbers $s_1, \ldots, s_l$ such that $\sum_{j=1}^{l} s_j = 1$ and $\sum_{j=1}^{l} s_j \boldsymbol{v}_j = \boldsymbol{v}_0$. We put $\Delta_j = \Delta_{\boldsymbol{v}_j}$ for each $1 \leq j \leq l$. Then $B = \Delta_{\boldsymbol{v}_0} = \sum_{j=1}^{l} s_j \Delta_j$. For every component Q of Σ_Y, the equation (1) implies that

$$\begin{aligned}
t(B, Q) &= \frac{1 - \mathrm{coeff}_{P_{(B,Q)}}(B)}{m_{P_{(B,Q)}}} && \text{(see (1))} \\
&= \frac{1 - \mathrm{coeff}_{P_{(B,Q)}}\left(\sum_{j=1}^{l} s_j \Delta_j\right)}{m_{P_{(B,Q)}}} && (B = \textstyle\sum_{j=1}^{l} s_j \Delta_j) \\
&= \textstyle\sum_{j=1}^{l} s_j \cdot \dfrac{1 - \mathrm{coeff}_{P_{(B,Q)}}(\Delta_j)}{m_{P_{(B,Q)}}} && (\textstyle\sum_{j=1}^{l} s_j = 1) \\
&= \textstyle\sum_{j=1}^{l} s_j \cdot t(\Delta_j, Q) && \text{(see (1))}.
\end{aligned}$$

Since $t(\Delta_j, Q) = 1 - \mathrm{coeff}_Q\left(\mathbf{B}_Y^{\Delta_j}\right)$ for every $1 \leq j \leq l$ and every irreducible component Q of Σ_Y, we see that $\mathbf{B}_Y = \sum_{j=1}^{l} s_j \mathbf{B}_Y^{\Delta_j}$.

We are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Fix any projective birational morphism $\sigma\colon Y' \to Y$ from a normal quasi-projective variety Y', and let

$$\begin{tikzcd}
(X', B_{X'}) \arrow[r, "\mu"] \arrow[d, "f'"'] & (X, B) \arrow[d, "f"] \\
Y' \arrow[r, "\sigma"'] & Y
\end{tikzcd}$$

be the induced basic $\mathbb{R}$-slc-trivial fibration (see Definition 3.4). It is sufficient to show that $\sigma^*(K_Y + \mathbf{B}_Y) = K_{Y'} + \mathbf{B}_{Y'}$ and $\mathbf{M}_Y$ is a potentially nef $\mathbb{R}$-divisor on Y with $\sigma^*\mathbf{M}_Y = \mathbf{M}_{Y'}$.

We pick $\mathbb{Q}$-divisors $B_1, \ldots, B_k$ on X, $\mathbb{Q}$-divisors $D_1, \ldots, D_k$ on Y and positive real numbers $r_1, \ldots, r_k$ as in Lemma 5.4. Then, the following properties hold.

- $\sum_{i=1}^{k} r_i = 1$ with $\sum_{i=1}^{k} r_i B_i = B$ and $\sum_{i=1}^{k} r_i D_i = D$,
- $\mathrm{Supp}\, B = \mathrm{Supp}\, B_i$ and $\mathrm{Supp}\, D = \mathrm{Supp}\, D_i$ hold for every i, and
- $K_X + B_i \sim_{\mathbb{Q}} f^*D_i$ holds for every i.

We put $D'_i = \sigma^* D_i$ and we define B'_i by $K_{X'} + B'_i = \mu^*(K_X + B_i)$ for any $1 \leq i \leq k$. Then $f'\colon (X', B'_i) \to Y'$ are basic $\mathbb{Q}$-slc-trivial fibrations with $K_{X'} + B'_i \sim_{\mathbb{Q}} f'^* D'_i$. As in Lemma 5.5, we put

$$\mathcal{P}' = \left\{ \sum_{i=1}^{k} t_i B'_i \;\middle|\; 0 \leq t_i \leq 1 \text{ for every } i \text{ with } \sum_{i=1}^{k} t_i = 1 \right\}.$$

We may assume that $f'\colon (X', \Delta) \to Y'$ is a basic $\mathbb{R}$-slc-trivial fibration for every $\Delta \in \mathcal{P}$. We define $\mathcal{P}'_{\mathbb{Q}}$ by

$$\mathcal{P}'_{\mathbb{Q}} := \left\{ \sum_{i=1}^{k} t_i B'_i \;\middle|\; t_i \in \mathbb{Q} \text{ and } 0 \leq t_i \leq 1 \text{ for every } i \text{ with } \sum_{i=1}^{k} t_i = 1 \right\}.$$

Note that $B_{X'} \in \mathcal{P}'$.

Pick any $\Delta = \sum_{i=1}^{k} t_i B'_i \in \mathcal{P}'_{\mathbb{Q}}$. Since $\mu_* B'_i = B_i$, we have $\mu_*\Delta = \sum_{i=1}^{k} t_i B_i$ such that $t_i \in \mathbb{Q}$. Therefore, the morphism $f\colon (X, \mu_*\Delta) \to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration such that $K_X + \mu_*\Delta \sim_{\mathbb{Q}} f^*\big(\sum_{i=1}^{k} t_i D_i\big)$. Let $\mathbf{B}^{\Delta}$ and $\mathbf{M}^{\Delta}$ be the discriminant $\mathbb{Q}$-b-divisor and the moduli $\mathbb{Q}$-b-divisor of the basic $\mathbb{Q}$-slc-trivial fibration $f\colon (X, \mu_*\Delta) \to Y$, respectively. Because we have $\operatorname{Supp}\big(\sum_{i=1}^{k} t_i D_i\big) \subset \operatorname{Supp} D$ and $\operatorname{Supp} \mu_*\Delta \subset \operatorname{Supp} B$, we may apply Theorem 4.1. Therefore, for every $\Delta \in \mathcal{P}'_{\mathbb{Q}}$ it follows that $\sigma^*(K_Y + \mathbf{B}^{\Delta}_Y) = K_{Y'} + \mathbf{B}^{\Delta}_{Y'}$ and $\mathbf{M}^{\Delta}_Y$ is a potentially nef $\mathbb{Q}$-divisor on Y with $\sigma^*\mathbf{M}^{\Delta}_Y = \mathbf{M}^{\Delta}_{Y'}$. It also follows from the construction that $f'\colon (X', \Delta) \to Y'$ is the basic $\mathbb{Q}$-slc-trivial fibration induced from $f\colon (X, \mu_*\Delta) \to Y$ such that $K_{X'} + \Delta \sim_{\mathbb{Q}} f'^*\big(\sum_{i=1}^{k} t_i D'_i\big)$. It is because $K_{X'} + \Delta = \mu^*(K_X + \mu_*\Delta)$ by construction.

We apply Lemma 5.5 to $f'\colon (X', B_{X'}) \to Y'$ and $\mathcal{P}'$. Then, we can find $\Delta_1, \dots, \Delta_l \in \mathcal{P}'_{\mathbb{Q}}$ and positive real numbers $s_1, \dots, s_l$ such that

- $\sum_{j=1}^{l} s_j = 1$ and $\sum_{j=1}^{l} s_j \Delta_j = B_{X'}$, and
- $\mathbf{B}_{Y'} = \sum_{j=1}^{l} s_j \mathbf{B}^{\Delta_j}_{Y'}$.

Since $\mathbf{B}$ and $\mathbf{B}^{\Delta_j}$ are $\mathbb{R}$-b-divisors, we have $\mathbf{B}_Y = \sum_{j=1}^{l} s_j \mathbf{B}^{\Delta_j}_Y$. Then

$$
\begin{aligned}
\sigma^*(K_Y + \mathbf{B}_Y) = \sigma^*\left(K_Y + \sum_{j=1}^{l} s_j \mathbf{B}^{\Delta_j}_Y\right) &= \sum_{j=1}^{l} s_j \sigma^*(K_Y + \mathbf{B}^{\Delta_j}_Y) \\
&= \sum_{j=1}^{l} s_j (K_{Y'} + \mathbf{B}^{\Delta_j}_{Y'}) = K_{Y'} + \sum_{j=1}^{l} s_j \mathbf{B}^{\Delta_j}_{Y'} \\
&= K_{Y'} + \mathbf{B}_{Y'}.
\end{aligned}
$$

Therefore, we have $\sigma^*(K_Y + \mathbf{B}_Y) = K_{Y'} + \mathbf{B}_{Y'}$, from which we see that

$$
\mathbf{K} + \mathbf{B} = \overline{\mathbf{K}_Y + \mathbf{B}_Y}.
$$

As in the third paragraph, for each j we define D'_{Δ_j} to be the $\mathbb{Q}$-divisor on Y' associated to the basic $\mathbb{Q}$-slc-trivial fibration $f'\colon (X', \Delta_j) \to Y'$. Note that $K_{X'} + \Delta_j \sim_{\mathbb{Q}} f'^* D'_{\Delta_j}$ for all j. Since $\sum_{j=1}^{l} s_j = 1$ and $\sum_{j=1}^{l} s_j \Delta_j = B_{X'}$, we have $\sigma^* D = \sum_{j=1}^{l} s_j D'_{\Delta_j}$. By the relation $\mathbf{B}_{Y'} = \sum_{j=1}^{l} s_j \mathbf{B}^{\Delta_j}_{Y'}$ and the definition of the moduli $\mathbb{R}$-b-divisors (see Definition 3.5), we have

$$
\mathbf{M}_{Y'} = \sum_{j=1}^{l} s_j \mathbf{M}^{\Delta_j}_{Y'} \qquad \text{and} \qquad \mathbf{M}_Y = \sum_{j=1}^{l} s_j \mathbf{M}^{\Delta_j}_Y
$$

Then $\mathbf{M}_Y$ is a potentially nef $\mathbb{R}$-divisor on Y and

$$
\sigma^*\mathbf{M}_Y = \sigma^*\left(\sum_{j=1}^{l} s_j \mathbf{M}^{\Delta_j}_Y\right) = \sum_{j=1}^{l} s_j \mathbf{M}^{\Delta_j}_{Y'} = \mathbf{M}_{Y'}.
$$

Here, we used $\sigma^*\mathbf{M}^{\Delta_j}_Y = \mathbf{M}^{\Delta_j}_{Y'}$ for every j, which follows from the third paragraph. We complete the proof.

The following result is essentially obtained in the proof of Theorem 5.1. We explicitly state it here for future use.

Theorem 5.6. *Let $f\colon (X,B)\to Y$ be a basic $\mathbb{R}$-slc-trivial fibration with $K_X+B\sim_{\mathbb{R}} f^*D$. Then there are $\mathbb{Q}$-divisors $B_1,\dots,B_l$ on X, $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $D_1,\dots,D_l$ on Y and positive real numbers $r_1,\dots,r_l$ satisfying the following properties.*

- $\sum_{j=1}^{l} r_j=1$ *with* $\sum_{j=1}^{l} r_jB_j=B$ *and* $\sum_{j=1}^{l} r_jD_j=D$,
- $\operatorname{Supp} B=\operatorname{Supp} B_j$, $\lfloor B^{>1}\rfloor=\lfloor B_j^{>1}\rfloor$, *and* $\lceil -(B^{<1})\rceil=\lceil -(B_j^{<1})\rceil$ *hold for every* j,
- *if* $\operatorname{coeff}_S(B)\in\mathbb{Q}$ *for a prime divisor* S *on* X, *then* $\operatorname{coeff}_S(B)=\operatorname{coeff}_S(B_j)$ *holds for every* j,
- $\operatorname{Supp} D=\operatorname{Supp} D_j$ *holds for every* j,
- *if* $\operatorname{coeff}_T(D)\in\mathbb{Q}$ *for a prime divisor* T *on* Y, *then* $\operatorname{coeff}_T(D)=\operatorname{coeff}_T(D_j)$ *holds for every* j,
- $K_X+B_j\sim_{\mathbb{Q}} f^*D_j$ *holds for every* j,
- $\mathbf{B}=\sum_{j=1}^{l} r_j\mathbf{B}_j$ *as b-divisors, where* $\mathbf{B}$ (*resp.* $\mathbf{B}_j$) *is the discriminant* $\mathbb{R}$-*b-divisor* (*resp. the discriminant* $\mathbb{Q}$-*b-divisor*) *of* $f\colon (X,B)\to Y$ (*resp.* $f\colon (X,B_j)\to Y$), *and*
- $\mathbf{M}=\sum_{j=1}^{l} r_i\mathbf{M}_j$ *as b-divisors, where* $\mathbf{M}$ (*resp.* $\mathbf{M}_j$) *is the moduli* $\mathbb{R}$-*b-divisor* (*the moduli* $\mathbb{Q}$-*b-divisor*) *associated to* $f\colon (X,B)\to Y$ (*resp.* $f\colon (X,B_j)\to Y$).

Sketch of Proof. It can be proved by Theorem 5.1, Lemma 5.4 and Lemma 5.5. We only outline the proof.

We note that the properties of Theorem 5.6 except the last two properties correspond to (1)–(6) of Lemma 5.4 respectively. By Lemma 5.4, we can find $\mathbb{Q}$-divisors $\widetilde{B}_1,\dots,\widetilde{B}_k$ on X, $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $\widetilde{D}_1,\dots,\widetilde{D}_k$ on Y and positive real numbers $\widetilde{r}_1,\dots,\widetilde{r}_k$ satisfying (1)–(6) of Lemma 5.4. Then $\widetilde{B}_i$, $\widetilde{D}_i$, and $\widetilde{r}_i$ satisfy all the properties of Theorem 5.6 except the last two properties. More specifically, $\widetilde{B}_i$, $\widetilde{D}_i$, and $\widetilde{r}_i$ satisfy

- $\sum_{i=1}^{k}\widetilde{r}_i=1$ with $\sum_{i=1}^{k}\widetilde{r}_i\widetilde{B}_i=B$ and $\sum_{i=1}^{k}\widetilde{r}_i\widetilde{D}_i=D$ (see (1) of Lemma 5.4),
- $\operatorname{Supp} B=\operatorname{Supp}\widetilde{B}_i$ and $\operatorname{Supp} D=\operatorname{Supp}\widetilde{D}_i$ hold for every i,
- $K_X+\widetilde{B}_i\sim_{\mathbb{Q}} f^*\widetilde{D}_i$ holds for every i (see (6) of Lemma 5.4),

and (2)–(5) in Lemma 5.4. We take a smooth higher model $\sigma\colon Y'\to Y$ so that the induced basic $\mathbb{R}$-slc-trivial fibration $f'\colon (X',B')\to Y'$ satisfies the property that there exists a simple normal crossing divisor Σ' on Y' such that $\operatorname{Supp}\sigma^*D\subset\Sigma'$ and that every stratum of $(X',\operatorname{Supp} B')$ is smooth over $Y'\setminus\Sigma'$. The morphism $X'\to X$ is denoted by μ. For each $1\le i\le k$, let $\widetilde{B}'_i$ be a $\mathbb{Q}$-divisor on X' defined by $K_{X'}+\widetilde{B}'_i=\mu^*(K_X+\widetilde{B}_i)$. Note that $K_{X'}+\widetilde{B}'_i\sim_{\mathbb{Q}} f'^*\sigma^*\widetilde{D}_i$. We may assume that $\operatorname{Supp}\sigma^*\widetilde{D}_i\subset\Sigma'$ and that every stratum of $(X',\operatorname{Supp}\widetilde{B}'_i)$ is smooth over $Y'\setminus\Sigma'$ for every i by taking $\sigma\colon Y'\to Y$ suitably. We define

$$\mathcal{P}=\left\{\sum_{i=1}^{k} t_i\widetilde{B}'_i \,\middle|\, 0\le t_i\le 1 \text{ for every } i \text{ with } \sum_{i=1}^{k} t_i=1\right\}.$$

By Lemma 5.5, we can find $B'_1,\dots,B'_l\in\mathcal{P}$ which are $\mathbb{Q}_{\ge 0}$-linear combinations of $\widetilde{B}'_1,\dots,\widetilde{B}'_l$ and positive real numbers $r_1,\dots,r_l$ such that

- $\sum_{j=1}^{l} r_j = 1$ and $\sum_{j=1}^{l} r_j B'_j = B'$, and
- $\mathbf{B}_{Y'} = \sum_{j=1}^{l} r_j \mathbf{B}_{jY'}$.

Here, $\mathbf{B}_j$ is the discriminant $\mathbb{Q}$-b-divisor associated to $f'\colon (X', B'_j) \to Y'$. By Theorem 5.1, we have $\mathbf{K}+\mathbf{B} = \overline{\mathbf{K}_{Y'}+\mathbf{B}_{Y'}}$ and $\mathbf{K}+\mathbf{B}_j = \overline{\mathbf{K}_{Y'}+\mathbf{B}_{jY'}}$. We put $B_j = \mu_* B'_j$ for each $1 \leq j \leq l$. Then we can find $\mathbb{Q}$-divisors $D_1, \ldots, D_l$ on Y such that $K_X + B_j \sim_{\mathbb{Q}} f^* D_j$ and $\sum_{j=1}^{l} r_j D_j = D$. By construction, we can easily see that $B_1 \ldots, B_l$, $D_1, \ldots, D_l$, and $r_1, \ldots, r_l$ constructed above satisfy the desired properties.

§6. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2, which is the main result of this paper. Then we will treat Theorem 1.1 and Corollary 1.4. We note that we will freely use the framework of quasi-log schemes in the proof of Theorem 1.2. For the details of quasi-log schemes, see [5, Chapter 6]. Let us start with the proof of Theorem 1.2.

Proof of Theorem 1.2. From Step 1 to Step 3, we will define a natural quasi-log scheme structure on Z. This part is essentially contained in [5, Chapter 6] and [7].

Step 1. In this step, we will give a natural quasi-log scheme structure on $W' := W \cup \mathrm{Nlc}(X, \Delta)$. This step is essentially the adjunction for quasi-log schemes (see [5, Theorem 6.3.5 (i)]).

We put $W' := W \cup \mathrm{Nlc}(X, \Delta)$ as above. We will sketch how to define a natural quasi-log scheme structure on W'. Let $f\colon Y \to X$ be a projective birational morphism from a smooth quasi-projective variety Y such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$ and that $\mathrm{Supp}\, \Delta_Y$ is a simple normal crossing divisor on Y. By taking some more blow-ups, we may assume that the union of all log canonical centers of (Y, Δ_Y) mapped to W' by f, which is denoted by V', is a union of some irreducible components of $\Delta_Y^{=1}$. As usual, we put $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$ and consider the following short exact sequence:

$$0 \to \mathcal{O}_Y(A - N - V') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{V'}(A - N) \to 0.$$

By taking $R^i f_*$, we obtain:

$$\begin{aligned} 0 &\longrightarrow f_*\mathcal{O}_Y(A-N-V') \longrightarrow f_*\mathcal{O}_Y(A-N) \longrightarrow f_*\mathcal{O}_{V'}(A-N) \\ &\xrightarrow{\delta} R^1 f_*\mathcal{O}_Y(A-N-V') \to \cdots \end{aligned}$$

The connecting homomorphism δ is zero since no associated prime of $R^1 f_*\mathcal{O}_Y(A-N-V')$ is contained in $W' = f(V')$ (see [4, theorem 6.3 (i)] and [5, Theorem 5.6.2 (i)]). Hence we have:

$$0 \to f_*\mathcal{O}_Y(A-N-V') \to f_*\mathcal{O}_Y(A-N) \to f_*\mathcal{O}_{V'}(A-N) \to 0.$$

Note that $\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_*\mathcal{O}_Y(A-N)$ by definition. We put $\mathcal{I}_{W'} = f_*\mathcal{O}_Y(A-N-V')$ and $\mathcal{I}_{W'_{-\infty}} = f_*\mathcal{O}_{V'}(A-N)$. We define $\Delta_{V'}$ by $(K_Y + \Delta_Y)|_{V'} = K_{V'} + \Delta_{V'}$. Then

$$\big(W', (K_X+\Delta)|_{W'}, f\colon (V', \Delta_{V'}) \to W'\big)$$

is a quasi-log scheme. By construction, $\mathrm{Nqlc}(W', (K_X+\Delta)|_{W'}) = \mathrm{Nlc}(X, \Delta)$ holds. By construction again, C is a qlc stratum of $[W', (K_X+\Delta)|_{W'}]$ if and only if C is a log canonical center of (X, Δ) included in W. We note that the above construction is independent of the choice of $f\colon Y \to X$ by [5, Proposition 6.3.1].

Step 2. In this step, we will give a natural quasi-log scheme structure on $[W, (K_X+\Delta)|_W]$. This step is essentially [7, Lemma 4.19].

In Step 1, we may further assume that the union of all strata of $(V', \Delta_{V'})$ mapped to $W\cap \mathrm{Nlc}(X, \Delta)$ is also a union of some irreducible components of V'. Let $\widehat{V}$ be the union of the irreducible components of V' mapped to W by f. We put $\Delta_{\widehat{V}}$ by $(K_Y+\Delta_Y)|_{\widehat{V}} = K_{\widehat{V}}+\Delta_{\widehat{V}}$. Then, by the proof of [7, Lemmas 4.18 and 4.19],

$$\left(W, (K_X+\Delta)|_W, f\colon (\widehat{V}, \Delta_{\widehat{V}})\to W\right)$$

is a quasi-log scheme. By [7, Lemma 4.19], we obtain that $\mathcal{I}_{W_{-\infty}} = \mathcal{I}_{W'_{-\infty}}$ holds and that C is a qlc stratum of $[W', (K_X+\Delta)|_{W'}]$ if and only if C is a qlc stratum of $[W, (K_X+\Delta)|_W]$. Hence $W\cap \mathrm{Nlc}(X, \Delta)=W_{-\infty}$ and

$$W\cap \left(\mathrm{Nlc}(X, \Delta)\cup \bigcup_{W\not\subset W^\dagger} W^\dagger\right) = \mathrm{Nqklt}(W, (K_X+\Delta)|_W)$$

hold set theoretically, where $W^\dagger$ runs over log canonical centers of (X, Δ) which do not contain W.

Step 3. In this step, we will give a natural quasi-log scheme structure on Z. This step is nothing but [7, Theorem 1.9].

In Step 2, we may further assume that the union of all strata of $(\widehat{V}, \Delta_{\widehat{V}})$ mapped to $\mathrm{Nqklt}(W, (K_X+\Delta)|_W)$ is a union of some irreducible components of $\widehat{V}$. Let V be the union of the irreducible components of $\widehat{V}$ which are dominant onto W. Then, by the proof of [7, Theorem 1.9], $f\colon V\to W$ factors through Z and

$$(Z, \nu^*(K_X+\Delta), f\colon (V, \Delta_V)\to Z)$$

becomes a quasi-log scheme, where Δ_V is defined by $(K_Y+\Delta_Y)|_V = K_V+\Delta_V$. By construction, we have $\nu_*\mathcal{I}_{\mathrm{Nqklt}(Z, \nu^*(K_X+\Delta))} = \mathcal{I}_{\mathrm{Nqklt}(W, (K_X+\Delta)|_W)}$. Hence

$$\mathrm{Nqklt}(Z, \nu^*(K_X+\Delta)) = \nu^{-1}\,\mathrm{Nqklt}(W, (K_X+\Delta)|_W)$$

holds.

Step 4. Then $f\colon (V, \Delta_V)\to Z$ is a basic $\mathbb{R}$-slc-trivial fibration. Hence we can apply Corollary 5.2 and Remark 5.3 to $f\colon (V, \Delta_V)\to Z$. We note that $f\colon (V, \Delta_V)\to Z$ is a basic $\mathbb{Q}$-slc-trivial fibration when $K_X+\Delta$ is $\mathbb{Q}$-Cartier. In that case, Theorem 3.6 with Theorem 3.7 is sufficient.

Step 5. By [7, Theorem 7.1] and Steps 1, 2, and 3 in its proof, we can construct a projective birational morphism $p\colon Z'\to Z$ from a smooth quasi-projective variety Z' satisfying (i), (ii), (iii), and (v). We note that we can directly apply Step 3 in the proof of [7, Theorem 7.1] to basic $\mathbb{R}$-slc-trivial fibrations by Corollary 5.2. We also note that $\mathbf{B}$ is a well-defined $\mathbb{R}$-b-divisor on Z and is independent of $f\colon Y\to X$ (see [10, Lemma 5.1]).

Step 6. (see [10, Theorem 5.4]) In this final step, we will prove (iv). This step is essentially [10, Theorem 5.4]. We explain it here for the reader's convenience.

Without loss of generality, we may assume that X is affine by taking a finite affine open cover of X. Let $g_{dlt}\colon X_{dlt} \to X$ be a good dlt blow-up of (X, Δ) such that $K_{X_{dlt}} + \Delta_{X_{dlt}} = g^*_{dlt}(K_X + \Delta)$ (see [10, Lemma 3.5]). We may assume that there is an irreducible component S of $\Delta^{=1}_{X_{dlt}}$ with $g_{dlt}(S) = W$. We put

$$D = \Delta^{\geq 1}_{X_{dlt}} - \operatorname{Supp} \Delta^{\geq 1}_{X_{dlt}} = \Delta^{>1}_{X_{dlt}} - \operatorname{Supp} \Delta^{>1}_{X_{dlt}}.$$

Then $-D$ is semi-ample over X and $\operatorname{Supp} D = \operatorname{Nlc}(X_{dlt}, \Delta_{X_{dlt}})$ holds set theoretically (see [10, Lemma 3.5]). By taking the contraction morphism $\varphi\colon X_{dlt} \to X_{lc}$ associated to $-D$ over X, we get a log canonical modification $g_{lc}\colon X_{lc} \to X$ with $K_{X_{lc}} + \Delta_{X_{lc}} = g^*_{lc}(K_X + \Delta)$ (see [10, Theorem 1.2]).

$$\begin{tikzcd} X_{dlt} \arrow[rr, "\varphi"] \arrow[dr, "g_{dlt}"'] & & X_{lc} \arrow[dl, "g_{lc}"] \\ & X & \end{tikzcd}$$

We put $D' = \varphi_* D$. Then $-D'$ is ample over X, and

$$g^{-1}_{lc} \operatorname{Nlc}(X, \Delta) = \operatorname{Nlc}(X_{lc}, \Delta_{X_{lc}}) = \operatorname{Supp} D'$$

holds set theoretically. We note that

$$\operatorname{Nlc}(X_{dlt}, \Delta_{X_{dlt}}) = \varphi^{-1} \operatorname{Nlc}(X_{lc}, \Delta_{X_{lc}}) = g^{-1}_{dlt} \operatorname{Nlc}(X, \Delta)$$

holds set theoretically. Let $\tilde{W}$ be the strict transform of W on X_{lc}. Let $\tilde{\nu}\colon \tilde{Z} \to \tilde{W}$ be the normalization. Then we can easily see that

$$\operatorname{Supp} \mathbf{B}^{>1}_{\tilde{Z}} = \widetilde{\nu}^* D' = \tilde{\nu}^{-1} \left(\operatorname{Nlc}(X_{lc}, \Delta_{X_{lc}}) \cap \tilde{W} \right) = (g_{lc} \circ \tilde{\nu})^{-1} \left(\operatorname{Nlc}(X, \Delta) \cap W \right)$$

holds set theoretically. We note that $\mathbf{B}^{>1} = 0$ over $X \setminus \operatorname{Nlc}(X, \Delta)$ by construction. Hence we obtain $\nu \circ p(\mathbf{B}^{>1}_{Z'}) = W \cap \operatorname{Nlc}(X, \Delta)$ set theoretically.

We finish the proof of Theorem 1.2.

Finally, we prove Theorem 1.1 and Corollary 1.4.

Proof of Theorem 1.1. Here, we use the same notation as in Theorem 1.2. We put $B_Z = \mathbf{B}_Z$ and $M_Z = \mathbf{M}_Z$ in Theorem 1.2. We note that $\mathbf{M}_{Z'}$ is a finite $\mathbb{R}_{>0}$-linear combination of potentially nef Cartier divisors on Z' with $p_* \mathbf{M}_{Z'} = M_Z$. Hence the desired statement follows from Theorem 1.2.

Proof of Corollary 1.4. By the definition of $\mathbf{B}$ in Theorem 1.2 (see the proof of Theorem 1.2 and Definition 1.3), we can easily check that B_Z is nothing but Shokurov's different (see [4, Section 14]) and $\nu^*(K_X + \Delta) = K_Z + B_Z$ holds, where $\nu\colon Z \to W$ is the normalization of W. In particular, we have $M_Z = 0$. By (A) in Theorem 1.1, we obtain that (X, Δ) is log canonical in a neighborhood of W if and only if (Z, B_Z) is log canonical in the usual sense. It recovers Kawakita's inversion of adjunction (see [14, Theorem]). By (B), we see that

(Z, B_Z) is kawamata log terminal if and only if (X, Δ) is log canonical in a neighborhood of W and W is a minimal log canonical center of (X, Δ) (see [4, Theorem 9.1] and [5, Theorem 6.3.11]). Note that (X, Δ) is purely log terminal in a neighborhood of W if and only if (X, Δ) is log canonical in a neighborhood of W and W is a minimal log canonical center of (X, Δ).

We close this section with the following remark which summarizes the construction of the $\mathbb{R}$-b-divisors $\mathbf{B}$ and $\mathbf{M}$ on Z.

Remark 6.1. Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let W be a log canonical center of (X, Δ) and let $\nu\colon Z \to W$ be the normalization of W.

We take a log resolution $f\colon Y \to X$ of (X, Δ) which is a sufficiently high birational model. We define Δ_Y by $K_Y + \Delta_Y = f^*(K_X + \Delta)$, and let V be the union of the irreducible components of $\Delta_Y^{=1}$ which map onto W. Let Δ_V be an $\mathbb{R}$-divisor on V defined by $K_V + \Delta_V = (K_Y + \Delta_Y)|_V$, then we get the morphism $f\colon (V, \Delta_V) \to Z$ which has the structure of a basic $\mathbb{R}$-slc-trivial fibration. Then $\mathbf{B}$ and $\mathbf{M}$ are defined to be the discriminant $\mathbb{R}$-b-divisor and the moduli $\mathbb{R}$-b-divisor as in Definition 3.5. By construction, we can easily check that the construction in the proof of Theorem 1.2 and the one in Definition 1.3 define the same $\mathbb{R}$-b-divisor $\mathbf{B}$ on Z (see [10, Lemma 5.1]). Precisely speaking, when $\dim W \leq \dim X - 2$, we consider the $\mathbb{R}$-line bundle $\mathcal{L}$ on X associated to $K_X + \Delta$. We fix an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on Z whose associated $\mathbb{R}$-line bundle is the pullback of $\mathcal{L}$. Then we put $\mathbf{M} = \overline{D} - \mathbf{K} - \mathbf{B}$, where $\overline{D}$ is the $\mathbb{R}$-Cartier closure of D and $\mathbf{K}$ is the canonical b-divisor of Z.

§7. Adjunction for codimension two log canonical centers

In this final section, we first discuss basic slc-trivial fibrations under some extra assumption and then prove adjunction for codimension two log canonical centers.

Theorem 7.1. *Let $f\colon (X, B) \to Y$ be a basic $\mathbb{R}$-slc-trivial fibration. Assume that there exists a stratum S of (X, B) such that the induced morphism $S \to Y$ is generically finite and surjective. Then there exists a proper birational morphism $p\colon Y' \to Y$ from a smooth quasi-projective variety Y' such that $\mathbf{M} = \overline{\mathbf{M}_{Y'}}$ with $\mathbf{M}_{Y'} \sim_{\mathbb{R}} 0$. In particular, $\mathbf{M}$ is b-semi-ample.*

Proof. By Theorem 5.6, we may assume that $f\colon (X, B) \to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration. Let $\nu\colon X^\nu \to X$ be the normalization. We define an $\mathbb{R}$-divisor B^ν on X^ν by $K_{X^\nu} + B^\nu = \nu^*(K_X + B)$. Note that after the reduction we may find a log canonical center S of (X^ν, B^ν) such that the induced morphism $S \to Y$ is generically finite and surjective. By [6, Lemma 4.12], we may further assume that Y is a complete variety. By replacing Y with a smooth higher birational model and $f\colon (X, B) \to Y$ with the induced basic $\mathbb{Q}$-slc-trivial fibration, we may assume that Y is a smooth projective variety, $\mathbf{M} = \overline{\mathbf{M}_Y}$, and $\mathbf{M}_Y$ is nef. The induced morphism $S \to Y$ is denoted by f_S. We define a $\mathbb{Q}$-divisor B_S on S by $K_S + B_S = (K_{X^\nu} + B^\nu)|_S$.

From now on, we will show that $-\mathbf{M}_Y$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor. We consider the divisor $\nu^* f^* \mathbf{M}_Y \sim_{\mathbb{Q}} K_{X^\nu} + B^\nu - \nu^* f^*(K_Y + \mathbf{B}_Y)$. By restricting it to S, we get the relation $f_S^* \mathbf{M}_Y \sim_{\mathbb{Q}} K_S + B_S - f_S^*(K_Y + \mathbf{B}_Y)$. Let $g\colon S \to T$ be the Stein factorization of f_S. The finite morphism $T \to Y$ is denoted by f_T. We put $B_T = g_* B_S$.

Then the relation $K_S+B_S=g^*(K_T+B_T)$ holds because K_S+B_S is $\mathbb{Q}$-linearly trivial over Y. We also have the relation

$$f_T^*\mathbf{M}_Y \sim_{\mathbb{Q}} K_T+B_T-f_T^*(K_Y+\mathbf{B}_Y).$$

To show that $-\mathbf{M}_Y$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor, it is sufficient to prove that $-\big(K_T+B_T-f_T^*(K_Y+\mathbf{B}_Y)\big)$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor.

By the definition of the discriminant $\mathbb{Q}$-b-divisor (see Definition 3.5), for every prime divisor P on Y, we have $\operatorname{coeff}_P(\mathbf{B}_Y)=1-b_P$ where b_P is the log canonical threshold of (X^ν,B^ν) with respect to ν^*f^*P over the generic point of P. Since f_T is finite, we may write $f_T^*P=\sum_{Q_i}m_iQ_i$, where Q_i runs over prime divisors on T with $f_T(Q_i)=P$ and m_i is the multiplicity of Q_i with respect to f_T. By the ramification formula, over a neighborhood of the generic point of P we may write

$$\begin{aligned} f_T^*(K_Y+\mathbf{B}_Y)=&f_T^*(K_Y+(1-b_P)P)\\ =&K_T-\sum_{Q_i}(m_i-1)Q_i+(1-b_P)\sum_{Q_i}m_iQ_i\\ =&K_T+\sum_{Q_i}(1-m_ib_P)Q_i. \end{aligned}$$

We define $E:=\sum_{Q_i}(\operatorname{coeff}_{Q_i}(B_T)-(1-m_ib_P))Q_i$. Then, over a neighborhood of the generic point of P, we have

$$f_T^*\mathbf{M}_Y \sim_{\mathbb{Q}} K_T+B_T-f_T^*(K_Y+\mathbf{B}_Y)=\sum_{Q_i}(\operatorname{coeff}_{Q_i}(B_T)-(1-m_ib_P))Q_i=E.$$

On the other hand, by the definition of b_P (see Definition 3.5) and the fact that S is a log canonical center of (X^ν,B^ν), the pair $(S,B_S+b_Pf_S^*P)$ is sub log canonical over the generic point of P. Since $g\colon S\to T$ is birational and $K_S+B_S=g^*(K_T+B_T)$, the pair $(T,B_T+b_Pf_T^*P)$ is sub log canonical over the generic point of P. This shows $\operatorname{coeff}_{Q_i}(B_T)+m_ib_P\le 1$ for all Q_i such that $f_T(Q_i)=P$. Thus, $-E$ is effective. Hence $-\mathbf{M}_Y$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor.

Finally, since $\mathbf{M}_Y$ is nef, we see that $\mathbf{M}_Y\sim_{\mathbb{Q}}0$.

We prove the b-semi-ampleness of $\mathbf{M}$ for basic slc-trivial fibrations of relative dimension one under some extra assumption.

THEOREM 7.2. *Let $f\colon (X,B)\to Y$ be a basic $\mathbb{R}$-slc-trivial fibration with* $\dim X=\dim Y+1$ *such that the horizontal part B^h of B is effective. Then the moduli $\mathbb{R}$-b-divisor* $\mathbf{M}$ *is b-semi-ample.*

Proof. By Theorem 5.6, we may assume that $f\colon (X,B)\to Y$ is a basic $\mathbb{Q}$-slc-trivial fibration. By [6, Lemma 4.12], we may further assume that Y is a complete variety. When X is reducible, by the definition of basic slc-trivial fibrations (see Definition 3.2), there is a stratum S of X such that the morphism $S\to Y$ is generically finite and surjective since $\dim X=\dim Y+1$. Thus, we can apply Theorem 7.1. By Theorem 7.1, the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-semi-ample when X is reducible. So we may assume that X is irreducible. Let F be a general fiber of f. Then $B|_F\ge 0$ by the assumption $B^h\ge 0$. If $(F,B|_F)$ is not kawamata log terminal, then there is a log canonical center S' of (X, B), that is, S' is a stratum of (X, B), such that the morphism $S' \to Y$ is generically finite and surjective. As in the reducible case, by applying Theorem 7.1, we see that the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-semi-ample. If $(F, B|_F)$ is kawamata log terminal, then the morphism $f\colon (X, B) \to Y$ satisfies [16, Assumption 7.11]. Therefore, by [16, Theorem 8.1], the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-semi-ample. In this way, in any case, the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-semi-ample.

By combining Theorem 7.2 with the proof of Theorem 1.2, we obtain the following result, which generalizes Kawamata's theorem (see [15, Theorem 1]).

Corollary 7.3. (Adjunction and Inversion of Adjunction in codimension two) *Under the same notation as in Theorem 1.2, we further assume that* $\dim W = \dim X - 2$. *Then* $\mathbf{M}$ *is b-semi-ample. Equivalently,* $M_{Z'}$ *is semi-ample. In particular, there exists an effective* $\mathbb{R}$*-divisor* Δ_Z *on* Z *such that*

- $\nu^*(K_X + \Delta) \sim_{\mathbb{R}} K_Z + \Delta_Z$,
- (Z, Δ_Z) *is log canonical if and only if* (X, Δ) *is log canonical near* W, *and*
- (Z, Δ_Z) *is kawamata log terminal if and only if* (X, Δ) *is log canonical near* W *and* W *is a minimal log canonical center of* (X, Δ).

When $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier, we further make* Δ_Z *an effective* $\mathbb{Q}$*-divisor on* Z *such that* $\nu^*(K_X + \Delta) \sim_{\mathbb{Q}} K_Z + \Delta_Z$ *in the above statement.*

Proof. We use the same notation as in Theorem 1.2. Note that W is a codimension two log canonical center of (X, Δ) by assumption. Let $f\colon Y \to X$ be a projective birational morphism from a smooth quasi-projective variety Y such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$ and that $\operatorname{Supp} \Delta_Y$ is a simple normal crossing divisor on Y. Without loss of generality, we may assume that $f^{-1}(W)$ is a simple normal crossing divisor on Y such that $f^{-1}(W) = \sum_i E_i$ is the irreducible decomposition. We put

$$E = \sum_{a(E_i, X, \Delta) = -1} E_i.$$

We define Δ_E by $K_E + \Delta_E = (K_Y + \Delta_Y)|_E$. In this situation, we can check that Δ_E is effective over the generic point of W. Indeed, if X is a surface then we can check this fact by using the minimal resolution. In the general case, by shrinking X and cutting X by general hyperplanes, we can reduce the problem to the case where X is a surface.

Let Z be the normalization of W. By the same arguments as in Steps 1, 2, and 3 in the proof of Theorem 1.2, we can construct a basic $\mathbb{R}$-slc-trivial fibration $f\colon (V, \Delta_V) \to Z$. Then $\dim V = \dim Z + 1$ because $\dim V = \dim X - 1$ and W is a codimension two log canonical center of (X, Δ). Furthermore, by the discussion in the first paragraph, we see that the horizontal part Δ^h_V of Δ_V with respect to $f\colon V \to Z$ is effective. By the same arguments as in Steps 4, 5, and 6 in the proof of Theorem 1.2, we get a projective birational morphism $p\colon Z' \to Z$ from a smooth quasi-projective variety Z' satisfying (i)–(v) of Theorem 1.2. Moreover, by Theorem 7.2, $\mathbf{M}$ is b-semi-ample, that is, $\mathbf{M}_{Z'}$ is semi-ample.

Let $N \sim_{\mathbb{R}} \mathbf{M}_{Z'}$ be a general effective $\mathbb{R}$-divisor such that N and $\mathbf{B}_{Z'}$ have no common components, $\operatorname{Supp}(N + \mathbf{B}_{Z'})$ is a simple normal crossing divisor on Z', and all the coefficients of N are less than one. We put $\Delta_Z = p_* N + \mathbf{B}_Z$. Then, it is easy to see that Δ_Z satisfies the desired three conditions of Corollary 7.3. By the above construction, we can make Δ_Z an effective $\mathbb{Q}$-divisor such that $K_Z + \Delta_Z \sim_{\mathbb{Q}} \nu^*(K_X + \Delta)$ when $K_X + \Delta$ is $\mathbb{Q}$-Cartier. So we are done.

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