RIMS-1969 revision

The degree 2 part of the LMO invariant of cyclic branched covers of knots obtained by plumbing the doubles of two knots

By

Kouki YAMAGUCHI

March 2023

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Abstract

The LMO invariant is a universal quantum invariant of 3-manifolds. In this paper, we present the degree 2 part of the LMO invariant of cyclic branched covers of knots by using the 3-loop invariant of knots, and we calculate it concretely for knots obtained by plumbing the doubles of two knots.

1 Introduction

The LMO invariant of 3-manifolds is an invariant derived from the Kontsevich invariant of knots, and it is universal to all perturbative invariants and finite type invariants of 3-manifolds. The LMO invariant takes its value in the space of Jacobi diagrams, which are some kinds of trivalent graphs. It is well-known that the degree 0 part of the LMO invariant is equal to the order of the first homology group, and the degree 1 part of the LMO invariant is equal to the Casson-Walker-Lescop invariant up to scalar multiplication. The value of the LMO invariant is presented by an infinite sum of Jacobi diagrams, and in general, it is difficult to determine all terms of it.

There are not so many examples of calculation of the LMO invariants of 3-manifolds so far. For example, the LMO invariants of Lens spaces are caluculated in [1], and the small degree parts of the LMO invariants of Seifert fibered rational homology spheres are calculated in [2].

In [4], Garoufalidis and Kricker found a formula for the LMO invariant of cyclic branched covers of knots by using the rational form of the Kontsevich invariant of knots, which is closely related to the loop expansion of the Kontsevich invariant of knots [3], [6]. Its 2-loop part is presented by the 2-loop polynomial, and its 3-loop part is presented by the 3-loop invariant. By using the formula of Garoufalidis and Kricker, we can calculate the degree 1 part of the LMO invariant (Casson-Walker invariant) of cyclic branched covers of a knot, via the 2-loop polynomial of the knot [4].

In this paper, we calculate the degree 2 part of the LMO invariant of cyclic branched covers of some knots, via the 3-loop invariant of knots. Further, in [11], the author calculated the 3-loop polynomial of knots obtained by plumbing the doubles of two knots, where this class of knots includes untwisted Whitehead doubles. By using this result and the formula in [4], we calculate the degree 2 part of the LMO invariant of cyclic branched covers of knots obtained by plumbing the doubles of two knots. This is a new example of the calculation of the degree 2 part of the LMO invariant.

This paper is organized as follows. In Section 2, we review the fundamental notions and the formula of Garoufalidis and Kricker in [4]. In Section 3, we state the main theorem and prove it. In Section 4, we consider the degree 2 part of the formula of the Garoufalidis and Kricker in detail.

The author would like to thank Advisor Tomotada Ohtsuki for encouragement and comments.

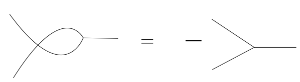
2 The LMO invariant of cyclic branched covers of knots

In this section, we review how to calculate the LMO invariant of the cyclic branched covers of knots. This is a work of Garoufalidis and Kricker [4].

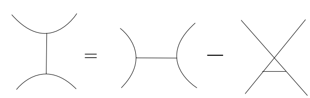
Let K be a knot in S^3, and let Σ_K^p is the *p-fold cyclic branched covers* of K. We call a knot K *p-regular* if Σ_K^p is a rational homology sphere, and we call a knot K *regular* if it is p-regular for all p. It is known that a knot K is p-regular if and only if its Alexander polynomial $\Delta_K(t)$ has no complex pth root of unity.

A *Jacobi diagram* on $\emptyset$ is a trivalent graph such that a cyclic order of the three edges around each trivalent vertex is fixed, in other words, each trivalent vertex is *vertex-oriented*. When we draw a Jacobi diagram on $\emptyset$, each trivalent vertex is vertex-oriented in the counterclockwise order. Furthermore, we define the *degree* of a Jacobi diagram to be half the number of all vertices of the graph of the Jacobi diagram. We define $\mathcal{A}(\emptyset)$ to be the quotient vector space spanned by Jacobi diagrams on $\emptyset$ subject to *the AS, IHX relations.*

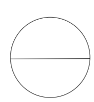
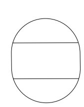
the AS relation :



the IHX relation :



The *LMO invariant* Z^{LMO} of a closed 3-manifold is defined to be in $\mathcal{A}(\emptyset)$ (Strictly speaking, it is defined to be in the completion of $\mathcal{A}(\emptyset)$ with respect to the degree). For a closed 3-manifold M, the LMO invariant is presented by

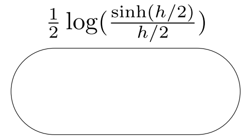
$Z^{LMO}(M) = \exp\Big(c_1(M)$  $+ c_2(M)$  $+ \text{(terms of connected diagrams of degree} > 2)\Big),$

where $c_i(M)$ is a scalar invariant of M. Note that $c_1(M)$ is equal to $(-1)^{b_1(M)}\lambda(M)/2$, where $\lambda(M)$ is the Casson-Walker-Lescop invariant and $b_1(M)$ is the first Betti number of M. For details, see for example [8], [9].

Let $\sigma_K : S^1 \to \mathbb{Z}$ be the signature function. It is defined for all complex numbers of absolute value 1, see for example [5].

In [4], Garoufalidis and Kricker showed that for all p and p-regular knot K, we have

$$Z^{LMO}(\Sigma_K^p) = e^{\sigma_K(p)\Theta/16}\mathrm{Lift}_p \circ \tau_{\alpha_p}^{\mathrm{rat}} \circ Z^{\mathrm{rat}}(K) \in \mathcal{A}(\emptyset). \tag{1}$$

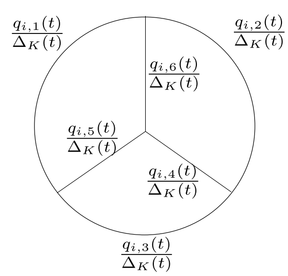
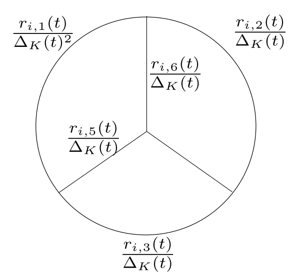
Here, Z^{rat} is the rational form of the Kontsevich invariant of a knot K, defined in [3]. Further, α_p is defined by $\alpha_p = \Omega^{(p-1)/p} \in \mathcal{A}(*)$, where $\Omega = \exp\Bigg($  $\Bigg)$ is the value of the Kontsevich invariant of the unknot in $\mathcal{A}(*)$. Here, $\mathcal{A}(*)$ is the quotient vector space spanned by open Jacobi diagrams subject to the AS, IHX relations, where an *open Jacobi diagram* is an uni-trivalent graph such that each trivalent vertex is vertex-oriented. The maps Lift_p and $\tau^{\mathrm{rat}}_{\alpha_p}$ are defined in [4].

By considering the degree 2 part (the 3-loop part) of $Z^{LMO}(\Sigma_K^p)$, we obtain the following proposition.

Proposition 2.1. *For all p and p-regular knot K, we have*

$$c_2(\Sigma_K^p) = \frac{1}{48p^2} \sum_{\omega_1^p=\omega_2^p=\omega_3^p=1} \Lambda_K(\omega_1, \omega_2, \omega_3, (\omega_1\omega_2\omega_3)^{-1}) + l_p^K.$$

Here, the rational form $\Lambda_K(t_1, t_2, t_3, t_4) \in \mathbb{Q}(t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1})/(t_1t_2t_3t_4 = 1)$ is the *3-loop invariant* for a knot K, which is a rational form presenting the 3-loop part of the Kontsevich invariant of knots. For a knot K, we present its 3-loop part of $Z^{\mathrm{rat}}(K)$ by

$Z^{\mathrm{rat}}(K)^{(3\text{-loop})} = \sum_i$  $+ \sum_i$  ,

where $q_{i,j}(t)$ and $r_{i,j}(t)$ are polynomials in $t^{\pm1}$. Then the 3-loop invariant $\Lambda_K(t_1, t_2, t_3, t_4)$ is defined by

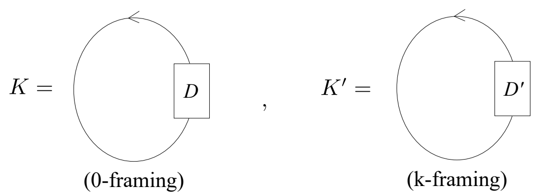
$$\begin{aligned}
&\Lambda_K(t_1, t_2, t_3, t_4) \\
&= \sum_{\substack{i \\ \tau \in \mathfrak{S}_4}} \frac{q_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,4}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(3)}^{-\mathrm{sgn}\tau}) q_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(1)}^{-\mathrm{sgn}\tau}) q_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})} \\
&+ \sum_{\substack{i \\ \tau \in \mathfrak{S}_4}} \frac{r_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(1)}^{-\mathrm{sgn}\tau}) r_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_{\tau(1)}t_{\tau(4)}^{-1})^2\Delta_K(t_{\tau(2)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(1)}^{-1})\Delta_K(t_{\tau(1)}t_{\tau(2)}^{-1})} \\
&\quad \in \mathbb{Q}(t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1})^1/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1),
\end{aligned}$$

where $\mathbb{Q}(t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1})^1$ is the ring of rational forms $\dfrac{v(t_1, t_2, t_3, t_4)}{u(t_1, t_2, t_3, t_4)}$ such that $u(1, 1, 1, 1) = 1$. In particular, if $\Delta_K(t) = 1$, then $\Lambda_K(t_1, t_2, t_3, t_4)$ is a polynomial, so in this case, we call it the *3-loop polynomial*. For details, see [10], [11]. Further, l_p^K is a scaler invariant of a knot K, which can be calculated by an equivariant linking matrix of a surgery link in $S^3\backslash K$. For details, see [4]. As shown later, l_p^K is not essential for our case. For the definition of l_p^K and proof of Proposition 2.1, see Section 4

3 The degree 2 part of $Z^{LMO}(\Sigma^p_{D(K,K')})$

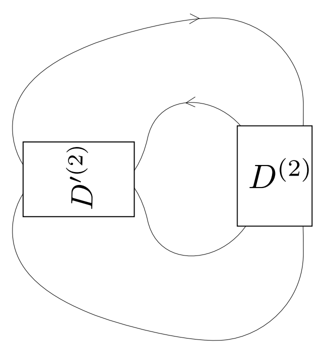
In this section, we state the main theorem of this paper and prove it.

Let K be a 0-framed knot, and let K' be a k-framed knot ($k \in \mathbb{Z}$). Let D, D' be 1-tangles whose closures are K, K', respectively, noting that isotopy classes of D and D' are uniquely determined by K and K'.



(2)

We define $D(K, K')$ to be the following knot,



where $D^{(2)}$ and $D'^{(2)}$ are the doubles of D and D', respectively. We can obtain $D(K, K')$ by *plumbing* of the doubles of K and K', noting that $D(K, K')$ is a genus 1 knot with trivial Alexander polynomial, hence, $D(K, K')$ is regular.

We denote that a_i is a degree i Vassiliev invariant of K, presented by

$$a_2 = -\frac{1}{2}c_2, \quad a_3 = -\frac{1}{24}j_3, \quad a_4 = \frac{1}{24}(-12c_4 + 6c_2^2 - c_2),$$

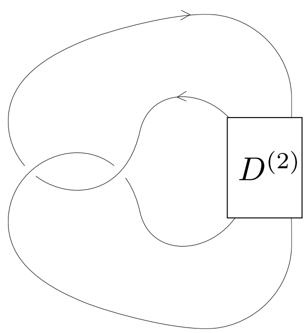
where c_n are coefficient of the Conway polynomial $\nabla_K(z) = \sum c_n z^n$ and j_n are coefficient of the Jones polynomial $J_K(e^t) = \sum j_n t^n$. Note that the Conway polynomial is defined by $\nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_K(t)$. The value a'_i is a degree i Vassiliev invariant of K', presented by in the same way.

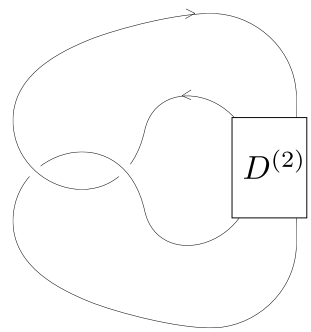
Now, we state the main theorem of this paper.

Theorem 3.1. *For all p, we have*

$$\begin{aligned} & c_2(\Sigma^p_{D(K,K')}) \\ & = \frac{1}{48p^2} \sum_{\omega_1^p=\omega_2^p=\omega_3^p=1} \Lambda_{D(K,K')}(\omega_1, \omega_2, \omega_3, (\omega_1\omega_2\omega_3)^{-1}) \\ & = \left(4a_2a_2' + \frac{1}{6}k^2a_2 + 2ka_3 + 10k^2a_4 + 6k^2a_2^2\right) p. \end{aligned}$$

In particular, we can obtain $c_2(\Sigma^p_{Wh^{\pm}(K)})$, where $Wh^{\pm}(K)$ denotes the untwisted Whitehead double of K.

$Wh^+(K) =$ 

, $Wh^-(K) =$ 

Here, D is a 1-tangle whose closure is K as shown in (2)

Corollary 3.2. *For all p, we have*

$$\begin{aligned} & c_2(\Sigma^p_{Wh^{\pm}(K)}) \\ & = \frac{1}{48p^2} \sum_{\omega_1^p=\omega_2^p=\omega_3^p=1} \Lambda_{Wh^{\pm}(K)}(\omega_1, \omega_2, \omega_3, (\omega_1\omega_2\omega_3)^{-1}) \\ & = \left(\frac{1}{6}a_2 \mp 2a_3 + 10a_4 + 6a_2^2\right) p. \end{aligned}$$

The corollary immediately follows from Theorem 3.1.

Proof of Theorem 3.1. By Proposition 4.1 in Section 4, we get $l_p^{D(K,K')} = 0$. Thus, by Proposition 2.1, we obtain that

$$c_2(\Sigma^p_{D(K,K')}) = \frac{1}{48p^2} \sum_{\omega_1^p=\omega_2^p=\omega_3^p=1} \Lambda_{D(K,K')}(\omega_1, \omega_2, \omega_3, (\omega_1\omega_2\omega_3)^{-1}). \tag{3}$$

On the other hand, the 3-loop polynomial of $D(K, K')$ is calculated in [11], as follows,

$$\begin{aligned}
&\Lambda_{D(K,K')}(t_1, t_2, t_3, t_4) \\
&= (-16a_2a_2' - k^2a_2 - 8ka_3)(u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4}) \\
&\quad + (-\frac{k^2a_2}{12} + 4k^2a_4)(u_{1,4}u_{2,4} + u_{1,4}u_{3,4} + u_{2,4}u_{3,4} + u_{1,3}u_{2,3} + u_{1,3}u_{4,3} + u_{2,3}u_{4,3} \\
&\qquad + u_{1,2}u_{3,2} + u_{1,2}u_{4,2} + u_{3,2}u_{4,2} + u_{2,1}u_{3,1} + u_{2,1}u_{4,1} + u_{3,1}u_{4,1}) \\
&\quad + 24k^2a_4(u_{1,2}u_{3,4} + u_{1,3}u_{2,4} + u_{1,4}u_{2,3}) \\
&\quad + 8k^2a_2^2(u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2) \\
&\quad - \frac{k^2a_2}{4}(v_{1,4}v_{2,4} + v_{1,4}v_{3,4} + v_{2,4}v_{3,4} + v_{1,3}v_{2,3} + v_{1,3}v_{4,3} + v_{2,3}v_{4,3} \\
&\qquad + v_{1,2}v_{3,2} + v_{1,2}v_{4,2} + v_{3,2}v_{4,2} + v_{2,1}v_{3,1} + v_{2,1}v_{4,1} + v_{3,1}v_{4,1}),
\end{aligned} \tag{4}$$

where $u_{i,j} = t_it_j^{-1} + t_i^{-1}t_j - 2$, and $v_{i,j} = t_it_j^{-1} - t_i^{-1}t_j$. By straightforward calculation, we get

$$e^{\frac{2\pi ik}{p}}e^{-\frac{2\pi il}{p}} + e^{-\frac{2\pi ik}{p}}e^{\frac{2\pi il}{p}} - 2 = 2\cos\frac{2(k-l)\pi}{p} - 2,$$
$$e^{\frac{2\pi ik}{p}}e^{-\frac{2\pi il}{p}} - e^{-\frac{2\pi ik}{p}}e^{\frac{2\pi il}{p}} = 2i\sin\frac{2(k-l)\pi}{p} \quad (0 \le k < p, 0 \le l < p).$$

Thus, by applying to (4) to (3), we obtain

$$\begin{aligned}
&c_2(\Sigma^p_{D(K,K')}) \\
&= \frac{1}{48p^2}\sum_{\substack{0\le k<p\\0\le l<p\\0\le m<p}} \Big((-16a_2a_2' - k^2a_2 - 8ka_3)(r_{k-l} + r_{k-m} + r_{l-m} + r_{2k+l+m} + r_{k+2l+m} + r_{k+l+2m}) \\
&\quad + (-\frac{k^2a_2}{12} + 4k^2a_4)(r_{k-l}r_{k-m} + r_{k-l}r_{l-m} + r_{k-m}r_{l-m} + r_{k-l}r_{2k+l+m} \\
&\qquad + r_{k-l}r_{k+2l+m} + r_{k-m}r_{2k+l+m} + r_{k-m}r_{k+l+2m} + r_{l-m}r_{k+2l+m} \\
&\qquad + r_{l-m}r_{k+l+2m} + r_{2k+l+m}r_{k+2l+m} + r_{2k+l+m}r_{k+l+2m} + r_{k+2l+m}r_{k+l+2m}) \\
&\quad + 24k^2a_4(r_{k-l}r_{k+l+2m} + r_{k-m}r_{k+2l+m} + r_{l-m}r_{2k+l+m}) \\
&\quad + 8k^2a_2^2(r_{k-l}^2 + r_{k-m}^2 + u_{l-m}^2 + r_{2k+l+m}^2 + r_{k+2l+m}^2 + r_{k+l+2m}^2) \\
&\quad - \frac{k^2a_2}{4}(-s_{k-l}s_{k-m} + s_{k-l}s_{l-m} - s_{k-m}s_{l-m} - s_{k-l}s_{2k+l+m} \\
&\qquad + s_{k-l}s_{k+2l+m} - s_{k-m}s_{2k+l+m} + s_{k-m}s_{k+l+2m} - s_{l-m}s_{k+2l+m} \\
&\qquad + s_{l-m}s_{k+l+2m} - s_{2k+l+m}s_{k+2l+m} - s_{2k+l+m}s_{k+l+2m} - s_{k+2l+m}s_{k+l+2m})\Big),
\end{aligned}$$

where $r_n = 2\cos\dfrac{2n\pi}{p} - 2,\ s_n = 2\sin\dfrac{2n\pi}{p}$. Further,

$$\sum_{\substack{0\le k<p\\0\le l<p\\0\le m<p}} (r_{k-l} + r_{k-m} + r_{l-m} + r_{2k+l+m} + r_{k+2l+m} + r_{k+l+2m}) = -12p^3,$$

$$\begin{aligned}\sum_{\substack{0\le k<p\\0\le l<p\\0\le m<p}} (r_{k-l}r_{k-m} + r_{k-l}r_{l-m} + r_{k-m}r_{l-m} + r_{k-l}r_{2k+l+m}\\ + r_{k-l}r_{k+2l+m} + r_{k-m}r_{2k+l+m} + r_{k-m}r_{k+l+2m} + r_{l-m}r_{k+2l+m}\\ + r_{l-m}r_{k+l+2m} + r_{2k+l+m}r_{k+2l+m} + r_{2k+l+m}r_{k+l+2m} + r_{k+2l+m}r_{k+l+2m}) = 48p^3,\end{aligned}$$

$$\sum_{\substack{0\le k<p\\0\le l<p\\0\le m<p}} (r_{k-l}r_{k+l+2m} + r_{k-m}r_{k+2l+m} + r_{l-m}r_{2k+l+m}) = 12p^3,$$

$$\sum_{\substack{0\le k<p\\0\le l<p\\0\le m<p}} (r_{k-l}^2 + r_{k-m}^2 + r_{l-m}^2 + r_{2k+l+m}^2 + r_{k+2l+m}^2 + r_{k+l+2m}^2) = 36p^3,$$

$$\begin{aligned}\sum_{\substack{0\le k<p\\0\le l<p\\0\le m<p}} (-s_{k-l}s_{k-m} + s_{k-l}s_{l-m} - s_{k-m}s_{l-m} - s_{k-l}s_{2k+l+m}\\ + s_{k-l}s_{k+2l+m} - s_{k-m}s_{2k+l+m} + s_{k-m}s_{k+l+2m} - s_{l-m}s_{k+2l+m}\\ + s_{l-m}s_{k+l+2m} - s_{2k+l+m}s_{k+2l+m} - s_{2k+l+m}s_{k+l+2m} - s_{k+2l+m}s_{k+l+2m}) = 0.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&c_2(\Sigma^p_{D(K,K')})\\ &= \frac{1}{48p^2}\big((-16a_2a_2' - k^2a_2 - 8ka_3)(-12p^3) + (-\frac{k^2a_2}{12} + 4k^2a_4)\cdot 48p^3\\ &\qquad + 24k^2a_4\cdot 12p^3 + 8k^2a_2^2(36p^3) - \frac{k^2a_2}{4}\cdot 0\big)\\ &= \left(4a_2a_2' + \frac{1}{6}k^2a_2 + 2ka_3 + 10k^2a_4 + 6k^2a_2^2\right)p.\end{aligned}$$

□

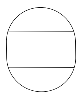
4 The degree 2 part of the formula of Garoufalidis and Kricker

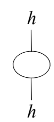
In this section, we learn more about the degree 2 part of the formula of Garoulafidis and Kricker.

Let K be a knot in S^3. In [4], for the 3-loop invariant $\Lambda_K(t_1,t_2,t_3,t_4)$, the value $\mathrm{Res}_p^{t_1,t_2,t_3,t_4}\Lambda_K(t_1,t_2,t_3,t_4)$ is defined by

$$\begin{aligned}\mathrm{Res}_p^{t_1,t_2,t_3,t_4}\Lambda_K(t_1,t_2,t_3,t_4) &= p^{\chi}\sum_{\omega_1^p=\omega_2^p=\omega_3^p=1}\Lambda_K(\omega_1,\omega_2,\omega_3,(\omega_1\omega_2\omega_3)^{-1}) \\ &= \frac{1}{p^2}\sum_{\omega_1^p=\omega_2^p=\omega_3^p=1}\Lambda_K(\omega_1,\omega_2,\omega_3,(\omega_1\omega_2\omega_3)^{-1}), \qquad (5)\end{aligned}$$

where χ is the Euler characteristic of the 3-loop graph, and its value equals to -2. Further, we define l_p^K by

l_p^K 

$$= \mathrm{Lift}_p\left(\left\langle \exp\Big(-\frac{1}{2}\mathrm{tr}^{\circlearrowleft}\log(EL(te^h)EL(t)^{-1})\Big), -\frac{p-1}{48p}\right.\right.$$ 

$$\left.\left.\right\rangle\right), \qquad (6)$$

where $EL(t)$ is a equivariant linking matrix of a surgery link in $S^3\backslash K$, and $\mathrm{tr}^{\circlearrowleft}$ is the wheel-valued trace, see for example [7]. The bracket $\langle\ \ ,\ \ \rangle$ is defined by

$$\langle C_1, C_2\rangle = \left(\begin{array}{l}\text{sum of all ways gluing the } h\text{-marked legs of } C_1 \\ \text{to the } h\text{-marked legs of } C_2\end{array}\right).$$

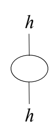
It can be shown that l_p^K does not change by Kirby moves in $S^3\backslash K$, hence it is an invariant K. For details, see [4].

Then, we prove Proposition 2.1.

Proof of Proposition 2.1. By [4], $\tau_{\alpha_p}^{\mathrm{rat}}\circ Z^{\mathrm{rat}}(K)$ is presented by

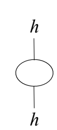
$$\tau_{\alpha_p}^{\mathrm{rat}}\circ Z^{\mathrm{rat}}(K) = \left\langle \exp\Big(-\frac{1}{2}\mathrm{tr}^{\circlearrowleft}\log(EL(te^h)EL(t)^{-1})\Big)\sqcup\big(Z^{\mathrm{rat}}(K)|_{t\to te^h}\big), \alpha_p(h)\right\rangle, \qquad (7)$$

where $EL(t)$ is a equivariant linking matrix, and

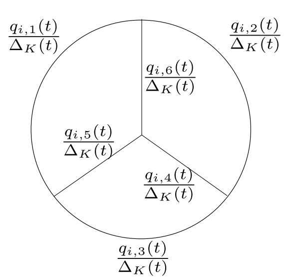
$$\alpha_p(h) = \Omega(h)^{(p-1)/p} = \emptyset + \frac{p-1}{48p}$$ 

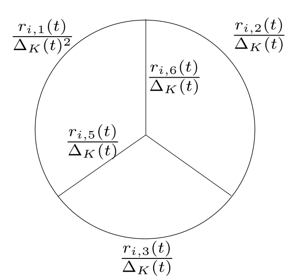
$$+ \text{(linear sum of higher terms)}.$$

However, the terms in (7) which contribute to the 3-loop part are

$$\left\langle \emptyset\sqcup\gamma_K^{(3)}, \emptyset\right\rangle, \quad \text{and} \quad \left\langle \exp\Big(-\frac{1}{2}\mathrm{tr}^{\circlearrowleft}\log(EL(te^h)EL(t)^{-1})\Big)\sqcup\emptyset, -\frac{p-1}{48p}\right.$$ 

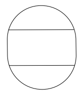
$$\left.\right\rangle,$$

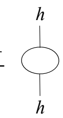
where $\gamma_K^{(3)} = \sum_i$ 

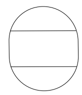
$+\sum_i$ 

is the 3-loop part of

$Z^{\mathrm{rat}}(K)$. Thus, by (1) and (6), we get

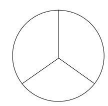
$$c_2(\Sigma_K^p)$$ 

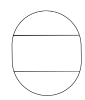
$$= \mathrm{Lift}_p\left(\left\langle \emptyset \sqcup \gamma_K^{(3)}, \emptyset \right\rangle + \left\langle \exp\left(-\frac{1}{2}\mathrm{tr}^{\circlearrowleft}\log(EL(te^h)EL(t)^{-1})\right) \sqcup \emptyset, -\frac{p-1}{48p}\right.\right.$$  $$\left.\left.\right\rangle\right)$$

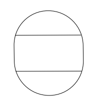
$$= \mathrm{Lift}_p(\gamma_K^{(3)}) + l_p^K$$  $$. \tag{8}$$

Note that $e^{\sigma_K(p)\Theta/16}$ does not contribute to the 3-loop part. However, by Theorem 7 in [4] and the argument in Section 4 in [11], we can get

$$\mathrm{Lift}_p(\gamma_K^{(3)})$$

$$= \mathrm{Res}_p^{t_1,t_2,t_3,t_4}\left(\frac{1}{|\mathfrak{S}_4|}\Lambda_K(t_1,t_2,t_3,t_4)\right)$$ 

$$= \frac{1}{48}\mathrm{Res}_p^{t_1,t_2,t_3,t_4}\Lambda_K(t_1,t_2,t_3,t_4)$$ 

$$= \frac{1}{48p^2}\sum_{\omega_1^p=\omega_2^p=\omega_3^p=1}\Lambda_K(\omega_1,\omega_2,\omega_3,(\omega_1\omega_2\omega_3)^{-1})$$  $$, \tag{9}$$

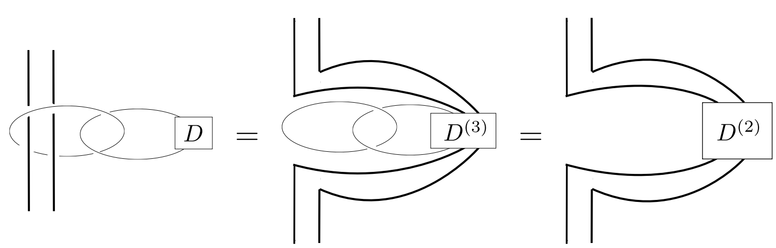
where we obtain the last equality by (5). By (8), (9), we obtain the required formula. □

Lastly, we calculate l_p^K for $D(K,K')$.

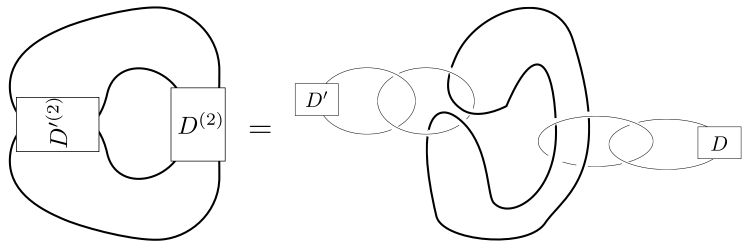
Proposition 4.1. *We get*

$$l_p^{D(K,K')} = 0.$$

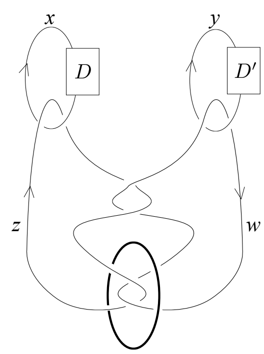
Proof. By handle slide, we can show that



(surgery along the link drawn by thin lines), so we get the following surgery presentation,



$=$



$= K_0 \cup L,$

where K_0 is depicted by a thick line, and L is depicted by thin lines. Thus, a equivariant linking matrix $EL(t)$ of $L \subset S^3 \backslash K_0$ is given by

$$EL(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & k & 0 & 1 \\ 1 & 0 & 0 & t-1 \\ 0 & 1 & t^{-1}-1 & 0 \end{pmatrix},$$

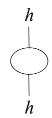
and we get

$$EL(t)^{-1} = \begin{pmatrix} k(t+t^{-1}-2) & -t+1 & 1 & k(t-1) \\ -t^{-1}+1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ k(t^{-1}-1) & 1 & 0 & -k \end{pmatrix}.$$

Hence,

$$\begin{aligned}
&\log(EL(te^h)EL(t)^{-1}) \\
&= \log \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -k(t-1)(e^h-1) & t(e^h-1) & 1 & kt(-e^h+1) \\ t^{-1}(e^{-h}-1) & 0 & 0 & 1 \end{pmatrix} \\
&= \log \Bigg(I_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k(t-1)h & th & 0 & -kth \\ -t^{-1}h & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}k(t-1)h^2 & \frac{1}{2}th^2 & 0 & -\frac{1}{2}kth^2 \\ \frac{1}{2}t^{-1}h^2 & 0 & 0 & 0 \end{pmatrix} \\
&\qquad + \big(\text{linear sum of matrices with (degree of } h) > 2\big)\Bigg)
\end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k(t-1)h & th & 0 & -kth \\ -t^{-1}h & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{2}k(t-1)h^2 & \frac{1}{2}th^2 & 0 & -\frac{1}{2}kth^2 \\ \frac{1}{2}t^{-1}h^2 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}kh^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$+ \big(\text{linear sum of matrices with (degree of } h) > 2\big).$$

Thus, the coefficients of h and h^2 of $\operatorname{tr}\log(EL(te^h)EL(t)^{-1})$ are equal to 0, and this implies that the coefficient of  in $\exp\Big(-\frac{1}{2}\operatorname{tr}^{\circlearrowleft}\log(EL(te^h)EL(t)^{-1})\Big)$ is also equal to 0. Therefore, by (6), we get that $l_p^{D(K,K')} = 0$. □

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Research Institute for Mathematical Science, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan
E-mail address: yamakou@kurims.kyoto-u.ac.jp