**Mathematics Subject Classification (2010)** Primary 14E30; Secondary 14J45, 32Q45

**Keywords** dlt blow-ups, log canonical modifications, inversion of adjunction, lengths of extremal rational curves, Mori hyperbolicity, quasi-log schemes

# Existence of log canonical modifications and its applications

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the date of receipt and acceptance should be inserted later

**Abstract** The main purpose of this paper is to establish some useful partial resolutions of singularities for pairs from the minimal model theoretic viewpoint. We first establish the existence of log canonical modifications of normal pairs under some suitable assumptions. Then we recover Kawakita's inversion of adjunction on log canonicity in full generality. We also discuss the existence of semi-log canonical modifications for demi-normal pairs and construct dlt blow-ups with several extra good properties. As an application, we study lengths of extremal rational curves.

**Keywords** dlt blow-ups · log canonical modifications · inversion of adjunction · lengths of extremal rational curves · Mori hyperbolicity · quasi-log schemes

## 1 Introduction

The main purpose of this paper is to establish some useful partial resolutions of singularities for pairs from the minimal model theoretic viewpoint.

Let us start with an elementary example. Let $X$ be a normal surface. Then it is well known that there exists a unique minimal resolution of singularities

The first author was partially supported by JSPS KAKENHI Grant Numbers JP16H03925, JP16H06337. The second author was partially supported by JSPS KAKENHI Grant Numbers JP16J05875, JP19J00046.

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$f\colon Y \to X$ of $X$. It plays a crucial role for the study of singularities of $X$. Let $g\colon Z \to X$ be any resolution of singularities of $X$. Then we can see $f\colon Y \to X$ as a relative minimal model of $Z$ over $X$. When $X$ is a higher-dimensional quasi-projective variety and $g\colon Z \to X$ is a resolution of singularities of $X$, we can always construct a relative minimal model $f\colon Y \to X$ of $Z$ over $X$ by running a minimal model program (see [3]). Unfortunately, however, $Y$ may be singular. In general, $Y$ has $\mathbb{Q}$-factorial terminal singularities. Since the singularities of $Y$ is milder than those of $X$, $f\colon Y \to X$ sometimes plays an important role as a partial resolution of singularities of $X$.

In the recent study of higher-dimensional algebraic varieties, we know that it is natural and useful to treat pairs. Let us consider a quasi-projective log canonical pair $(X, \Delta)$. Based on [3], Hacon constructed a projective birational morphism $f\colon (Y, \Delta_Y) \to (X, \Delta)$ from a $\mathbb{Q}$-factorial divisorial log terminal pair $(Y, \Delta_Y)$ with $K_Y + \Delta_Y = f^*(K_X + \Delta)$ (see [5] and [14]). We usually call $f\colon (Y, \Delta_Y) \to (X, \Delta)$ a *dlt blow-up* of $(X, \Delta)$. By dlt blow-ups, many problems on log canonical pairs can be reduced to those on $\mathbb{Q}$-factorial divisorial log terminal pairs. We can see $f\colon (Y, \Delta_Y) \to (X, \Delta)$ as a partial resolution of singularities of $(X, \Delta)$ from the minimal model theoretic viewpoint.

In this paper, we are mainly interested in pairs whose singularities are not necessarily log canonical.

## 1.1 Existence of log canonical modifications

We first establish the existence of log canonical modifications, which is a kind of partial resolution of singularities of pairs from the minimal model theoretic viewpoint.

**Theorem 1.1** (Log canonical modifications)**.** *Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $B$ be an $\mathbb{R}$-divisor on $X$ such that the coefficients of $B$ belong to* $[0, 1]$*, $\Delta - B$ is effective and* $\mathrm{Supp}B = \mathrm{Supp}\Delta$*. Then there exists a log canonical modification of $X$ and $B$, that is, a log canonical pair $(Y, B_Y)$ and a projective birational morphism $f\colon Y \to X$ such that*

(i) *the divisor $B_Y$ is the sum of $f_*^{-1}B$ and the reduced $f$-exceptional divisor $E$, that is, $E = \sum_j E_j$ where $E_j$ are the $f$-exceptional prime divisors on $Y$, and*
(ii) *the divisor $K_Y + B_Y$ is $f$-ample.*

Let us see an application of Theorem 1.1.

**Example 1.2.** Theorem 1.1 shows that every pair consisting of a quasi-projective variety $X$ and a boundary $\mathbb{R}$-divisor $\Delta_X$ always has an effective $\mathbb{R}$-divisor $D$ on $X$ whose coefficients are arbitrarily small such that there exists a log canonical modification of $X$ and $\Delta_X + D$. We note that $K_X + \Delta_X$ is not assumed to be $\mathbb{R}$-Cartier. Indeed, we can pick $A \geq 0$ so that $K_X + \Delta_X + A$ is an ample $\mathbb{R}$-Cartier divisor with $\mathrm{Supp}(\Delta_X + A) \neq \mathrm{Supp}\Delta_X$. Then, for any

$\epsilon > 0$, we may find $D \geq 0$ such that $\mathrm{Supp}(\Delta_X + D) = \mathrm{Supp}(\Delta_X + A)$, $A \geq D$, the coefficients of $\Delta_X + D$ belong to $[0, 1]$, and the coefficients of $D$ are less than $\epsilon$. Then, by Theorem 1.1, we see that there exists a log canonical modification of $X$ and $\Delta_X + D$.

The following special case of Theorem 1.1 is important for applications of this paper.

**Theorem 1.3.** *Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We put*

$$B = \Delta^{<1} + \mathrm{Supp}\Delta^{\geq 1}.$$

*Then there exists a log canonical modification of $X$ and $B$.*

Theorem 1.1 is a generalization of [18, Theorem 1.2]. Odaka and Xu proved Theorem 1.1 under the extra assumption that $\Delta = B$ and $B$ is a $\mathbb{Q}$-divisor. By Theorem 1.3, we can recover Kawakita's inversion of adjunction on log canonicity (see Corollary 5.5).

**Theorem 1.4** (see [12] and Corollary 5.5)**.** *Let $(X, S + B)$ be a normal pair such that $S$ is a reduced divisor, $B$ is effective, and $S$ and $B$ have no common irreducible components. Let $\nu\colon S^\nu \to S$ be the normalization of $S$. We put $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$. Then $(X, S + B)$ is log canonical near $S$ if and only if $(S^\nu, B_{S^\nu})$ is log canonical.*

Kawakita's original proof of Theorem 1.4 in [12] does not use the minimal model program. There are some attempts to recover Kawakita's inversion of adjunction by using the minimal model program under extra assumptions (see [18] and [10]). We note that, in Theorem 1.4, the divisor $B$ is an effective $\mathbb{R}$-divisor which may not be a boundary $\mathbb{R}$-divisor. Hence, this is the first time to recover Kawakita's inversion of adjunction on log canonicity in full generality as an application of the minimal model program.

For equidimensional reduced and reducible schemes, Kollár and Shepherd-Barron constructed minimal semi-resolutions of surfaces (see [16, Proposition 4.10]). As a higher-dimensional generalization, Fujita (see [4]) established semi-terminal modifications of demi-normal pairs. Here, we note that a *demi-normal scheme* means an equidimensional reduced scheme which satisfies Serre's $S_2$ condition and is normal crossing in codimension one. On the other hand, Odaka and Xu treated semi-log canonical modifications of demi-normal pairs in [18, Corollary 1.2]. The following theorem is a generalization of [18, Corollary 1.2] and Theorem 1.3 for $\mathbb{Q}$-divisors.

**Theorem 1.5** (see Theorem 4.4)**.** *Let $X$ be a demi-normal scheme, and let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that* $\mathrm{Supp}\Delta$ *does not contain any codimension one singular loci and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We put*

$$B = \Delta^{<1} + \mathrm{Supp}\Delta^{\geq 1}.$$

*Then $X$ equipped with $B$ has a semi-log canonical modification, that is, a semi-log canonical pair $(Y, B_Y)$ and a projective birational morphism $f\colon Y \to X$ such that*

(i) *$f$ is an isomorphism over the generic point of any codimension one singular locus,*
(ii) *$B_Y$ is the sum of the birational transform of $B$ on $Y$ and the reduced $f$-exceptional divisor, and*
(iii) *$K_Y + B_Y$ is $f$-ample.*

We remark that $K_X + B$ in Theorem 1.5 is not necessarily $\mathbb{Q}$-Cartier. As in [18, Example 3.1], there is an example of demi-normal pairs having no semi-log canonical modifications. In our proof of Theorem 1.5, the $\mathbb{R}$-Cartier property of $K_X + \Delta$ is crucial to apply the gluing theory of Kollár. For the details, see Remark 4.5.

### 1.2 Special crepant models

By combining the idea of the proof of Theorem 1.1 with the minimal model theory for $\mathbb{Q}$-factorial divisorial log terminal pairs, we obtain Theorem 1.6, which is a generalization of [7, Lemma 3.10]. Note that the morphism $g\colon (Y, \Delta_Y) \to (X, \Delta)$ in Theorem 1.6 is a kind of dlt blow-up with some extra good properties. Here, we call it a *special crepant model* of $(X, \Delta)$.

**Theorem 1.6** (Special crepant models)**.** *Let $X$ be a normal quasi-projective variety and let $\Delta$ an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then we can construct a crepant model $g\colon (Y, \Delta_Y) \to (X, \Delta)$, that is, a projective birational morphism $g\colon Y \to X$ from a normal $\mathbb{Q}$-factorial variety $Y$ and an effective $\mathbb{R}$-divisor $\Delta_Y$ on $Y$ such that*

(i) $K_Y + \Delta_Y = g^*(K_X + \Delta)$,
(ii) *the pair $(Y, \Delta'_Y)$ is dlt, where $\Delta'_Y = \Delta_Y^{<1} + \mathrm{Supp}\Delta_Y^{\geq 1}$, such that $K_Y + \Delta'_Y$ is $g$-semi-ample,*
(iii) *every $g$-exceptional prime divisor is a component of $(\Delta'_Y)^{=1}$,*
(iv) *$g^{-1}(\mathrm{Nklt}(X, \Delta))$ coincides with $\mathrm{Nklt}(Y, \Delta_Y)$ and $\mathrm{Nklt}(Y, \Delta'_Y)$ set theoretically,*
(v) *$g^{-1}(\mathrm{Nlc}(X, \Delta))$ coincides with $\mathrm{Nlc}(Y, \Delta_Y)$ and $\mathrm{Supp}\Delta_Y^{>1}$ set theoretically, and*
(vi) *there is an effective $\mathbb{R}$-divisor $\Gamma_Y$ on $Y$ such that*
   (a) $\mathrm{Supp}\Gamma_Y = g^{-1}(\mathrm{Nklt}(X, \Delta)) = \mathrm{Supp}\Delta_Y^{\geq 1}$ *set theoretically,*
   (b) *$-\Gamma_Y$ is $g$-semi-ample, and*
   (c) *$\Delta_Y - \Gamma_Y$ is effective and $(Y, \Delta_Y - \Gamma_Y)$ is klt.*

The main difference between Theorem 1.6 and the usual notion of dlt blow-ups is (ii). The $g$-semi-ampleness of $K_Y + \Delta'_Y$ is highly nontrivial. We note that $\Delta_Y \neq \Delta'_Y$ holds when $(X, \Delta)$ is not log canonical.

We also note that we only need the minimal model program essentially obtained in [3] for the proof of [7, Lemma 3.10]. On the other hand, the proof of Theorems 1.1 and 1.6 is much harder because it heavily depends on the minimal model theory for log canonical pairs discussed in [11]. Although Theorem 1.6 may look artificial, it seems to have many applications.

### 1.3 Extremal rational curves

As an application of Theorem 1.6, we prove:

**Theorem 1.7.** *Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X\to S$ be a projective morphism onto a scheme $S$ such that $-(K_X+\Delta)$ is $\pi$-ample. We assume that*

$$\pi\colon \mathrm{Nklt}(X,\Delta)\to\pi(\mathrm{Nklt}(X,\Delta))$$

*is finite. Let $P$ be a closed point of $S$ such that there exists a curve $C^\dagger\subset\pi^{-1}(P)$ with $\mathrm{Nklt}(X,\Delta)\cap C^\dagger\neq\emptyset$. Then there exists a non-constant morphism*

$$f\colon \mathbb{A}^1\longrightarrow (X\setminus \mathrm{Nklt}(X,\Delta))\cap\pi^{-1}(P)$$

*such that the curve $C$, the closure of $f(\mathbb{A}^1)$ in $X$, is a (possibly singular) rational curve satisfying $C\cap\mathrm{Nklt}(X,\Delta)\neq\emptyset$ with*

$$0<-(K_X+\Delta)\cdot C\leq 1.$$

Theorem 1.7 is a kind of generalization of [7, Theorem 1.8]. We note that a log canonical pair, any union of some log canonical centers of a log canonical pair, and a quasi-projective semi-log canonical pair have natural quasi-log scheme structures. Therefore, the theory of quasi-log schemes can be seen as a framework to treat all the above objects on an equal footing. For the details of the theory of quasi-log schemes, see [6, Chapter 6] and [7]. We will quickly explain the basic definitions in Section 7. By combining Theorem 1.7 with the framework of quasi-log schemes discussed in [7], we have:

**Theorem 1.8.** *Let $[X,\omega]$ be a quasi-log scheme and let $\pi\colon X\to S$ be a projective morphism between schemes such that $-\omega$ is $\pi$-ample and that*

$$\pi\colon \mathrm{Nqklt}(X,\omega)\to\pi(\mathrm{Nqklt}(X,\omega))$$

*is finite. Let $P$ be a closed point of $S$ such that there exists a curve $C^\dagger\subset\pi^{-1}(P)$ with $\mathrm{Nqklt}(X,\omega)\cap C^\dagger\neq\emptyset$. Then there exists a non-constant morphism*

$$f\colon \mathbb{A}^1\longrightarrow (X\setminus \mathrm{Nqklt}(X,\omega))\cap\pi^{-1}(P)$$

*such that $C$, the closure of $f(\mathbb{A}^1)$ in $X$, satisfies $C\cap\mathrm{Nqklt}(X,\omega)\neq\emptyset$ with*

$$0<-\omega\cdot C\leq 1.$$

Theorem 1.8 completely solves the first author's conjecture (see [7, Conjecture 1.15]). As an easy direct consequence of Theorem 1.8, we establish:

**Theorem 1.9** (Lengths of extremal rational curves for quasi-log schemes)**.** *Let $[X,\omega]$ be a quasi-log scheme and let $\pi\colon X\to S$ be a projective morphism between schemes. Let $R_j$ be an $\omega$-negative extremal ray of $\overline{NE}(X/S)$ that are rational and relatively ample at infinity and let $\varphi_{R_j}$ be the contraction morphism associated to $R_j$. Let $U_j$ be any open qlc stratum of $[X,\omega]$ such that $\varphi_{R_j}\colon\overline{U_j}\to\varphi_{R_j}(\overline{U_j})$ is not finite and that $\varphi_{R_j}\colon W^\dagger\to\varphi_{R_j}(W^\dagger)$ is finite for* *every qlc center* $W^\dagger$ *of* $[X,\omega]$ *with* $W^\dagger \subsetneq \overline{U_j}$*, where* $\overline{U_j}$ *is the closure of* $U_j$ *in* $X$*. Let* $P$ *be a closed point of* $\varphi_{R_j}(U_j)$*. If there exists a curve* $C^\dagger$ *such that* $\varphi_{R_j}(C^\dagger)=P$*,* $C^\dagger \not\subset U_j$*, and* $C^\dagger \subset \overline{U_j}$*, then there exists a non-constant morphism*

$$f_j\colon \mathbb{A}^1 \longrightarrow U_j \cap \varphi^{-1}_{R_j}(P)$$

*such that* $C_j$*, the closure of* $f_j(\mathbb{A}^1)$ *in* $X$*, spans* $R_j$ *in* $N_1(X/S)$ *and satisfies* $C_j \not\subset U_j$ *with*

$$0 < -\omega\cdot C_j \leq 1.$$

Note that Theorem 1.9 supplements [7, Theorem 1.6 (iii)]. We also note that the above results generalize [17, Theorem 3.1] completely. The following example may help the reader understand Theorem 1.9.

**Example 1.10.** This example shows that the condition $C^\dagger \not\subset U_j$ is necessary for the estimate of the length of $C_j$ in Theorem 1.9. Let $H_1, \dots, H_n$ be general hyperplanes on $X=\mathbb{P}^n$. We put $\Delta=\sum_{i=1}^{n} H_i$ and $\Delta'=\sum_{i=1}^{n-1} H_i$. Let us consider the structure morphism $\pi\colon X\to S=\mathrm{Spec}(\mathbb{C})$. We note that $(X,\Delta)$ and $(X,\Delta')$ are log canonical and that $-(K_X+\Delta)$ and $-(K_X+\Delta')$ are $\pi$-ample. Since the Picard number of $X$ is one, $\pi\colon X\to S$ is an extremal contraction. Let $C$ be any one-dimensional lc center of $(X,\Delta)$. Then it is easy to see that $C\simeq\mathbb{P}^1$, $-(K_X+\Delta)\cdot C=1$, and the open lc center associated to $C$ is isomorphic to $\mathbb{A}^1$. On the other hand, there are no zero-dimensional lc centers of $(X,\Delta')$ and $-(K_X+\Delta')\cdot C'\geq 2$ holds for every curve $C'$ on $X$.

We summarize the contents of this paper. In Section 2, we collect some basic definitions and results for the reader's convenience. Section 3 is the main part of this paper. We prove Theorems 1.1, 1.3, and 1.6 by using the minimal model theory for log canonical pairs. The main ingredient of this section is the second author's theorem: Theorem 3.1, which was obtained in [11]. In Section 4, we discuss semi-log canonical modifications for demi-normal pairs. We prove Theorem 1.5 by using Theorem 1.3 and Kollár's gluing theory in [13]. The readers who are interested only in normal pairs can skip this section. In Section 5, we treat inversion of adjunction on log canonicity. We first prove a slight generalization of Hacon's inversion of adjunction on log canonicity for log canonical centers. Then we recover Kawakita's inversion of adjunction in full generality (see Theorem 1.4) as a special case. Section 6 is devoted to the proof of Theorem 1.7, which heavily depends on the minimal model program for normal pairs. In Section 7, we quickly review some basic definitions in the theory of quasi-log schemes. In Section 8, we prove Theorems 1.8 and 1.9 by using Theorem 1.7 and the framework of quasi-log schemes. We note that we need quasi-log schemes only in Sections 7 and 8.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. In this paper, a *scheme* means a separated scheme of finite type over $\mathbb{C}$. A *variety* means an integral scheme, that is, an irreducible and reduced separated scheme of finite type over $\mathbb{C}$.

## 2 Preliminaries

In this paper, we use the theory of minimal models for higher-dimensional log canonical pairs. Here we collect some definitions and results for the reader's convenience. For the details, see [5], [6], [13], and [15].

**Definition 2.1** (Singularities of pairs)**.** Let $X$ be a variety and let $E$ be a prime divisor on $Y$ for some birational morphism $f\colon Y \to X$ from a normal variety $Y$. Then $E$ is called a divisor *over* $X$. A *normal pair* $(X, \Delta)$ consists of a normal variety $X$ and an $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $(X, \Delta)$ be a normal pair and let $f\colon Y \to X$ be a projective birational morphism from a normal variety $Y$. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

with

$$f_*\left(\sum_E a(E, X, \Delta)E\right) = -\Delta,$$

where $E$ runs over prime divisors on $Y$. We call $a(E, X, \Delta)$ the *discrepancy* of $E$ with respect to $(X, \Delta)$. Note that we can define the discrepancy $a(E, X, \Delta)$ for any prime divisor $E$ over $X$ by taking a suitable resolution of singularities of $X$. If $a(E, X, \Delta) \geq -1$ (resp. $> -1$) for every prime divisor $E$ over $X$, then $(X, \Delta)$ is called *sub log canonical* (resp. *sub kawamata log terminal*). We further assume that $\Delta$ is effective. Then $(X, \Delta)$ is called *log canonical* (*lc*, for short) and *kawamata log terminal* (*klt*, for short) if it is sub log canonical and sub kawamata log terminal, respectively.

Let $(X, \Delta)$ be a log canonical pair. If there exists a projective birational morphism $f\colon Y \to X$ from a smooth variety $Y$ such that both $\mathrm{Exc}(f)$, the exceptional locus of $f$, and $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}\Delta$ are simple normal crossing divisors on $Y$ and that $a(E, X, \Delta) > -1$ holds for every $f$-exceptional divisor $E$ on $Y$, then $(X, \Delta)$ is called *divisorial log terminal* (*dlt*, for short).

**Definition 2.2** (Non-klt loci, non-lc loci, and lc centers)**.** Let $(X, \Delta)$ be a normal pair. If there exist a projective birational morphism $f\colon Y \to X$ from a normal variety $Y$ and a prime divisor $E$ on $Y$ such that $(X, \Delta)$ is sub log canonical in a neighborhood of the generic point of $f(E)$ and that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* (an *lc center*, for short) of $(X, \Delta)$.

From now on, we further assume that $\Delta$ is effective. Then the *non-klt locus* of $(X, \Delta)$, denoted by $\mathrm{Nklt}(X, \Delta)$, is the smallest closed subset $Z$ of $X$ whose complement $(X \setminus Z, \Delta|_{X \setminus Z})$ is a klt pair. Similarly, the *non-lc locus* of $(X, \Delta)$, denoted by $\mathrm{Nlc}(X, \Delta)$, is the smallest closed subset $Z$ of $X$ such that the complement $(X \setminus Z, \Delta|_{X \setminus Z})$ is log canonical.

**Definition 2.3.** Let $X$ be an equidimensional reduced scheme and let $D = \sum_i d_i D_i$ be an $\mathbb{R}$-divisor on $X$ such that $d_i$ is a real number and $D_i$ is an irreducible reduced closed subscheme of $X$ of pure codimension one for every $i$ with $D_i \neq D_j$. We put

$$D^{<1} = \sum_{d_i<1} d_i D_i, \qquad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \qquad D^{=1} = \sum_{d_i=1} D_i,$$
$$D^{\geq 1} = \sum_{d_i \geq 1} d_i D_i, \qquad \text{and} \qquad D^{>1} = \sum_{d_i>1} d_i D_i.$$

We also put

$$\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i, \quad \lceil D \rceil = -\lfloor -D \rfloor, \quad \text{and} \quad \{D\} = D - \lfloor D \rfloor,$$

where $\lfloor d_i \rfloor$ is the integer defined by $d_i - 1 < \lfloor d_i \rfloor \leq d_i$. We say that $D$ is a *boundary divisor* if $D$ is effective and $D = D^{\leq 1}$. We say that $D$ is a *reduced divisor* if $D = D^{=1}$.

**Notation 2.4.** Let $f\colon X \dashrightarrow X'$ be a birational map of normal varieties and let $D$ be an $\mathbb{R}$-divisor on $X$. If there is no risk of confusion, $D_{X'}$ denotes the sum of $f_*D$ and the reduced $f^{-1}$-exceptional divisor $E$ on $X'$, that is, $E = \sum_j E_j$ where $E_j$ are the $f^{-1}$-exceptional prime divisors on $X'$.

**Definition 2.5.** Let $p\colon V \to W$ be a projective surjective morphism from a normal variety $V$ to a variety $W$ and let $D_1$ and $D_2$ be $\mathbb{R}$-Cartier divisors on $V$. Then $D_1 \sim_{\mathbb{R},W} D_2$ means that there exists an $\mathbb{R}$-Cartier divisor $D$ on $W$ such that $D_1 - D_2 \sim_{\mathbb{R}} p^*D$. We say that $D_1$ is $\mathbb{R}$-linearly equivalent to $D_2$ over $W$ when $D_1 \sim_{\mathbb{R},W} D_2$.

In this paper, we adopt the following definition of *models*.

**Definition 2.6** (Models)**.** Let $(X, \Delta)$ be a log canonical pair and $X \to Z$ a projective morphism to a variety $Z$. Let $X' \to Z$ be a projective morphism from a normal variety and let $\phi\colon X \dashrightarrow X'$ be a birational map over $Z$. Let $E$ be the reduced $\phi^{-1}$-exceptional divisor on $X'$, that is, $E = \sum_j E_j$ where $E_j$ are the $\phi^{-1}$-exceptional prime divisors on $X'$. Put $\Delta' = \phi_*\Delta + E$. If $K_{X'} + \Delta'$ is $\mathbb{R}$-Cartier, then the pair $(X', \Delta')$ is called a *log birational model* of $(X, \Delta)$ over $Z$. A log birational model $(X', \Delta')$ of $(X, \Delta)$ over $Z$ is called a *good minimal model* if

(i) $X'$ is $\mathbb{Q}$-factorial,
(ii) $K_{X'} + \Delta'$ is semi-ample over $Z$, and
(iii) for any prime divisor $D$ on $X$ which is exceptional over $X'$, we have

$$a(D, X, \Delta) < a(D, X', \Delta').$$

A log birational model $(X', \Delta')$ of $(X, \Delta)$ over $Z$ is called a *Mori fiber space* if $X'$ is $\mathbb{Q}$-factorial and there is a contraction $X' \to W$ over $Z$ with $\dim W < \dim X'$ such that

(iv) the relative Picard number $\rho(X'/W)$ is one and $-(K_{X'} + \Delta')$ is ample over $W$, and

(v) for any prime divisor $D$ over $X$, we have

$$a(D, X, \Delta) \leq a(D, X', \Delta')$$

and strict inequality holds if $D$ is a divisor on $X$ and is exceptional over $X'$.

We make two important remarks on the minimal model program for log canonical pairs.

**Remark 2.7.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial dlt pair and $\pi\colon X \to Z$ a projective morphism from a normal quasi-projective variety $X$ to a quasi-projective variety $Z$. If $(X, \Delta)$ has a good minimal model or a Mori fiber space over $Z$ as in Definition 2.6, then all $(K_X + \Delta)$-minimal model programs over $Z$ with scaling of an ample divisor terminate (see [2, Theorem 4.1]).

**Remark 2.8.** Let $\pi\colon X \to Z$ be a projective morphism from a normal quasi-projective variety $X$ to a quasi-projective variety $Z$. Let $(X, \Delta)$ and $(X, \Delta')$ be two $\mathbb{Q}$-factorial dlt pairs such that $K_X + \Delta' \sim_{\mathbb{R},Z} t(K_X + \Delta)$ for a positive real number $t$. Suppose that $(X, \Delta)$ has a good minimal model over $Z$. By Remark 2.7, there exists a $(K_X + \Delta)$-minimal model program over $Z$ with scaling of an ample divisor that terminates after finitely many steps. Because any $(K_X + \Delta)$-minimal model program over $Z$ with scaling of an ample divisor is also a $(K_X + \Delta')$-minimal model program over $Z$ with scaling of an ample divisor, we see that there is a $(K_X + \Delta')$-minimal model program over $Z$ terminating with a good minimal model. Thus, we see that $(X, \Delta')$ has a good minimal model over $Z$.

**Definition 2.9** (Log canonical modifications)**.** Let $X$ be a normal variety and let $B$ be a boundary $\mathbb{R}$-divisor on $X$. Then, a *log canonical modification of $X$ and $B$* is a log canonical pair $(Y, B_Y)$ and a projective birational morphism $f\colon Y \to X$ such that

(i) the divisor $B_Y$ is the sum of $f_*^{-1}B$ and the reduced $f$-exceptional divisor $E$, that is, $E = \sum_j E_j$ where $E_j$ are the $f$-exceptional prime divisors on $Y$, and
(ii) the divisor $K_Y + B_Y$ is $f$-ample.

In this paper, if there is no risk of confusion, then the notation $f\colon (Y, B_Y) \to (X, B)$ denotes the structure of a log canonical modification when there is a log canonical modification of $X$ and $B$.

In this paper, we will freely use the existence of *dlt blow-ups*, which was obtained in [7]. Note that a dlt blow-up is sometimes called a *dlt modification* in the literature.

**Theorem 2.10** (Dlt blow-ups, see [7, Theorem 3.9])**.** *Let $X$ be a normal quasi-projective variety and let $\Delta = \sum_i d_i \Delta_i$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. In this case, we can construct a projective birational morphism $f\colon Y \to X$ from a normal quasi-projective variety $Y$ with the following properties.*

(i) *$Y$ is $\mathbb{Q}$-factorial.*
(ii) *$a(E, X, \Delta) \le -1$ for every $f$-exceptional divisor $E$ on $Y$.*
(iii) *We put*

$$\Delta^\dagger := \sum_{0<d_i<1} d_i f_*^{-1}\Delta_i + \sum_{d_i \ge 1} f_*^{-1}\Delta_i + \sum_{E:\ f\text{-exceptional}} E.$$

*Then $(Y, \Delta^\dagger)$ is dlt and the following equality*

$$K_Y + \Delta^\dagger = f^*(K_X + \Delta) + \sum_{a(E,X,\Delta)<-1} (a(E, X, \Delta) + 1)E$$

*holds.*

Note that $\Delta$ is not necessarily a boundary divisor in Theorem 2.10. We close this section with an important remark on Theorem 2.10.

**Remark 2.11.** Let us recall how to construct $f\colon Y \to X$ in Theorem 2.10. In the proof of [7, Theorem 3.9], we first take a suitable resolution of singularities of $X$ and then run a minimal model program over $X$. After finitely many flips and divisorial contractions over $X$, we get a desired birational map $f\colon Y \to X$. Hence we may further assume that $f\colon Y \to X$ is the identity map over some nonempty Zariski open subset of $X$ in Theorem 2.10. More precisely, in Theorem 2.10, let $U$ be the largest Zariski open subset of $X$ such that $(U, \Delta|_U)$ has only $\mathbb{Q}$-factorial kawamata log terminal singularities. Then we can make $f$ the identity map over $U$.

## 3 Proof of Theorems 1.1, 1.3, and 1.6

In this section, we prove Theorems 1.1, 1.3, and 1.6. One of the main ingredients of this section is the second author's following result on the minimal model program for log canonical pairs.

**Theorem 3.1** ([11, Corollary 3.6])**.** *Let $\pi\colon X \to Z$ be a projective morphism of normal quasi-projective varieties and let $(X, B)$ be a log canonical pair. Suppose that there is an effective $\mathbb{R}$-divisor $D$ on $X$ such that*

(a) *$-(K_X + B + D)$ is nef over $Z$, and*
(b) *$(X, B + aD)$ is log canonical for some positive real number $a$.*

*Then, $(X, B)$ has a good minimal model or a Mori fiber space over $Z$.*

Before we prove Theorem 1.1, we prepare an elementary lemma.

**Lemma 3.2.** *Let $X$ be a normal variety and $B$ a boundary $\mathbb{R}$-divisor on $X$. Suppose that there are two log canonical modifications $f\colon (Y, B_Y) \to (X, B)$ and $f'\colon (Y', B_{Y'}) \to (X, B)$ of $X$ and $B$. Then the induced birational map $\phi := f'^{-1} \circ f\colon Y \dashrightarrow Y'$ is an isomorphism and $\phi_* B_Y = B_{Y'}$.*

*Proof.* Let $h\colon W \to Y$ and $h'\colon W \to Y'$ be a common resolution of $\phi$. We define an $\mathbb{R}$-divisor $E$ on $W$ by

$$E := h^*(K_Y + B_Y) - h'^*(K_{Y'} + B_{Y'}).$$

Since $\phi$ is a birational map over $X$, every component $D$ of $E$ is exceptional over $X$. If a component $D$ of $E$ is not $h$-exceptional, then $h_*D$ is exceptional over $X$. Thus we have $a(D, Y, B_Y) = -\mathrm{coeff}_{h_*D}(B_Y) = -1$. On the other hand, we have $a(D, Y', B_{Y'}) \geq -1$ because $(Y', B_{Y'})$ is log canonical. So, we obtain $\mathrm{coeff}_D(E) \geq 0$. Applying the negativity lemma ([3, Lemma 3.6.2 (2)]) to $h\colon W \to Y$ and $E$, we have $E \geq 0$. We apply the same argument to $-E$, then we obtain $-E \geq 0$. Therefore, it follows that $E = 0$. Since $K_Y + B_Y$ and $K_{Y'} + B_{Y'}$ are both ample over $X$, $\phi$ is an isomorphism and $\phi_*B_Y = B_{Y'}$. □

**Remark 3.3.** Let $X$ be a smooth projective variety and let $g\colon X \to X$ be any automorphism of $X$. Then $g\colon X \to X$ is a log canonical modification of $X$ and $B = 0$ by definition.

In some geometric applications, we implicitly require that a log canonical modification $f\colon (Y, B_Y) \to (X, B)$ satisfies the extra assumption that $f$ is the identity morphism over some nonempty Zariski open subset of $X$. Under this extra assumption, by Lemma 3.2, the log canonical modification $f\colon (Y, B_Y) \to (X, B)$ of $X$ and $B$ is unique if it exists.

Let us prove Theorem 1.1.

*Proof of Theorem 1.1.* In Step 1, we will prove Theorem 1.1 under the extra assumption that $X$ is quasi-projective. Then, in Step 2, we will treat the general case.

**Step 1.** In this step, we will prove Theorem 1.1 under the extra assumption that $X$ is quasi-projective. Hence, from now on, we assume that $X$ is quasi-projective.

We take a dlt blow-up $g\colon Z \to X$ with $K_Z + \Delta_Z = g^*(K_X + \Delta)$ as in Theorem 2.10, that is, $g$ is a projective birational morphism such that every $g$-exceptional prime divisor $F$ satisfies $a(F, X, \Delta) \leq -1$ and that $(Z, \Delta_Z^{<1} + \mathrm{Supp}\Delta_Z^{\geq 1})$ is a $\mathbb{Q}$-factorial dlt pair. Note that we may further assume that $g$ is the identity morphism over some nonempty Zariski open subset of $X$ by Remark 2.11.

We define an $\mathbb{R}$-divisor $B_Z$ on $Z$ to be the sum of $g_*^{-1}B$ and the reduced $g$-exceptional divisor (Notation 2.4). Then the relations

$$B_Z \geq 0 \qquad \text{and} \qquad \left(\Delta_Z^{<1} + \mathrm{Supp}\Delta_Z^{\geq 1}\right) - B_Z \geq 0$$

hold since the coefficients of $B$ belong to $[0, 1]$ and $\Delta - B$ is effective. This implies that the pair $(Z, B_Z)$ is a $\mathbb{Q}$-factorial dlt pair. We will prove that $(Z, B_Z)$ has a good minimal model over $X$. We put

$$D_Z = \Delta_Z - B_Z.$$

We have $\Delta_Z - \left(\Delta_Z^{<1} + \mathrm{Supp}\Delta_Z^{\geq 1}\right) \geq 0$ by construction, so

$$D_Z = \Delta_Z - B_Z \geq \left(\Delta_Z^{<1} + \mathrm{Supp}\Delta_Z^{\geq 1}\right) - B_Z \geq 0,$$

from which $D_Z$ is an effective $\mathbb{R}$-divisor on $Z$. Furthermore, recalling $\mathrm{Supp}B = \mathrm{Supp}\Delta$ and that $B_Z$ is the sum of $g_*^{-1}B$ and the reduced $g$-exceptional divisor, it follows that $\mathrm{Supp}\Delta_Z = \mathrm{Supp}B_Z$. Thus, we see that

$$\mathrm{Supp}D_Z \subset \mathrm{Supp}\Delta_Z = \mathrm{Supp}B_Z.$$

We can find a real number $t > 0$ such that $B_Z - tD_Z \geq 0$. Then the pair $(Z, B_Z - tD_Z)$ is dlt because $(Z, B_Z)$ is a dlt pair and $D_Z$ is effective. Since $K_Z + \Delta_Z = g^*(K_X + \Delta)$, we have

$$K_Z + B_Z = K_Z + \Delta_Z - D_Z \sim_{\mathbb{R},X} -D_Z.$$

By this relation, we obtain

$$K_Z + B_Z - tD_Z \sim_{\mathbb{R},X} -(1+t)D_Z \sim_{\mathbb{R},X} (1+t)(K_Z + B_Z).$$

By Remark 2.8, the existence of a good minimal model of $(Z, B_Z)$ over $X$ follows from the existence of a good minimal model of $(Z, B_Z - tD_Z)$ over $X$. We put

$$\tilde{B}_Z = B_Z - tD_Z.$$

Then $K_Z + \tilde{B}_Z + (1+t)D_Z \sim_{\mathbb{R},X} 0$ and $(Z, \tilde{B}_Z + tD_Z)$ is dlt since $\tilde{B}_Z + tD_Z = B_Z$ by definition. By Theorem 3.1, $(Z, \tilde{B}_Z)$ has a good minimal model over $X$. Therefore, $(Z, B_Z)$ also has a good minimal model over $X$.

By running a minimal model program over $X$, we get a good minimal model $(Z', B_{Z'})$ of $(Z, B_Z)$ over $X$ (see Remark 2.7). Let $Z' \to Y$ be the contraction over $X$ induced by $K_{Z'} + B_{Z'}$. We define $B_Y$ to be the birational transform of $B_{Z'}$ on $Y$. Then it is easy to check that $(Y, B_Y)$ is a log canonical pair and the induced morphism $f \colon Y \to X$ is the desired birational morphism. By construction, we may assume that $f \colon Y \to X$ is the identity morphism over some nonempty Zariski open subset of $X$.

**Step 2.** In this step, we will treat the general case, that is, $X$ is not necessarily quasi-projective.

We take a finite affine open covering $X = \bigcup_i U_i$. By Step 1, there exist log canonical modifications $f_i \colon (V_i, B_{V_i}) \to (U_i, B|_{U_i})$ of $U_i$ and $B|_{U_i}$ such that $f_i$ is the identity morphism over some nonempty Zariski open subset of $U_i$ for all $i$. By Lemma 3.2 (see also Remark 3.3), $f_i \colon (V_i, B_{V_i}) \to (U_i, B|_{U_i})$ coincides with $f_j \colon (V_j, B_{V_j}) \to (U_j, B|_{U_j})$ over $U_i \cap U_j$ for every $j \neq i$. Therefore, we get a log canonical modification of $X$ and $B$ by gluing them.

We finish the proof of Theorem 1.1. □

*Proof of Theorem 1.3.* It is a special case of Theorem 1.1. □

The following remark easily follows from the definition of log canonical modifications. It is very useful for various geometric applications.

**Remark 3.4.** Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. We put $B=\Delta^{<1}+\mathrm{Supp}\Delta^{\geq 1}$. Let $f\colon (Y,B_Y)\to (X,B)$ be a log canonical modification of $X$ and $B$. We give two important remarks.

We put $U=X\setminus f(\mathrm{Exc}(f))$. Then, any point $x$ of $X$ is contained in $U$ if and only if $K_X+B$ is $\mathbb{R}$-Cartier and $(X,B)$ is log canonical near $x$.

We define $\Delta_Y$ by $K_Y+\Delta_Y=f^*(K_X+\Delta)$, and we put $\Gamma_Y=\Delta_Y-B_Y$. Then, it follows that $\Gamma_Y$ is effective, $-\Gamma_Y$ is ample over $X$, and we have $\mathrm{Exc}(f)\subset \mathrm{Supp}\Gamma_Y=\mathrm{Supp}\Delta_Y^{>1}$.

By the same argument as in the proof of Theorem 1.1, we obtain:

**Lemma 3.5** (Good dlt blow-ups)**.** *Let $X$ be a normal quasi-projective variety and let $\Delta=\sum_i d_i\Delta_i$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Then there exists a projective birational morphism $f\colon Y\to X$ as in Theorem 2.10 such that $K_Y+\Delta^\dagger$ in Theorem 2.10 is $f$-semi-ample.*

*Proof.* We put $K_Y+\Delta_Y=f^*(K_X+\Delta)$. Then

$$\Delta^\dagger=\Delta_Y^{<1}+\mathrm{Supp}\Delta_Y^{\geq 1}$$

holds. Therefore, as in the proof of Theorem 1.1 (put $B=\Delta^{<1}+\mathrm{Supp}\Delta^{\geq 1}$ in the proof of Theorem 1.1), by Theorem 3.1, the dlt pair

$$(Y,\Delta^\dagger)=(Y,\Delta_Y^{<1}+\mathrm{Supp}\Delta_Y^{\geq 1})$$

has a good minimal model over $X$. Hence, after finitely many flips and divisorial contractions, we can make $K_Y+\Delta^\dagger$ $f$-semi-ample. $\square$

**Remark 3.6.** As in Remark 2.11, by construction, we may further assume that $f$ is the identity morphism over some nonempty Zariski open subset of $X$ in Theorems 1.1, 1.3, and Lemma 3.5.

Let us prove Theorem 1.6.

*Proof of Theorem 1.6.* Let $f\colon (Z,\Delta_Z)\to (X,\Delta)$ be a good dlt blow-up as in Lemma 3.5. This means that $f\colon Z\to X$ is a projective birational morphism from a normal $\mathbb{Q}$-factorial variety $Z$ satisfying (i)–(iii). If necessary, then we may further assume that $f$ is the identity morphism over some nonempty Zariski open subset of $X$ (see Remark 3.6).

**Step 1.** In this step, we will run a $(K_Z+\Delta_Z^{\leq 1}+(1-\epsilon)\mathrm{Supp}\Delta_Z^{\geq 1})$-minimal model program over $X$ and make $K_Z+\Delta_Z^{\leq 1}+(1-\epsilon)\mathrm{Supp}\Delta_Z^{\geq 1}$ semi-ample over $X$ for some small number $\varepsilon$.

We can always take $\epsilon>0$ such that, for any $(K_Z+\Delta_Z^{\leq 1}+(1-\epsilon)\mathrm{Supp}\Delta_Z^{\geq 1})$-minimal model program over $X$, the divisor $K_Z+\Delta_Z^{\leq 1}+\mathrm{Supp}\Delta_Z^{\geq 1}$ is numerically trivial over all extremal contractions of the steps of the minimal model program. This fact follows from the well-known estimate of lengths of extremal rational curves (see, for example, [1, Proposition 3.2]). More explicitly, if $\varepsilon$ is sufficiently small, then we can use [1, Proposition 3.2 (4) and (5)] to prove that $K_Z + \Delta_Z^{<1} + \mathrm{Supp}\Delta_Z^{\geq 1}$ is numerically trivial on each step of any $(K_Z + \Delta_Z^{<1} + (1-\varepsilon)\mathrm{Supp}\Delta_Z^{\geq 1})$-minimal model program since $K_Z + \Delta_Z^{<1} + \mathrm{Supp}\Delta_Z^{\geq 1}$ is nef over $X$. Since $(Z, \Delta_Z^{<1} + (1-\epsilon)\mathrm{Supp}\Delta_Z^{\geq 1})$ is klt, we can run a $(K_Z + \Delta_Z^{<1} + (1-\epsilon)\mathrm{Supp}\Delta_Z^{\geq 1})$-minimal model program over $X$ and finally obtain a good minimal model $(Z', \Delta_{Z'}^{<1} + (1-\epsilon)\mathrm{Supp}\Delta_{Z'}^{\geq 1})$ over $X$ by [3]. By the choice of $\epsilon$, the divisor $K_{Z'} + \Delta_{Z'}^{<1} + \mathrm{Supp}\Delta_{Z'}^{\geq 1}$ is semi-ample over $X$.

Therefore, for any $u \in [0, \epsilon]$, the divisor

$$K_{Z'} + \Delta_{Z'}^{<1} + (1-u)\mathrm{Supp}\Delta_{Z'}^{\geq 1}$$

is semi-ample over $X$. Note that the divisor $-(\Delta_{Z'}^{\geq 1} - (1-\epsilon)\mathrm{Supp}\Delta_{Z'}^{\geq 1})$ is also semi-ample over $X$ because

$$K_{Z'} + \Delta_{Z'}^{<1} + (1-\epsilon)\mathrm{Supp}\Delta_{Z'}^{\geq 1} \sim_{\mathbb{R},X} -(\Delta_{Z'}^{\geq 1} - (1-\epsilon)\mathrm{Supp}\Delta_{Z'}^{\geq 1})$$

holds. By the above construction of $(Z', \Delta_{Z'})$, the pair $(Z', \Delta_{Z'}^{<1} + \mathrm{Supp}\Delta_{Z'}^{\geq 1})$ is lc and

$$\mathrm{Nklt}\left(Z', \Delta_{Z'}^{<1} + \mathrm{Supp}\Delta_{Z'}^{\geq 1}\right) = \mathrm{Supp}\Delta_{Z'}^{\geq 1}$$

holds set theoretically.

**Step 2.** The morphism $Z' \to X$ is denoted by $\alpha'$. Then we take a dlt blow-up $\beta\colon Y \to Z'$ of $(Z', \Delta_{Z'}^{<1} + \mathrm{Supp}\Delta_{Z'}^{\geq 1})$ such that $a(E, Z', \Delta_{Z'}^{<1} + \mathrm{Supp}\Delta_{Z'}^{\geq 1}) = -1$ holds for every $\beta$-exceptional divisor $E$ on $Y$ (see Theorem 2.10). We set $g = \alpha' \circ \beta$ and define $\Delta_Y$ by $K_Y + \Delta_Y = \beta^*\alpha'^*(K_X + \Delta)$. By the definition of $\Delta_Y$, $K_Y + \Delta_Y = g^*(K_X + \Delta)$ obviously holds. This means that $g\colon (Y, \Delta_Y) \to (X, \Delta)$ satisfies (i). By the construction of $\alpha'\colon Z' \to X$, $a(E, X, \Delta) \leq -1$ holds for every $\alpha'$-exceptional divisor $E$ on $Z'$. By the construction of $\beta\colon Y \to Z'$, $a(E, X, \Delta) = a(E, Z', \Delta_{Z'}) \leq a(E, Z', \Delta_{Z'}^{<1} + \mathrm{Supp}\Delta_{Z'}^{\geq 1}) = -1$ holds for every $\beta$-exceptional divisor $E$ on $Y$. This means that every $g$-exceptional prime divisor is a component of $(\Delta_Y')^{=1}$. Therefore, $g\colon (Y, \Delta_Y) \to (X, \Delta)$ satisfies (iii). By the construction of the dlt blow-up $\beta\colon Y \to Z'$ of $(Z', \Delta_{Z'}^{<1} + \mathrm{Supp}\Delta_{Z'}^{\geq 1})$,

$$K_Y + \Delta_Y' = \beta^*(K_{Z'} + \Delta_{Z'}^{<1} + \mathrm{Supp}\Delta_{Z'}^{\geq 1})$$

holds. Hence, $K_Y + \Delta_Y'$ is semi-ample over $X$. Thus, $g\colon (Y, \Delta_Y) \to (X, \Delta)$ satisfies (ii).

From now on, we will explain how to check (iv). We put

$$E_{Z'} = \Delta_{Z'}^{\geq 1} - (1-\epsilon)\mathrm{Supp}\Delta_{Z'}^{\geq 1}.$$

Then $\mathrm{Supp}E_{Z'} = \mathrm{Supp}\Delta_{Z'}^{\geq 1}$, and $-E_{Z'}$ is semi-ample over $Z$ by Step 1. We can also see that

$$\mathrm{Supp}\beta^*E_{Z'} = \mathrm{Supp}\Delta_Y^{\geq 1} = \mathrm{Nklt}(Y, \Delta_Y)$$

holds and $-\beta^*E_{Z'}$ is semi-ample over $X$. Now $g^{-1}(\mathrm{Nklt}(X, \Delta)) \supset \mathrm{Supp}\Delta_Y^{\geq 1}$ is obvious and it is easy to check that $g(\mathrm{Supp}\Delta_Y^{\geq 1}) = \mathrm{Nklt}(X, \Delta)$ holds set theoretically. If $g^{-1}(\mathrm{Nklt}(X,\Delta)) \supsetneq \mathrm{Supp}\Delta_Y^{\geq 1}$, then there is a curve $C \subset Y$ such that $g(C) \in \mathrm{Nklt}(X,\Delta)$ and $(C \cdot \beta^* E_{Z'}) > 0$ since $g(\mathrm{Supp}\Delta_Y^{\geq 1}) = \mathrm{Nklt}(X,\Delta)$ and $g$ has connected fibers. This is a contradiction because $-\beta^* E_{Z'}$ is nef over $X$. Hence we see that $g^{-1}(\mathrm{Nklt}(X,\Delta)) = \mathrm{Supp}\Delta_Y^{\geq 1}$ holds. This means that $g\colon (Y,\Delta_Y) \to (X,\Delta)$ satisfies (iv).

For (v), we note that $\Delta_Y - \Delta'_Y$ is effective and that $-(\Delta_Y - \Delta'_Y) \sim_{\mathbb{R},X} K_Y + \Delta'_Y$ is $g$-semi-ample. By the definition of $\Delta'_Y$, $\mathrm{Supp}(\Delta_Y - \Delta'_Y) = \mathrm{Supp}\Delta_Y^{>1}$ holds. By the same argument as in the proof of (iv) above, we can check that

$$g^{-1}(\mathrm{Nlc}(X,\Delta)) = \mathrm{Supp}\Delta_Y^{>1} = \mathrm{Nlc}(Y,\Delta_Y)$$

holds set theoretically.

Finally, we will construct $\Gamma_Y$ as in (vi). Since the pair $(Z', \Delta_{Z'}^{<1} + (1-u)\mathrm{Supp}\Delta_{Z'}^{\geq 1})$ is klt and $K_{Z'} + \Delta_{Z'}^{<1} + (1-u)\mathrm{Supp}\Delta_{Z'}^{\geq 1}$ is semi-ample over $X$ for every $u \in (0,\epsilon]$, by the construction of $\beta\colon Y \to Z'$, we can find a positive real number $u$ such that if we set $\Delta_Y^u$ by

$$K_Y + \Delta_Y^u = \beta^*(K_{Z'} + \Delta_{Z'}^{<1} + (1-u)\mathrm{Supp}\Delta_{Z'}^{\geq 1}),$$

then $\Delta_Y^u$ is effective, $(Y,\Delta_Y^u)$ is klt, and $K_Y + \Delta_Y^u$ is semi-ample over $X$. Fix such $u > 0$ and put

$$\Gamma_Y := \Delta_Y - \Delta_Y^u = \beta^*\left(\Delta_{Z'} - (\Delta_{Z'}^{<1} + (1-u)\mathrm{Supp}\Delta_{Z'}^{\geq 1})\right).$$

Note that $\Gamma_Y = (K_Y + \Delta_Y) - (K_Y + \Delta_Y^u) \sim_{\mathbb{R},X} -(K_Y + \Delta_Y^u)$ holds. Hence $-\Gamma_Y$ is semi-ample over $X$. It is clear that $(Y, \Delta_Y - \Gamma_Y)$ is klt because $\Delta_Y - \Gamma_Y = \Delta_Y^u$. Since

$$\mathrm{Supp}\left(\Delta_{Z'} - (\Delta_{Z'}^{<1} + (1-u)\mathrm{Supp}\Delta_{Z'}^{\geq 1})\right) = \mathrm{Supp}\Delta_{Z'}^{\geq 1} = \mathrm{Supp}E_{Z'},$$

we have $\mathrm{Supp}\Gamma_Y = \mathrm{Supp}\beta^* E_{Z'}$, thus

$$\mathrm{Supp}\Gamma_Y = \mathrm{Supp}\Delta_Y^{\geq 1} = g^{-1}(\mathrm{Nklt}(X,\Delta))$$

holds. In this way, we see that $\Gamma_Y$ satisfies all the desired conditions in (vi).

We complete the proof of Theorem 1.6. □

**Remark 3.7.** By construction (see also Remarks 2.11 and 3.6), we may further assume that $g\colon (Y,\Delta_Y) \to (X,\Delta)$ in Theorem 1.6 is the identity morphism over some nonempty Zariski open subset of $X$. Hence we can see $g\colon (Y,\Delta_Y) \to (X,\Delta)$ as a partial resolution of singularities of the pair $(X,\Delta)$. More precisely, in Theorem 1.6, let $U$ be the largest Zariski open subset of $X$ such that $(U, \Delta|_U)$ has only $\mathbb{Q}$-factorial kawamata log terminal singularities. Then we can make $g$ the identity map over $U$.

## 4 On semi-log canonical modifications of demi-normal pairs

A *demi-normal scheme* $X$ is a reduced and equidimensional scheme which satisfies Serre's $S_2$ condition and is normal crossing in codimension one. For basic definitions and properties of demi-normal pairs and semi-log canonical pairs, see [13, Sections 5.1 and 5.2]. In this section, we prove the existence of semi-log canonical modifications of demi-normal pairs (see Theorem 4.4). Let us start with the following lemma.

**Lemma 4.1.** *Let $(X, S+B)$ be a log canonical pair such that $B$ is an effective $\mathbb{R}$-divisor and $S$ is a prime divisor with the normalization $S^\nu$. Let $\Gamma$ be an effective $\mathbb{R}$-Cartier divisor on $X$ such that* $\mathrm{Supp}\Gamma \subset \lfloor B \rfloor$*. We define an effective $\mathbb{R}$-divisor $B_{S^\nu}$ on $S^\nu$ by applying adjunction to $(X, S+B)$ and $S^\nu$. We put $\Gamma_{S^\nu}$ as the pullback of $\Gamma$ to $S^\nu$. If $\Gamma_{S^\nu} \neq 0$, then for any component $P_{S^\nu}$ of $\Gamma_{S^\nu}$ we have* $\mathrm{coeff}_{P_{S^\nu}}(B_{S^\nu}) = 1$*. In particular, if* $\mathrm{Supp}\Gamma$ *intersects $S$, then the pair $(S^\nu, B_{S^\nu} + \Gamma_{S^\nu})$ is not log canonical.*

*Proof.* Note that $\Gamma_{S^\nu}$ is well-defined as an effective $\mathbb{R}$-Cartier divisor on $S^\nu$ because $S$ is not a component of $\lfloor B \rfloor$ and $\mathrm{Supp}\Gamma \subset \lfloor B \rfloor$. Since the problem is local, by shrinking $X$, we may assume that $X$ is quasi-projective.

Let $P_S$ be the image of $P_{S^\nu}$ on $X$. Then $P_S$ is a subvariety of $X$ of codimension two and $P_S \subset S \cap \mathrm{Supp}\Gamma$. We take a dlt blow-up $f \colon (Y, T+B_Y) \to (X, S+B)$ (see Theorem 2.10), where $T = f_*^{-1}S$ and $B_Y$ is the sum of $f_*^{-1}B$ and the reduced $f$-exceptional divisor. We put $\Gamma_Y = f^*\Gamma$. Note that $T$ is not a component of $\Gamma_Y$.

The facts $P_S \subset S \cap \mathrm{Supp}\Gamma$, $f(T) = S$, and $\mathrm{Supp}\Gamma_Y = f^{-1}(\mathrm{Supp}\Gamma)$ show the inclusion $P_S \subset f(T \cap \mathrm{Supp}\Gamma_Y)$. Because $P_S$ and all irreducible components of $T \cap \mathrm{Supp}\Gamma_Y$ have the same dimension, we can find an irreducible component $D_T$ of $T \cap \mathrm{Supp}\Gamma_Y$ such that $f(D_T) = P_S$. Furthermore, every component of $\Gamma_Y$ is a component of $\lfloor B_Y \rfloor$. It is because $\mathrm{Supp}\Gamma \subset \lfloor B \rfloor$ and all $f$-exceptional prime divisors on $Y$ are components of $\lfloor B_Y \rfloor$. Thus, it follows that $D_T$ is an irreducible component of $T \cap \lfloor B_Y \rfloor$. We define $B_T$ to be the $\mathbb{R}$-divisor on $T$ with $K_T + B_T = (K_Y + T + B_Y)|_T$. Since $(Y, T+B_Y)$ is a $\mathbb{Q}$-factorial dlt pair, we have $\mathrm{coeff}_{D_T}(B_T) = 1$. Since $f(D_T) = P_S$ and $B_{S^\nu}$ is the birational transform of $B_T$ on $S^\nu$, we obtain $\mathrm{coeff}_{P_{S^\nu}}(B_{S^\nu}) = 1$.

If $\mathrm{Supp}\Gamma$ intersects $S$, then $\Gamma_{S^\nu} \neq 0$ and any irreducible component $P_{S^\nu}$ of $\Gamma_{S^\nu}$ satisfies

$$\mathrm{coeff}_{P_{S^\nu}}(B_{S^\nu} + \Gamma_{S^\nu}) > \mathrm{coeff}_{P_{S^\nu}}(B_{S^\nu}) = 1$$

by the above discussion. Therefore, the pair $(S^\nu, B_{S^\nu} + \Gamma_{S^\nu})$ is not log canonical. □

**Lemma 4.2.** *Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $S$ be a component of $\Delta^{=1}$ with the normalization $S^\nu$. We put $B = \Delta^{<1} + \mathrm{Supp}\Delta^{\geq 1}$, and let $f \colon (Y, B_Y) \to (X, B)$ be a log canonical modification of $X$ and $B$. We put $T = f_*^{-1}S$ with the* *normalization $T^\nu$, and let $\bar{f}\colon T^\nu \to S^\nu$ be the birational morphism induced by $f$.*

$$\begin{tikzcd} T^\nu \arrow[r] \arrow[d, "\bar{f}"'] & T \arrow[d] & \subset & Y \arrow[d, "f"] \\ S^\nu \arrow[r] & S & \subset & X \end{tikzcd}$$

*Let $\Delta_{S^\nu}$ be the effective $\mathbb{R}$-divisor on $S^\nu$ defined by applying adjunction to $(X,\Delta)$ and $S^\nu$, and let $B_{T^\nu}$ be the effective $\mathbb{R}$-divisor on $T^\nu$ defined by applying adjunction to $(Y,B_Y)$ and $T^\nu$. We put $B_{S^\nu} = \Delta_{S^\nu}^{<1} + \mathrm{Supp}\Delta_{S^\nu}^{\geq 1}$.*

*Then, the relation $B_{S^\nu} = \bar{f}_* B_{T^\nu}$ holds and $\bar{f}\colon (T^\nu, B_{T^\nu}) \to (S^\nu, B_{S^\nu})$ is a log canonical modification of $S^\nu$ and $B_{S^\nu}$.*

*Proof.* As in Step 2 in the proof of Theorem 1.1, it is sufficient to take a finite affine open covering $X = \bigcup_i U_i$ and prove this lemma on each open subset $U_i$. Therefore, we may assume that $X$ is quasi-projective. It is obvious from construction that the pair $(T^\nu, B_{T^\nu})$ is log canonical and $K_{T^\nu} + B_{T^\nu}$ is ample over $S^\nu$. Thus, it is enough to prove that $B_{T^\nu}$ is the sum of $\bar{f}_*^{-1} B_{S^\nu}$ and the reduced $\bar{f}$-exceptional divisor.

We define $\Delta_Y$ by $K_Y + \Delta_Y = f^*(K_X+\Delta)$, and we define $\Gamma_Y$ by $\Gamma_Y = \Delta_Y - B_Y$. By Remark 3.4, $\Gamma_Y$ is effective and $\mathrm{Supp}\Gamma_Y = \mathrm{Supp}\Delta_Y^{>1} \supset \mathrm{Exc}(f)$. Let $\Delta_{T^\nu}$ be the effective divisor on $T^\nu$ defined by applying adjunction to $(Y,\Delta_Y)$ and $T^\nu$, and let $\Gamma_{T^\nu}$ be the pullback of $\Gamma_Y$ to $T^\nu$. Then $\Delta_{T^\nu} = B_{T^\nu} + \Gamma_{T^\nu}$ and $\bar{f}_*\Delta_{T^\nu} = \Delta_{S^\nu}$.

We pick a prime divisor $P_{T^\nu}$ on $T^\nu$. If $\Gamma_{T^\nu} \neq 0$ and $P_{T^\nu}$ is a component of $\Gamma_{T^\nu}$, then we can apply Lemma 4.1 to $(Y,B_Y)$, $T$ and $\Gamma_Y$. In this way, we obtain $\mathrm{coeff}_{P_{T^\nu}}(B_{T^\nu}) = 1$ and $\mathrm{coeff}_{P_{T^\nu}}(\Delta_{T^\nu}) > 1$. On the other hand, if $P_{T^\nu}$ is not a component of $\Gamma_{T^\nu}$, then we have $\mathrm{coeff}_{P_{T^\nu}}(\Delta_{T^\nu}) = \mathrm{coeff}_{P_{T^\nu}}(B_{T^\nu}) \leq 1$. From these discussions, we obtain

$$B_{T^\nu} = \Delta_{T^\nu}^{\leq 1} + \mathrm{Supp}\Delta_{T^\nu}^{>1} = \Delta_{T^\nu}^{<1} + \mathrm{Supp}\Delta_{T^\nu}^{\geq 1}.$$

Since $B_{S^\nu} = \Delta_{S^\nu}^{<1} + \mathrm{Supp}\Delta_{S^\nu}^{\geq 1}$ which is the hypothesis of Lemma 4.2, we obtain

$$\bar{f}_* B_{T^\nu} = \bar{f}_*(\Delta_{T^\nu}^{<1} + \mathrm{Supp}\Delta_{T^\nu}^{\geq 1}) = B_{S^\nu}.$$

Furthermore, since $\mathrm{Supp}\Delta_Y^{>1} \supset \mathrm{Exc}(f)$ by Remark 3.4, every $\bar{f}$-exceptional prime divisor $E$ on $T^\nu$ is a component of $\Gamma_{T^\nu}$, hence $\mathrm{coeff}_E(B_{T^\nu}) = 1$. Therefore, we see that $B_{T^\nu}$ is the sum of $\bar{f}_*^{-1} B_{S^\nu}$ and the reduced $\bar{f}$-exceptional divisor. It follows that $\bar{f}\colon (T^\nu, B_{T^\nu}) \to (S^\nu, B_{S^\nu})$ is a log canonical modification of $S^\nu$ and $B_{S^\nu} = \Delta_{S^\nu}^{<1} + \mathrm{Supp}\Delta_{S^\nu}^{\geq 1}$. □

**Lemma 4.3.** *Let $X$ be a normal variety and $B$ a boundary $\mathbb{R}$-divisor on $X$. Suppose that there is a log canonical modification $f\colon (Y,B_Y) \to (X,B)$ and an involution $\tau$ of $X$, that is, $\tau$ is an isomorphism of $X$ such that $\tau^2$ is the identity morphism, such that $\tau_* B = B$. Then $\tau$ lifts to an involution $\tau'$ of $Y$ such that $\tau'_* B_Y = B_Y$.*

*Proof.* The morphism $\tau \circ f\colon (Y, B_Y) \to (X, B)$ is also a log canonical modification of $X$ and $B$. Therefore, the induced birational map $f^{-1} \circ \tau \circ f\colon Y \to Y$ is an isomorphism by Lemma 3.2. We put $\tau' = f^{-1} \circ \tau \circ f$. By Lemma 3.2 again, we have $\tau'_* B_Y = B_Y$. Moreover, the isomorphism $\tau'^2\colon Y \to Y$ is the identity on an open subset of $Y$, so $\tau'^2$ is the identity. In this way, $\tau'$ is an involution of $Y$ such that $\tau'_* B_Y = B_Y$ and $f \circ \tau' = \tau \circ f$. □

The following theorem is the main result of this section.

**Theorem 4.4.** *Let $X$ be a demi-normal scheme, and let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that* $\mathrm{Supp}\Delta$ *does not contain any codimension one singular loci and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We put*

$$B = \Delta^{<1} + \mathrm{Supp}\Delta^{\geq 1}.$$

*Then $X$ equipped with $B$ has a semi-log canonical modification, that is, a semi-log canonical pair $(Y, B_Y)$ and a projective birational morphism $f\colon Y \to X$ such that*

(i) *$f$ is an isomorphism over the generic point of any codimension one singular locus,*
(ii) *$B_Y$ is the sum of the birational transform of $B$ on $Y$ and the reduced $f$-exceptional divisor, and*
(iii) *$K_Y + B_Y$ is $f$-ample.*

*Proof.* We follow the proof of [18, Corollary 1.2]. Let $\nu\colon \bar{X} \to X$ be the normalization. We may write $K_{\bar{X}} + \bar{D} + \bar{\Delta} = \nu^*(K_X + \Delta)$, where $\bar{D}$ is the conductor and $\bar{\Delta}$ is an effective $\mathbb{Q}$-divisor. We decompose $\bar{X} = \amalg_i \bar{X}_i$ into irreducible components, and we set $\bar{D}_i = \bar{D}|_{\bar{X}_i}$ and $\bar{\Delta}_i = \bar{\Delta}|_{\bar{X}_i}$. Then $K_{\bar{X}_i} + \bar{D}_i + \bar{\Delta}_i$ is $\mathbb{Q}$-Cartier. We put

$$\bar{B}_i = \bar{\Delta}_i^{<1} + \mathrm{Supp}\bar{\Delta}_i^{\geq 1}.$$

Then there exists a log canonical modification $g_i\colon (Y_i, T_{Y_i} + B_{Y_i}) \to (\bar{X}_i, \bar{D}_i + \bar{B}_i)$ of $\bar{X}_i$ and $\bar{D}_i + \bar{B}_i$, where $T_{Y_i} = g_{i*}^{-1}\bar{D}_i$ and $B_{Y_i}$ is the sum of $g_{i*}^{-1}\bar{B}_i$ and the reduced $g_i$-exceptional divisor.

Fix an index $i$. We pick an irreducible component $\bar{D}_{i,j}$ of $\bar{D}_i$. Let $\bar{D}^{\nu}_{i,j}$ and $\bar{D}^{\nu}_i$ be the normalizations of $\bar{D}_{i,j}$ and $\bar{D}_i$, respectively. Then $\bar{D}^{\nu}_{i,j}$ is an irreducible component of $\bar{D}^{\nu}_i$. We put $T_{i,j} = g_{i*}^{-1}\bar{D}_{i,j}$ and let $T^{\nu}_{i,j}$ be the normalization of $T_{i,j}$.

$$\begin{array}{ccccc} T^{\nu}_{i,j} & \longrightarrow & T_{i,j} & \subset & Y_i \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle g_i} \\ \bar{D}^{\nu}_{i,j} & \longrightarrow & \bar{D}_{i,j} & \subset & \bar{X}_i \end{array}$$

We define $\Delta_{\bar{D}^{\nu}_{i,j}}$, $B_{T^{\nu}_{i,j}}$, and $B_{\bar{D}^{\nu}_{i,j}}$ as follows:

(a) $\Delta_{\bar{D}^{\nu}_{i,j}}$ is the $\mathbb{Q}$-divisor on $\bar{D}^{\nu}_{i,j}$ defined by adjunction for $(\bar{X}_i, \bar{D}_i + \bar{\Delta}_i)$ and $\bar{D}^{\nu}_{i,j}$,

(b) $B_{T^\nu_{i,j}}$ is the $\mathbb{Q}$-divisor on $T^\nu_{i,j}$ defined by adjunction for $(Y_i, T_{Y_i}+B_{Y_i})$ and $T^\nu_{i,j}$, and

(c) $B_{\bar{D}^\nu_{i,j}}$ is a $\mathbb{Q}$-divisor on $\bar{D}^\nu_{i,j}$ defined by $B_{\bar{D}^\nu_{i,j}} = \Delta^{<1}_{\bar{D}^\nu_{i,j}} + \mathrm{Supp}\Delta^{\geq 1}_{\bar{D}^\nu_{i,j}}$.

By Lemma 4.2, the morphism $(T^\nu_{i,j}, B_{T^\nu_{i,j}}) \to (\bar{D}^\nu_{i,j}, B_{\bar{D}^\nu_{i,j}})$ is a log canonical modification of $\bar{D}^\nu_{i,j}$ and $B_{\bar{D}^\nu_{i,j}}$.

We freely use the notations in the previous paragraph. Recall that $\bar{D}$ is the conductor. Let $\bar{D}^\nu$ be the normalization of $\bar{D}$. By construction, we have

$$\bar{D}^\nu = \amalg_{i,j}\bar{D}^\nu_{i,j}.$$

The construction of $\Delta_{\bar{D}^\nu_{i,j}}$ shows that $\sum_{i,j}\Delta_{\bar{D}^\nu_{i,j}}$ is the effective $\mathbb{Q}$-divisor on $\bar{D}^\nu$ defined by adjunction ([13, Definition 4.2]) for $(\bar{X}, \bar{D}+\bar{\Delta})$ and $\bar{D}^\nu$. Since $\bar{D}$ is the conductor of $X$, the normalization $\bar{D}^\nu$ has an involution $\tau\colon \bar{D}^\nu \to \bar{D}^\nu$ (see [13, 5.2]). Furthermore, [13, Proposition 5.12] shows that the relation $\tau_*\left(\sum_{i,j}\Delta_{\bar{D}^\nu_{i,j}}\right) = \sum_{i,j}\Delta_{\bar{D}^\nu_{i,j}}$ holds. Since we have defined $B_{\bar{D}^\nu_{i,j}} = \Delta^{<1}_{\bar{D}^\nu_{i,j}} + \mathrm{Supp}\Delta^{\geq 1}_{\bar{D}^\nu_{i,j}}$, we see that $\sum_{i,j}B_{\bar{D}^\nu_{i,j}}$ is an effective $\mathbb{Q}$-divisor on $\bar{D}^\nu = \amalg_{i,j}\bar{D}^\nu_{i,j}$ and

$$\tau_*\Big(\sum_{i,j}B_{\bar{D}^\nu_{i,j}}\Big) = \sum_{i,j}B_{\bar{D}^\nu_{i,j}}.$$

Thus, $\tau$ is an involution of $\bar{D}^\nu = \amalg_{i,j}\bar{D}^\nu_{i,j}$ such that $\tau_*\left(\sum_{i,j}B_{\bar{D}^\nu_{i,j}}\right) = \sum_{i,j}B_{\bar{D}^\nu_{i,j}}$. Since $(T^\nu_{i,j}, B_{T^\nu_{i,j}}) \to (\bar{D}^\nu_{i,j}, B_{\bar{D}^\nu_{i,j}})$ is a log canonical modification, $\tau$ lifts to an involution $\tau'$ of $\amalg_{i,j}T^\nu_{i,j}$ such that $\tau'_*\left(\sum_{i,j}B_{T^\nu_{i,j}}\right) = \sum_{i,j}B_{T^\nu_{i,j}}$ by Lemma 4.3.

We have constructed the following objects over the normalization $\bar{X} = \amalg_i\bar{X}_i$ of $X$.

(d) a log canonical pair $(Y_i, T_{Y_i}+B_{Y_i})$ and a projective birational morphism $Y_i \to \bar{X}_i$ such that $K_{Y_i}+T_{Y_i}+B_{Y_i}$ is ample over $\bar{X}_i$, and

(e) an involution $\tau'$ of $\amalg_{i,j}T^\nu_{i,j}$ such that $\tau'_*\left(\sum_{i,j}B_{T^\nu_{i,j}}\right) = \sum_{i,j}B_{T^\nu_{i,j}}$, where $\amalg_{i,j}T^\nu_{i,j}$ is the normalization of $\amalg_i T_{Y_i}$.

Using the gluing theory by Kollár ([13, Corollary 5.37, Corollary 5.33, and Theorem 5.38]), we get a semi-log canonical pair $(Y, B_Y)$ over $X$ whose normalization is $\amalg_i(Y_i, T_{Y_i}+B_{Y_i})$ and the conductor is $\sum_i T_{Y_i}$. More precisely, by [13, Corollary 5.37] we see that the set theoretical equivalence relation (see [13, Definition 9.1]) defined with $\tau'$ is finite. Then, by [13, Corollary 5.33] we get a demi-normal pair $(Y, B_Y)$ over $X$ whose normalization is $\amalg_i(Y_i, T_{Y_i}+B_{Y_i})$ and the conductor is $\sum_i T_{Y_i}$. Finally, by [13, Theorem 5.38] we see that $(Y, B_Y)$ is a semi-log canonical pair. By the construction of $(Y, B_Y)$, the morphism $(Y, B_Y) \to X$ satisfies all the conditions of Theorem 4.4. Indeed, the first condition of Theorem 4.4 follows from that the involution $\tau'$ of $\amalg_{i,j}T^\nu_{i,j}$ is the lift of the involution $\tau$ of $\bar{D}^\nu$, the normalization of the conductor of $X$. The second condition of Theorem 4.4 follows from the definition of $B_{Y_i}$ (see the first paragraph of this proof), and the third condition of Theorem 4.4 is obvious because $K_{Y_i} + T_{Y_i} + B_{Y_i}$ is ample over $\bar{X}_i$ and $\bar{X}_i \to X$ is a finite morphism. In this way, we can get a semi-log canonical modification of $X$ and $B$. □

We close this section with a remark.

**Remark 4.5.** A key ingredient for applying the gluing theory of Kollár is Lemma 4.2, which says that constructing a log canonical modification is compatible with adjunction. In Lemma 4.2, the $\mathbb{R}$-Cartier property of $K_X + \Delta$ is crucial for the proof. Therefore, the hypothesis of Theorem 4.4 that $K_X + \Delta$ is $\mathbb{Q}$-Cartier is necessary for the proof. In general, as shown in [18, Example 3.1], there is a demi-normal scheme $X$ having no semi-log canonical modification. If we take the normalization $\bar{X}$ and the conductor $\bar{D}$ of the demi-normal scheme $X$ in [18, Example 3.1], then the divisor $K_{\bar{X}} + \bar{D}$ is not $\mathbb{Q}$-Cartier and $K_{\bar{X}} + a\bar{D}$ is not $\mathbb{R}$-Cartier for any $a > 1$. See [18, Example 3.1] for details.

## 5 On inversion of adjunction on log canonicity

In this section, we treat inversion of adjunction on log canonicity for log canonical centers. In order to state the main result of this section (see Theorem 5.4), we prepare some definitions.

Let $(X, \Delta)$ be a normal pair such that $\Delta$ is effective, and let $V$ be a log canonical center with the normalization $V^\nu$. For any birational morphism $W \to V^\nu$ from a normal variety $W$, we define an $\mathbb{R}$-divisor $B_W$ on $W$ as follows: Let $f\colon Y \to X$ be a log resolution of $(X, \Delta)$ such that there is an induced surjective morphism from a component $T$ of $\Delta_Y^{=1}$ to $W$, where $\Delta_Y$ is defined by $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Note that such a log resolution always exists since $V$ is a log canonical center. We put $\Delta_T = (\Delta_Y - T)|_T$. Then we obtain a projective surjective morphism $f_T\colon T \to W$, which is induced by $f\colon Y \to X$, such that $K_T + \Delta_T \sim_{\mathbb{R},W} 0$. For any prime divisor $P$ on $W$ with the generic point $P_\eta$, we define $\alpha_{P,T}$ by

$$\alpha_{P,T} = \sup\{\lambda \in \mathbb{R} \mid (T, \Delta_T + \lambda f_T^* P) \text{ is sub log canonical over } P_\eta\}.$$

We note that we may assume that $P$ is a Cartier divisor on $W$ by shrinking $W$ suitably in the above definition of $\alpha_{P,T}$. Then we define an $\mathbb{R}$-divisor $B_W$ on $W$ by

$$B_W = \sum_P (1 - \inf_T \alpha_{P,T}) P$$

where $P$ runs over prime divisors on $W$ and $T$ runs over prime divisors over $X$ such that $a(T, X, \Delta) = -1$ and the image of $T$ on $X$ is $V$.

**Lemma 5.1.** *In the above notation, $B_W$ is a well-defined $\mathbb{R}$-divisor on $W$. Moreover, if $W = V^\nu$, then $B_{V^\nu}$ is effective.*

*Proof.* We take a log resolution $f\colon Y \to X$ as above. Let $D$ be the union of the irreducible components of $\Delta_Y^{=1}$ that are dominant onto $V$ by $f$. Without loss of generality, by replacing $Y$ with a higher model, we may assume that the induced dominant rational map $D \dashrightarrow W$ is a morphism. Moreover, we may assume that there exists a simple normal crossing divisor $\Sigma$ on $Y$ such that the support of the union of $\Sigma$ and $\mathrm{Supp}\Delta_Y$ is a simple normal crossing divisor on $Y$ and that $D \cap \Sigma = \mathrm{Supp} f_D^* P$, where $f_D := f|_D \colon D \to W$. Let $f' \colon Y' \to X$ be another log resolution such that $f' \colon Y' \to X$ and $D'$ satisfies the same condition. If $f' \colon Y' \to X$ factors through $f \colon Y \to X$, then we can directly check that

$$\min_S \alpha_{P,S} = \min_{S'} \alpha_{P,S'}$$

holds, where $S$ (resp. $S'$) runs over irreducible components of $D$ (resp. $D'$). This implies that

$$\inf_T \alpha_{P,T} = \min_S \alpha_{P,S} \in \mathbb{R}$$

holds, where $S$ runs over irreducible components of $D$. Hence $B_W$ is a well-defined $\mathbb{R}$-divisor on $W$.

From now on, we assume that $W = V^\nu$. Let $E$ be the reduced $f$-exceptional divisor on $Y$. By running a $(K_Y + f_*^{-1}\Delta^{<1} + \mathrm{Supp} f_*^{-1}\Delta^{\geq 1} + E)$-minimal model program over $X$, we obtain a dlt blow-up $f' \colon Y' \to X$ (see Theorem 2.10). Note that no components of $D$ are contracted in the above minimal model program. We put $K_{Y'} + \Delta_{Y'} = f'^*(K_X + \Delta)$. Then $\Delta_{Y'}$ is effective by construction. Let $D'$ be the birational transform of $D$ on $Y'$. Since every irreducible component of $S'$ of $D'$ is normal, we obtain a projective surjective morphism $f'_{S'} \colon S' \to V^\nu$, which is induced by $f' \colon Y' \to X$, such that $K_{S'} + \Delta_{S'} = (K_{Y'} + \Delta_{Y'})|_{S'}$ and $K_{S'} + \Delta_{S'} \sim_{\mathbb{R}, V^\nu} 0$. By adjunction, $\Delta_{S'}$ is effective. Hence, we can easily see that

$$\inf_T \alpha_{P,T} = \min_S \alpha_{P,S} \leq 1.$$

This implies that $B_{V^\nu} = \sum_P (1 - \inf_T \alpha_{P,T}) P \geq 0$. $\square$

By the above construction of $B_W$, we obtain an $\mathbb{R}$-$b$-divisor $\mathbf{B}$ such that $\mathbf{B}_W = B_W$. Following [10], we say that $(V^\nu, \mathbf{B})$ is *log canonical* if $(W, \mathbf{B}_W)$ is sub log canonical for all sufficiently higher model $W \to V^\nu$, equivalently, all coefficients of $\mathbf{B}$ are not greater than one.

**Remark 5.2.** If $\dim V = \dim X - 1$, then $B_{V^\nu}$ is nothing but Shokurov's different. Moreover, by definition, we can easily check that $K_W + B_W = \mu^*(K_{V^\nu} + B_{V^\nu})$ holds for every proper birational morphism $\mu \colon W \to V^\nu$ from a normal variety $W$. Hence $(V^\nu, B_{V^\nu})$ is log canonical in the usual sense if and only if $(V^\nu, \mathbf{B})$ is log canonical.

**Remark 5.3.** Our construction of $\mathbf{B}$ is slightly different from that of Hacon's b-divisor $\mathbf{B}(V; X, \Delta)$ in [10] because we take the infimum of $\alpha_{P,T}$ among $T$. By definition, it is clear that $\mathbf{B}$ is greater than or equal to the $b$-divisor $\mathbf{B}(V; X, \Delta)$ defined in [10]. We can prove that $\mathbf{B}$ coincides with Hacon's $\mathbf{B}(V; X, \Delta)$. For the details, see [9].

We are ready to state the main theorem of this section.

**Theorem 5.4** (Log canonical inversion of adjunction, cf. [10])**.** *With notation as above,* $(X, \Delta)$ *is log canonical near* $V$ *if and only if* $(V^\nu, \mathbf{B})$ *is log canonical.*

*Proof.* Since the problem is local, by shrinking $X$, we may assume that $X$ is quasi-projective. If $(X, \Delta)$ is log canonical near $V$, then it is easy to see that $(V^\nu, \mathbf{B})$ is log canonical. Suppose that $(V^\nu, \mathbf{B})$ is log canonical. By Lemma 3.5, we get a crepant model $f\colon (Y, \Delta_Y) \to (X, \Delta)$ satisfying the following properties:

(i) $a(E, X, \Delta) \le -1$ for every $f$-exceptional divisor $E$ on $Y$.

(ii) We define

$$\Delta^\dagger := \Delta_Y^{<1} + \mathrm{Supp}\Delta_Y^{\geq 1} \qquad \text{and} \qquad \Gamma_Y := \Delta_Y^{\geq 1} - \mathrm{Supp}\Delta_Y^{\geq 1}.$$

Then $(Y, \Delta^\dagger)$ is a $\mathbb{Q}$-factorial dlt pair, $\Gamma_Y$ is effective, and the following equality

$$K_Y + \Delta^\dagger = f^*(K_X + \Delta) - \Gamma_Y$$

holds.

(iii) The divisor $K_Y + \Delta^\dagger \sim_{\mathbb{R},X} -\Gamma_Y$ is semi-ample over $X$.

Since $V$ is an lc center of $(X, \Delta)$, we have $f(\mathrm{Supp}\Delta_Y^{>1}) \not\supset V$. We may further assume that there exists a component $S$ of $\Delta_Y^{=1}$ such that $f(S) = V$. We note that $(V^\nu, \mathbf{B})$ is log canonical by assumption. Suppose that $S \cap \mathrm{Supp}\Gamma_Y \neq \emptyset$ holds. We put $K_S + \Delta_S = (K_Y + \Delta_Y)|_S$ by adjunction. Since $(Y, \Delta^\dagger)$ is a $\mathbb{Q}$-factorial dlt pair, $S$ is normal and $\mathrm{coeff}_P(\Delta_S) > 1$ holds for every irreducible component $P$ of $S \cap \mathrm{Supp}\Gamma_Y$ (see also Lemma 4.1). By taking an appropriate birational model $W \to V^\nu$ of $V^\nu$, we may assume that the image of an irreducible component of $S \cap \mathrm{Supp}\Gamma_Y$ by the induced rational map $S \dashrightarrow W$ is a codimension one point of $W$. In this case, we can easily check $\mathbf{B}_W^{>1} \neq 0$. This is a contradiction. Hence we have $S \cap \mathrm{Supp}\Gamma_Y = \emptyset$.

Let $g\colon Y \to Z$ be the contraction over $X$ induced by $K_Y + \Delta^\dagger$. We put $\Gamma_Z = g_*\Gamma_Y$, and we put $h\colon Z \to X$ as the induced birational morphism. By construction, the morphism $(Z, g_*\Delta^\dagger) \to X$ is an lc modification of $X$ and $\Delta^{<1} + \mathrm{Supp}\Delta^{\geq 1}$. Because $\Gamma_Y = g^*\Gamma_Z$, the divisor $-\Gamma_Z$ is ample over $X$ and $\mathrm{Supp}\Gamma_Y = g^{-1}(\mathrm{Supp}\Gamma_Z)$. This fact and $S \cap \mathrm{Supp}\Gamma_Y = \emptyset$ imply that $g(S) \cap \mathrm{Supp}\Gamma_Z = \emptyset$. Furthermore, the inclusion $\mathrm{Exc}(h) \subset \mathrm{Supp}\Gamma_Z$ holds by Remark 3.4. Therefore, $h\colon Z \to X$ is an isomorphism on $Z \setminus \mathrm{Supp}\Gamma_Z$ which contains $g(S)$.

We have proved that the lc modification

$$h\colon (Z, g_*\Delta^\dagger) \to (X, \Delta^{<1} + \mathrm{Supp}\Delta^{\geq 1})$$

is an isomorphism near $g(S)$. Since $h(g(S)) = f(S) = V$, we see that $(X, \Delta)$ is log canonical near $V$. $\square$

We will treat a more precise version of adjunction and inversion of adjunction for log canonical centers of arbitrary codimension in [8]. We strongly recommend the interested reader to see [8].

Kawakita's inversion of adjunction on log canonicity is a very special case of Theorem 5.4.

**Corollary 5.5** (see [12]). *Let $(X, S+B)$ be a normal pair such that $S$ is a reduced divisor, $B$ is effective, and $S$ and $B$ have no common irreducible components. Let $\nu\colon S^\nu \to S$ be the normalization of $S$. We put $K_{S^\nu}+B_{S^\nu}=\nu^*(K_X+S+B)$. Then $(X, S+B)$ is log canonical near $S$ if and only if $(S^\nu, B_{S^\nu})$ is log canonical.*

*Proof.* It is a direct consequence of Theorem 5.4 (see also Remark 5.2). □

## 6 Proof of Theorem 1.7

This section is devoted to the proof of Theorem 1.7. Before proving Theorem 1.7, we introduce some lemmas.

**Lemma 6.1.** *Let $X$ be a normal quasi-projective variety and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X\to S$ be a projective morphism onto a scheme $S$ such that $-(K_X+\Delta)$ is $\pi$-ample. Suppose that*

$$\pi\colon \mathrm{Nklt}(X,\Delta)\to \pi(\mathrm{Nklt}(X,\Delta))$$

*is finite. Let $g\colon (Y,\Delta_Y)\to (X,\Delta)$ and $\Gamma_Y$ be as in Theorem 1.6. We consider a sequence of finite steps of a $(K_Y+\Delta_Y-\Gamma_Y)$-minimal model program over $S$*

$$(Y,\Delta_Y-\Gamma_Y)\dashrightarrow\cdots\dashrightarrow (Y',\Delta_{Y'}-\Gamma_{Y'}).$$

*Let $C'\subset Y'$ be a curve contained in a fiber of $Y'\to S$ such that $(K_{Y'}+\Delta_{Y'}-\Gamma_{Y'})\cdot C'<0$ and let $U$ be a Zariski open subset of $S$ containing $\pi_{Y'}(C')$, where $\pi_{Y'}\colon Y'\to S$. Suppose that the birational map $Y\dashrightarrow Y'$ is an isomorphism on an open subset containing*

$$\mathrm{Supp}\Gamma_Y\cap \pi_Y^{-1}(U)=g^{-1}(\mathrm{Nklt}(X,\Delta))\cap\pi_Y^{-1}(U),$$

*where $\pi_Y=\pi\circ g\colon Y\to S$. Then, the following properties hold true:*

(i) $\mathrm{Supp}\Gamma_{Y'}\not\supset C'$, *and*
(ii) $(K_{Y'}+\Delta_{Y'})\cdot C'<0$.

*Proof.* If $C'\cap\mathrm{Supp}\Gamma_{Y'}$ is empty, then it is obvious that $\mathrm{Supp}\Gamma_{Y'}\not\supset C'$ and $(K_{Y'}+\Delta_{Y'})\cdot C'=(K_{Y'}+\Delta_{Y'}-\Gamma_{Y'})\cdot C'<0$ holds. Therefore, we may assume that $C'$ intersects $\mathrm{Supp}\Gamma_{Y'}$. In the argument below, we can shrink $S$ and assume that $S=U$.

We take a common resolution $\phi\colon W\to Y$ and $\phi'\colon W\to Y'$ of the birational map $Y\dashrightarrow Y'$. Since $Y\dashrightarrow Y'$ is the restriction of a $(K_Y+\Delta_Y-\Gamma_Y)$-minimal model program and $Y\dashrightarrow Y'$ is an isomorphism on an open subset containing $\mathrm{Supp}\Gamma_Y$, there is an effective divisor $F$ on $W$ such that

$$\begin{aligned}\phi^*(K_Y+\Delta_Y-\Gamma_Y)&=\phi'^*(K_{Y'}+\Delta_{Y'}-\Gamma_{Y'})+F,\quad\text{and}\\ \phi^*(K_Y+\Delta_Y)&=\phi'^*(K_{Y'}+\Delta_{Y'})+F.\end{aligned}\tag{6.1}$$

Since $C'$ intersects $\mathrm{Supp}\Gamma_{Y'}$ and the birational map $Y\dashrightarrow Y'$ is an isomorphism on an open subset containing $\mathrm{Supp}\Gamma_Y$, we see that $C'$ intersects an open subset $U' \subset Y'$ on which $Y' \dashrightarrow Y$ is an isomorphism. Hence we can find a curve $C_W$ on $W$ and a curve $C$ on $Y$ such that $\phi(C_W) = C$, $\phi'(C_W) = C'$, and $(F \cdot C_W) \geq 0$. By (6.1), we have $(K_Y + \Delta_Y - \Gamma_Y) \cdot C \geq (K_{Y'} + \Delta_{Y'} - \Gamma_{Y'}) \cdot C'$ and $(K_Y + \Delta_Y) \cdot C \geq (K_{Y'} + \Delta_{Y'}) \cdot C'$. Furthermore, since $Y \dashrightarrow Y'$ is an isomorphism on an open subset containing $\mathrm{Supp}\Gamma_Y$, the condition $C' \subset \mathrm{Supp}\Gamma_{Y'}$ is equivalent to $C \subset \mathrm{Supp}\Gamma_Y$. From these facts, it is sufficient to show that

(a) $\mathrm{Supp}\Gamma_Y \not\supset C$, and
(b) $(K_Y + \Delta_Y) \cdot C < 0$.

We recall that $g\colon Y \to X$ is the birational morphism as in Theorem 1.6. Therefore, $-\Gamma_Y$ is $g$-nef and $K_Y + \Delta_Y = g^*(K_X + \Delta)$. By hypothesis, $-(K_X + \Delta)$ is ample over $S$.

**Step 1.** In this step, we will prove that $g(C)$ cannot be a point.

Suppose by contradiction that $g(C)$ is a point. Then $-\Gamma_Y \cdot C \geq 0$ because $-\Gamma_Y$ is $g$-nef. On the other hand, by recalling that $C'$ intersects $\mathrm{Supp}\Gamma_{Y'}$ and $Y \dashrightarrow Y'$ is an isomorphism on an open subset containing $\mathrm{Supp}\Gamma_Y$, we see that $C$ intersects $\mathrm{Supp}\Gamma_Y$. Since $\mathrm{Supp}\Gamma_Y = g^{-1}(\mathrm{Nklt}(X, \Delta))$, we have $g(C) \in \mathrm{Nklt}(X, \Delta)$. Therefore

$$C \subset g^{-1}(g(C)) \subset g^{-1}(\mathrm{Nklt}(X, \Delta)) = \mathrm{Supp}\Gamma_Y.$$

This shows that $Y \dashrightarrow Y'$ is an isomorphism on an open subset containing $C$. Thus, we have $(K_Y + \Delta_Y - \Gamma_Y) \cdot C = (K_{Y'} + \Delta_{Y'} - \Gamma_{Y'}) \cdot C'$. Then we obtain

$$\begin{aligned} 0 = g^*(K_X + \Delta) \cdot C &= (K_Y + \Delta_Y) \cdot C \\ &\leq (K_Y + \Delta_Y - \Gamma_Y) \cdot C \\ &= (K_{Y'} + \Delta_{Y'} - \Gamma_{Y'}) \cdot C' < 0. \end{aligned}$$

This is a contradiction. Therefore, we see that $g(C)$ cannot be a point.

**Step 2.** By Step 1, we may assume that $g(C)$ is a curve. By construction, $\pi(g(C))$ is a point. Let us recall that $\mathrm{Supp}\Gamma_Y = g^{-1}(\mathrm{Nklt}(X, \Delta))$ holds and that $\pi\colon \mathrm{Nklt}(X, \Delta) \to \pi(\mathrm{Nklt}(X, \Delta))$ is finite. Therefore, if $\mathrm{Supp}\Gamma_Y \supset C$, then $\pi(g(C))$ is not a point, which is a contradiction. Thus, we see that $\mathrm{Supp}\Gamma_Y \not\supset C$, which is the first property we wanted to prove. Since $g(C)$ is a curve, $\pi(g(C))$ is a point, and $-(K_X + \Delta)$ is ample over $S$, we have $(K_X + \Delta) \cdot g(C) < 0$. Hence we obtain

$$(K_Y + \Delta_Y) \cdot C = g^*(K_X + \Delta) \cdot C < 0,$$

which is the second property we wanted to prove.

From the above arguments, we obtain that $\mathrm{Supp}\Gamma_{Y'} \not\supset C'$ and $(K_{Y'} + \Delta_{Y'}) \cdot C' < 0$. We finish the proof of Lemma 6.1. □

Although the following lemma is more or less well known to the experts, we state it here explicitly for the benefit of the reader.

**Lemma 6.2** (Relative Kawamata–Viehweg vanishing theorem). *Let $V$ be a normal variety and let $\Delta_V$ be an effective $\mathbb{R}$-divisor on $V$ such that $K_V + \Delta_V$ is $\mathbb{R}$-Cartier and that $(V, \{\Delta_V\})$ is klt. Let $p\colon V \to W$ be a projective surjective morphism between normal varieties with connected fibers. Assume that $-(K_V + \Delta_V)$ is $p$-ample. Then $R^i p_*\mathcal{O}_V(-\lfloor \Delta_V \rfloor) = 0$ holds for every $i > 0$. This implies that $\lfloor \Delta_V \rfloor$ is connected in a neighborhood of any fiber of $p$. In particular, if $(V, \Delta_V)$ is klt, then $R^i p_*\mathcal{O}_V = 0$ for every $i > 0$.*

*Proof.* Since

$$-\lfloor \Delta_V \rfloor - (K_V + \{\Delta_V\}) = -(K_V + \Delta_V)$$

is $p$-ample, we have $R^i p_*\mathcal{O}_V(-\lfloor \Delta_V \rfloor) = 0$ for every $i > 0$ by the relative Kawamata–Viehweg vanishing theorem (see [6, Corollary 5.7.7]). We consider the following short exact sequence

$$0 \to \mathcal{O}_V(-\lfloor \Delta_V \rfloor) \to \mathcal{O}_V \to \mathcal{O}_{\lfloor \Delta_V \rfloor} \to 0.$$

Since $R^1 p_*\mathcal{O}_V(-\lfloor \Delta_V \rfloor) = 0$, we obtain the following short exact sequence:

$$0 \to p_*\mathcal{O}_V(-\lfloor \Delta_V \rfloor) \to \mathcal{O}_W \to p_*\mathcal{O}_{\lfloor \Delta_V \rfloor} \to 0.$$

This implies that $\operatorname{Supp}\lfloor \Delta_V \rfloor$ is connected in a neighborhood of any fiber of $p$. If we further assume that $(V, \Delta_V)$ is klt, then $\lfloor \Delta_V \rfloor = 0$. Therefore, $R^i p_*\mathcal{O}_V = 0$ for every $i > 0$ when $(V, \Delta_V)$ is klt. $\square$

We are ready to prove Theorem 1.7.

*Proof of Theorem 1.7.* By shrinking $S$ suitably, we may assume that $X$ and $S$ are both quasi-projective. Moreover, we may further assume that $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$ by taking the Stein factorization. By Theorem 1.6, we can construct a projective birational morphism $g\colon Y \to X$ from a normal $\mathbb{Q}$-factorial variety $Y$ and an effective $\mathbb{R}$-divisor $\Gamma_Y$ on $Y$ satisfying (i)–(vi) in Theorem 1.6. Since $K_Y + \Delta_Y = g^*(K_X + \Delta)$, $(K_Y + \Delta_Y)|_{\mathrm{Nklt}(Y, \Delta_Y)}$ is nef over $S$ by Theorem 1.6 (iv). Let us consider $\pi_Y := \pi \circ g\colon Y \to S$. We run a $(K_Y + \Delta_Y - \Gamma_Y)$-minimal model program over $S$ with scaling of an ample divisor. Then we have a sequence of flips and divisorial contractions

$$Y =: Y_0 \overset{\phi_0}{\dashrightarrow} Y_1 \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} Y_i \overset{\phi_i}{\dashrightarrow} \cdots$$

over $S$. As usual, we put $(Y_0, \Delta_{Y_0} - \Gamma_{Y_0}) := (Y, \Delta_Y - \Gamma_Y)$, $\Delta_{Y_{i+1}} = \phi_{i*}\Delta_{Y_i}$, $\Gamma_{Y_{i+1}} = \phi_{i*}\Gamma_{Y_i}$, and $\pi_{Y_i}\colon Y_i \to S$ for every $i$.

If $\dim S < \dim X$, then $K_Y + \Delta_Y - \Gamma_Y$ is not pseudo-effective over $S$ since $-(K_X + \Delta)$ is $\pi$-ample. Hence, the above minimal model program terminates at a Mori fiber space $p\colon (Y_k, \Delta_{Y_k} - \Gamma_{Y_k}) \to Z$ over $S$ (see [3]).

If $\dim S = \dim X$, then $K_Y + \Delta_Y - \Gamma_Y$ is big over $S$ and $(Y, \Delta_Y - \Gamma_Y)$ is klt by (vi) in Theorem 1.6. Therefore, the minimal model program terminates at a good minimal model $(Y_k, \Delta_{Y_k} - \Gamma_{Y_k})$ over $S$ (see [3]).

**Case 1.** In this case, we assume that $\dim S = \dim X$ and that there exists a Zariski open neighborhood $U$ of $P$ such that $Y = Y_0 \dashrightarrow Y_k$ is an isomorphism on some open subset containing $\mathrm{Supp}\varGamma_Y \cap \pi_Y^{-1}(U)$.

Since $\dim S = \dim X$, $(Y_k, \varDelta_{Y_k} - \varGamma_{Y_k})$ is a good minimal model over $S$. In particular, $K_{Y_k} + \varDelta_{Y_k} - \varGamma_{Y_k}$ is nef over $S$. We can take a curve $C_0$ on $Y_0 = Y$ such that $g(C_0) = C^\dagger$ and $C_0 \cap \mathrm{Supp}\varGamma_Y = C_0 \cap \mathrm{Nklt}(Y, \varDelta_Y) \neq \emptyset$. Since $-(K_X + \varDelta) \cdot C^\dagger > 0$, $-(K_Y + \varDelta_Y) \cdot C_0 > 0$ holds. Since $g(C_0) = C^\dagger$, we obtain $C_0 \not\subset \mathrm{Supp}\varGamma_Y$ because $\pi\colon \mathrm{Nklt}(X, \varDelta) \to \pi(\mathrm{Nklt}(X, \varDelta))$ is finite and $\pi(C^\dagger) = P$. Hence we have $C_0 \cdot \varGamma_Y > 0$. Therefore, $-(K_Y + \varDelta_Y - \varGamma_Y) \cdot C_0 > 0$ holds. By assumption, we can easily see that $Y = Y_0 \dashrightarrow Y_k$ is an isomorphism at the generic point of $C_0$. Thus, by the negativity lemma, we can check that

$$0 < -(K_Y + \varDelta_Y - \varGamma_Y) \cdot C_0 \leq -(K_{Y_k} + \varDelta_{Y_k} - \varGamma_{Y_k}) \cdot C_k$$

holds, where $C_k$ is the strict transform of $C_0$ on $Y_k$. This is a contradiction because $K_{Y_k} + \varDelta_{Y_k} - \varGamma_{Y_k}$ is nef over $S$. Hence this case never happens.

**Case 2.** In this case, we assume that $\dim S < \dim X$ and that there exists a Zariski open neighborhood $U$ of $P$ such that $Y = Y_0 \dashrightarrow Y_k$ is an isomorphism on some open subset containing $\mathrm{Supp}\varGamma_Y \cap \pi_Y^{-1}(U)$.

Since $\dim S < \dim X$, the $(K_Y + \varDelta_Y - \varGamma_Y)$-minimal model program terminates at a Mori fiber space $p\colon (Y_k, \varDelta_{Y_k} - \varGamma_{Y_k}) \to Z$ over $S$.

$$\begin{tikzcd}
Y = Y_0 \arrow[r, dashed, "\phi_0"] \arrow[ddrr, "\pi_Y = \pi\circ g"'] & Y_1 \arrow[r, dashed, "\phi_1"] & \cdots \arrow[r, dashed, "\phi_{k-1}"] & Y_k \arrow[d, "p"] \\
 & & & Z \arrow[dl, "\pi_Z"] \\
 & & S &
\end{tikzcd}$$

We note that $P \in \pi_{Y_k}(\mathrm{Supp}\varGamma_{Y_k})$ since $P \in \pi(\mathrm{Nklt}(X, \varDelta)) = \pi_Y(\mathrm{Supp}\varGamma_Y)$. Hence we can take a curve $C_k$ on $Y_k$ such that $p(C_k)$ is a point, $\pi_{Y_k}(C_k) = P$, and $C_k \cap \mathrm{Supp}\varGamma_{Y_k} \neq \emptyset$. Then, by Lemma 6.1, $-(K_{Y_k} + \varDelta_{Y_k}) \cdot C_k > 0$ and $C_k \not\subset \mathrm{Supp}\varGamma_{Y_k}$. In particular, $\varGamma_{Y_k}$ is $p$-ample. Since $(Y_k, \varDelta_{Y_k} - \varGamma_{Y_k})$ is klt and $-(K_{Y_k} + \varDelta_{Y_k} - \varGamma_{Y_k})$ is $p$-ample, we have $R^i p_* \mathcal{O}_{Y_k} = 0$ for every $i > 0$ by Lemma 6.2. We put $\pi_Z\colon Z \to S$. Since

$$-\lfloor \varDelta_{Y_k} \rfloor - (K_{Y_k} + \{\varDelta_{Y_k}\}) = -(K_{Y_k} + \varDelta_{Y_k})$$

is $p$-ample and $(Y_k, \{\varDelta_{Y_k}\})|_{\pi_{Y_k}^{-1}(U)}$ is klt, we obtain that $R^i p_* \mathcal{O}_{Y_k}(-\lfloor \varDelta_{Y_k} \rfloor) = 0$ holds on $\pi_Z^{-1}(U)$ for every $i > 0$ and that $\mathrm{Supp}\lfloor \varDelta_{Y_k} \rfloor = \mathrm{Supp}\varGamma_{Y_k}$ is connected in a neighborhood of any fiber of $p$ on $\pi_Z^{-1}(U)$ by Lemma 6.2. By Lemma 6.1, we see that $\mathrm{Supp}\varGamma_{Y_k} \cap \pi_{Y_k}^{-1}(U)$ is finite over $\pi_Z^{-1}(U)$. Hence, as in Case 1 in the proof of [7, Proposition 9.1], $\dim p^{-1}(z) = 1$ for every closed point $z \in \pi_Z^{-1}(U)$. Then, by [7, Lemma 8.2], $C_k \simeq \mathbb{P}^1$, $C_k \cap \mathrm{Supp}\varGamma_{Y_k}$ is a point, and

$0 < -(K_{Y_k} + \Delta_{Y_k}) \cdot C_k \le 1$ holds. By using the negativity lemma, we can check that

$$-(K_{Y_0} + \Delta_{Y_0}) \cdot C_0 \le -(K_{Y_k} + \Delta_{Y_k}) \cdot C_k \le 1$$

holds, where $C_0$ is the strict transform of $C_k$ on $Y_0 = Y$. Note that $C_0 \cap \mathrm{Nklt}(Y_0, \Delta_{Y_0}) = C_0 \cap \mathrm{Supp}\Gamma_Y$ is a point since $Y = Y_0 \dashrightarrow Y_k$ is an isomorphism in a neighborhood of $\mathrm{Supp}\Gamma_Y \cap \pi_Y^{-1}(U)$. Therefore, $C = g(C_0)$ is a curve on $X$ such that $C \cap \mathrm{Nklt}(X, \Delta)$ is a point by Theorem 1.6 (iv) with $0 < -(K_X + \Delta) \cdot C \le 1$. Hence we can construct a morphism

$$f\colon \mathbb{A}^1 \longrightarrow (X \setminus \mathrm{Nklt}(X, \Delta)) \cap \pi^{-1}(P)$$

such that $f(\mathbb{A}^1) = C \cap (X \setminus \mathrm{Nklt}(X, \Delta))$. This is a desired morphism.

**Case 3.** By Cases 1 and 2, it is sufficient to treat the following situation. There exist a Zariski open neighborhood $U$ of $P$ and $m \ge 0$ such that

(i) for any $i \le m$, the map $Y \dashrightarrow Y_i$ is an isomorphism on some open subset containing $\mathrm{Supp}\Gamma_Y \cap \pi_Y^{-1}(U)$, and
(ii) there is a curve $C' \subset Y_m$ contracted by the extremal birational contraction of the $(K_Y + \Delta_Y - \Gamma_Y)$-minimal model program over $S$ such that $C' \cap \mathrm{Supp}\Gamma_{Y_m} \ne \emptyset$ and $\pi_{Y_m}(C') = P$.

Essentially the same argument as in Case 2 above works with some minor modifications. Let us see it more precisely. Let $\varphi\colon Y_m \to Z$ be the extremal birational contraction in (ii). Let $\pi_Z\colon Z \to S$ be the structure morphism. Then, by Lemma 6.1, $\Gamma_{Y_m}$ is ample over $\pi_Z^{-1}(U)$ and $\mathrm{Supp}\Gamma_{Y_m} \cap \pi_{Y_m}^{-1}(U)$ is finite over $\pi_Z^{-1}(U)$. By Lemmas 6.1 and 6.2, we see that $\mathrm{Supp}\Gamma_{Y_m}$ is connected in a neighborhood of any fiber of $\varphi$ on $\pi_Z^{-1}(U)$. Therefore, $C' \cap \mathrm{Supp}\Gamma_{Y_m}$ is a point. By Lemma 6.1 again, $\dim \varphi^{-1}(z) \le 1$ holds for every closed point $z \in \pi_Z^{-1}(U)$. By Lemma 6.2 , $R^i\varphi_*\mathcal{O}_{Y_m} = 0$ holds on $\pi_Z^{-1}(U)$ for every $i > 0$. Thus, by [7, Lemma 8.2], $C' \simeq \mathbb{P}^1$ with $-(K_{Y_m} + \Delta_{Y_m}) \cdot C' \le 1$. By the negativity lemma, we can check that

$$-(K_{Y_0} + \Delta_{Y_0}) \cdot C_0 \le -(K_{Y_m} + \Delta_{Y_m}) \cdot C' \le 1$$

holds, where $C_0$ is the strict transform of $C'$ on $Y_0 = Y$. We note that $C_0 \cap \mathrm{Nklt}(Y_0, \Delta_{Y_0}) = C_0 \cap \mathrm{Supp}\Gamma_Y$ is a point since $Y = Y_0 \dashrightarrow Y_m$ is an isomorphism in a neighborhood of $\mathrm{Supp}\Gamma_Y \cap \pi_Y^{-1}(U)$. Hence, by the same argument as in Case 2 above, we get a desired morphism

$$f\colon \mathbb{A}^1 \longrightarrow (X \setminus \mathrm{Nklt}(X, \Delta)) \cap \pi^{-1}(P).$$

We finish the proof of Theorem 1.7. □

We close this section with the following generalization of [7, Theorem 9.2]. We will use it in the proof of Theorem 1.8.

**Theorem 6.3.** *Let $\pi\colon X \to S$ be a proper surjective morphism from a normal quasi-projective variety $X$ onto a scheme $S$. Let $\mathcal{P}$ be an $\mathbb{R}$-Cartier divisor on $X$ and let $H$ be an ample Cartier divisor on $X$. Let $\Sigma$ be a closed subset of $X$ and let $P$ be a closed point of $S$ such that there exists a curve $C^\dagger \subset \pi^{-1}(P)$ with $\Sigma \cap C^\dagger \neq \emptyset$. Assume that $-\mathcal{P}$ is $\pi$-ample and that $\pi\colon \Sigma \to \pi(\Sigma)$ is finite. We further assume*

(i) *$\{\varepsilon_i\}_{i=1}^\infty$ is a set of positive real numbers with $\varepsilon_i \searrow 0$ for $i \nearrow \infty$, and*
(ii) *for every $i$, there exists an effective $\mathbb{R}$-divisor $\Delta_i$ on $X$ such that*

$$\mathcal{P} + \varepsilon_i H \sim_{\mathbb{R}} K_X + \Delta_i$$

*and that*

$$\Sigma = \mathrm{Nklt}(X, \Delta_i)$$

*holds set theoretically.*

*Then there exists a non-constant morphism*

$$f\colon \mathbb{A}^1 \longrightarrow (X \setminus \Sigma) \cap \pi^{-1}(P)$$

*such that the curve $C$, the closure of $f(\mathbb{A}^1)$ in $X$, is a rational curve with*

$$0 < -\mathcal{P} \cdot C \leq 1$$

*Proof.* The proof of [7, Theorem 9.2] works as well in this case by replacing [7, Theorem 1.8] in the proof of [7, Theorem 9.2] with Theorem 1.7. □

## 7 Quick review of quasi-log schemes

In this section, we collect some basic definitions of the theory of quasi-log schemes. For the details, see [6, Chapter 6] and [7]. Let us start with the definition of *globally embedded simple normal crossing pairs*.

**Definition 7.1** (Globally embedded simple normal crossing pairs, see [6, Definition 6.2.1])**.** Let $Y$ be a simple normal crossing divisor on a smooth variety $M$ and let $B$ be an $\mathbb{R}$-divisor on $M$ such that $\mathrm{Supp}(B+Y)$ is a simple normal crossing divisor on $M$ and that $B$ and $Y$ have no common irreducible components. We put $B_Y = B|_Y$ and consider the pair $(Y, B_Y)$. We call $(Y, B_Y)$ a *globally embedded simple normal crossing pair* and $M$ the *ambient space* of $(Y, B_Y)$. A *stratum* of $(Y, B_Y)$ is a log canonical center of $(M, Y + B)$ that is contained in $Y$.

Let us recall the definition of *quasi-log schemes*.

**Definition 7.2** (Quasi-log schemes, see [6, Definition 6.2.2])**.** A *quasi-log scheme* is a scheme $X$ endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) $\omega$ on $X$, a closed subscheme $X_{-\infty} \subsetneq X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of $X$ such that there is a proper morphism $f\colon (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of reduced and irreducible subschemes $\{C\}$ coincides with the images of the strata of $(Y, B_Y)$ that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

if there is no risk of confusion. Note that a quasi-log scheme $[X, \omega]$ is the union of $\{C\}$ and $X_{-\infty}$. The reduced and irreducible subschemes $C$ are called the *qlc strata* of $[X, \omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$, and $f\colon (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ or

$$\mathrm{Nqlc}(X, \omega, f\colon (Y, B_Y) \to X)$$

to denote $X_{-\infty}$. If a qlc stratum $C$ of $[X, \omega]$ is not an irreducible component of $X$, then it is called a *qlc center* of $[X, \omega]$.

**Definition 7.3** (Open qlc strata)**.** Let $W$ be a qlc stratum of a quasi-log scheme $[X, \omega]$. We put

$$U := W \setminus \left\{ (W \cap \mathrm{Nqlc}(X, \omega)) \cup \bigcup_{W'} W' \right\},$$

where $W'$ runs over qlc centers of $[X, \omega]$ strictly contained in $W$, and call it the *open qlc stratum* of $[X, \omega]$ associated to $W$.

**Definition 7.4** ($\mathrm{Nqklt}(X, \omega)$)**.** Let $[X, \omega]$ be a quasi-log scheme. The union of $\mathrm{Nqlc}(X, \omega)$ and all qlc centers of $[X, \omega]$ is denoted by $\mathrm{Nqklt}(X, \omega)$. Note that if $\mathrm{Nqklt}(X, \omega) \neq \mathrm{Nqlc}(X, \omega)$ then $[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X, \omega)}]$ naturally becomes a quasi-log scheme by adjunction (see [6, Theorem 6.3.5 (i)] and [7, Theorem 4.6 (i)]).

Although we do not treat applications of the theory of quasi-log schemes to normal pairs here, the following remark is very important.

**Remark 7.5.** Let $(X, \Delta)$ be a normal pair such that $\Delta$ is effective. Then $[X, K_X + \Delta]$ naturally becomes a quasi-log scheme such that $\mathrm{Nqlc}(X, K_X + \Delta)$ coincides with $\mathrm{Nlc}(X, \Delta)$ and that $C$ is a qlc center of $[X, K_X + \Delta]$ if and only if $C$ is a log canonical center of $(X, \Delta)$. Hence $\mathrm{Nqklt}(X, K_X + \Delta)$ corresponds to $\mathrm{Nklt}(X, \Delta)$. For the details, see [6, 6.4.1] and [7, Example 4.10].

## 8 Proof of Theorems 1.8 and 1.9

In this section, we prove Theorems 1.8 and 1.9. Let us start with the proof of Theorem 1.8.

*Proof of Theorem 1.8.* By Steps 1, 2, 3, and 4 in the proof of [7, Theorem 1.6], we can reduce the problem to the case where $X$ is a normal variety such that $-\omega$ is $\pi$-ample and that $\pi\colon \mathrm{Nqklt}(X,\omega) \to \pi(\mathrm{Nqklt}(X,\omega))$ is finite. By taking the Stein factorization, we may further assume that $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$. We put $\Sigma = \mathrm{Nqklt}(X,\omega)$. It is sufficient to find a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow (X \setminus \Sigma) \cap \pi^{-1}(P)$$

such that the curve $C$, the closure of $f(\mathbb{A}^1)$ in $X$, is a (possibly singular) rational curve satisfying $C \cap \Sigma \neq \emptyset$ with

$$0 < -\omega \cdot C \leq 1.$$

Without loss of generality, we may assume that $X$ and $S$ are quasi-projective by shrinking $S$ suitably. Hence we have the following properties:

(a) $\pi\colon X \to S$ is a projective morphism from a normal quasi-projective variety $X$ to a scheme $S$,
(b) $-\omega$ is $\pi$-ample, and
(c) $\pi\colon \Sigma \to \pi(\Sigma)$ is finite, where $\Sigma := \mathrm{Nqklt}(X,\omega)$.

Let $H$ be an ample Cartier divisor on $X$ and let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a set of positive real numbers such that $\varepsilon_i \searrow 0$ for $i \nearrow \infty$. Then, by [7, Theorem 1.10], we have:

(d) there exists an effective $\mathbb{R}$-divisor $\Delta_i$ on $X$ such that

$$K_X + \Delta_i \sim_{\mathbb{R}} \omega + \varepsilon_i H$$

with

$$\mathrm{Nklt}(X, \Delta_i) = \Sigma$$

for every $i$.

Thus, by Theorem 6.3, we have a desired non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow (X \setminus \mathrm{Nqklt}(X,\omega)) \cap \pi^{-1}(P).$$

We complete the proof. □

Finally, we prove Theorem 1.9.

*Proof of Theorem 1.9.* We put $X' = \overline{U_j} \cup \mathrm{Nqlc}(X, \omega)$. Then $[X', \omega']$ naturally becomes a quasi-log scheme by adjunction, where $\omega' = \omega|_{X'}$ (see [6, Theorem 6.3.5 (i)] and [7, Theorem 4.6 (i)]). The induced morphism $\varphi_{R_j} \colon X' \to \varphi_{R_j}(X')$ is denoted by $\pi' \colon X' \to S'$. Then, $-\omega'$ is $\pi'$-ample,

$$\pi' \colon \mathrm{Nqklt}(X', \omega') \to \pi'(\mathrm{Nqklt}(X', \omega'))$$

is finite, and there is a curve $C^\dagger \subset (\pi')^{-1}(P)$ with $\mathrm{Nqklt}(X', \omega') \cap C^\dagger \neq \emptyset$. Hence, by Theorem 1.8, there exists a non-constant morphism

$$f_j \colon \mathbb{A}^1 \longrightarrow U_j \cap \varphi_{R_j}^{-1}(P)$$

with the desired properties. □

**Remark 8.1.** We use the same notation as in the proof of Theorem 1.9. Since $\varphi := \varphi_{R_j} \colon X \to V := \varphi_{R_j}(X)$ is a contraction morphism associated to $R_j$, the natural isomorphism $\varphi_* \mathcal{O}_X \simeq \mathcal{O}_V$ holds (see [6, Theorem 6.7.3 (ii)] and [7, Theorem 4.17 (ii)]). Let $\mathcal{I}_{X'}$ be the defining ideal sheaf of $X'$ on $X$. Then, by the vanishing theorem (see [6, Theorem 6.3.5 (ii)] and [7, Theorem 4.6 (ii)]), we have $R^i \varphi_* \mathcal{I}_{X'} = 0$ for every $i > 0$ since $-\omega$ is $\varphi$-ample. Thus we obtain the following short exact sequence

$$0 \to \varphi_* \mathcal{I}_{X'} \to \varphi_* \mathcal{O}_X \simeq \mathcal{O}_V \to \varphi_* \mathcal{O}_{X'} \to 0.$$

This means that $\varphi_{R_j} \colon X' \to \varphi_{R_j}(X')$ has connected fibers. Therefore, if $Q$ is a close point of $\pi'\left(\mathrm{Nqklt}(X', \omega')\right)$ with $\dim \pi'^{-1}(Q) \geq 1$, then we can always find a curve $\widetilde{C}$ such that $\varphi_{R_j}(\widetilde{C}) = Q$, $\widetilde{C} \not\subset U_j$, and $\widetilde{C} \subset \overline{U_j}$.

**Acknowledgements** The authors thank Christopher Hacon very much for answering their question. They also thank the referee for many useful comments and suggestions.

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