Geometric Satake correspondence in mixed characteristic and Springer correspondence

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1 Introduction

In this article, we explain the relation between geometric Satake correspondence in mixed characteristic and Springer correspondence. In the case of equal characteristic, the corresponding result was obtained by Achar–Rider–Henderson [AHR15]. Our results extend it.

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2 Notaitions

Let k be an algebraic closed field of characteristic $p \geq 0$. If $p > 0$, then let F be $k((t))$ or a totally ramified finite extension of the fraction field $W(k)[1/p]$ of the ring $W(k)$ of witt vectors. If $p = 0$, then put $F = k((t))$. Let G be a (split) reductive group over the ring $\mathcal{O}_F$ of integers in F. Fix a maximal split torus $T \subset G$ and a Borel subgroup $B \subset G$ containing T. Let $\bar{G}$ (resp. $\bar{B}, \bar{T}$) be the reduction of G (resp. B, T) to k. We write W_G for a Weyl group of G. Let $X_*(T)$ be a cocharacter lattice of T, and $X_*^+(T)$ its subset of dominant cocharacters.

Example 2.1. *If $G = \mathrm{GL}_n$, the Borel subgroup B is the subgroup of upper triangular matrices and the torus T is the subgroup of diagonal matrices, then $X_*(T)$ is naturally identified with $\mathbb{Z}^n$. With this identification,*

$$X_*^+(T) = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n \mid m_1 \geq \ldots \geq m_n\}.$$

3 Geometric Satake equivalence

In this section, we explain the definition of the geometric Satake equivalence. The results of this section follows Mirković–Vilonen [MV07] in the case of equal characteristic and Zhu [Zhu17] in the case of mixed characteristic.

Definition 3.1. *(i) We define L^+G as a functor*

$$L^+G\colon \text{Perf-}k\text{-Alg} \to \text{Sets}$$
$$L^+G(R) = \begin{cases} G(R[[t]]) & \text{if } F = k((t)) \\ G(W(R) \otimes_{W(k)} \mathcal{O}_F) & \text{if } F \text{ is of mixed characteristic} \end{cases}$$

from the category Perf-k-Alg *of perfect k-algebras to the category of sets. The functor L^+G is represented by a perfect scheme.*

(ii) We define LG as a functor

$$LG\colon \text{Perf-}k\text{-Alg} \to \text{Sets}$$
$$LG(R) = \begin{cases} G(R((t))) & \text{if } F = k((t)) \\ G(W(R) \otimes_{W(k)} F) & \text{if } F \text{ is of mixed characteristic.} \end{cases}$$

The functor LG is represented by a perfect ind-scheme.

Definition 3.2 (Affine Grassmannian)**.** *We define Gr_G as a fpqc quotient stack*

$$\mathrm{Gr}_G = [LG/L^+G].$$

This is represented by an inductive limit of perfection of projective schemes ([BS17]).

The space Gr_G has the following stratification:

Definition 3.3. *Fix a uniformizer $\varpi \in F$. For $\mu \in X_*^+(T)$, a k-valued point $\varpi^\mu \in \mathrm{Gr}_G(k)$ is defined as the image of ϖ under*

$$F^\times = L\mathbb{G}_m(k) \xrightarrow{L\mu} LG(k) \to \mathrm{Gr}_G(k).$$

Then left L^+G-orbit of ϖ^μ in Gr_G, which does not depend on the choice of ϖ, is denoted by

$$\mathrm{Gr}_{G,\mu}$$

and called Schubert cell. Then

$$\mathrm{Gr}_{G,\leq\mu} = \bigcup_{\mu'\leq\mu} \mathrm{Gr}_{G,\mu'}$$

is closed in Gr_G, and $\mathrm{Gr}_{G,\mu}$ is open in $\mathrm{Gr}_{G,\leq\mu}$. The space $\mathrm{Gr}_{G,\leq\mu}$ is called Schubert variety. It holds that

$$\mathrm{Gr}_G = \bigcup_{\mu\in X_*^+(T)} \mathrm{Gr}_{G,\leq\mu}.$$

We write $\check{G}$ for the Langlands dual group of G over $\overline{\mathbb{Q}}_\ell$.

It is known that there is a monoidal categorical equivalence called geometric Satake equivalence

$$\mathscr{S}_G\colon \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell).$$

Here the category $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell)$ is the category of what is called L^+G-equivariant étale perverse sheaves on Gr_G. The category $\mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)$ is the category of $\check{G}$-representations on fintie dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces. This equivalence was proved by Mirković–Vilonen [MV07] in the case of equal characteristic, and by Zhu [Zhu17] in the case of mixed characteristic.

Explicitly, $\mathscr{S}_G$ satisfies the following: Let $\mathrm{IC}_\mu \in \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell)$ be a what is called the intersection cohomology sheaf on $\mathrm{Gr}_{G,\leq\mu}$. Then

$$\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell)$$

is a semisimple category with simple objects $\{\mathrm{IC}_\mu \mid \mu \in X_*^+(T)\}$, and

$$\mathscr{S}_G(\mathrm{IC}_\mu)$$

is an irreducible $\check{G}$-representation whose highest weight is μ.

4 Springer correspondence

In this section, we explain the definition of the Springer correspondence introduced by Springer. Let $\bar{U}$ be a unipotent radical of $\bar{B}$. We write $\mathfrak{g}$ and $\mathfrak{u}$ for the Lie algebras of $\bar{G}$ and $\bar{U}$, respectively. Let $\mathcal{N} := \mathcal{N}_G \subset \mathfrak{g}$ be a nilpotent cone. The Springer resolution is a map

$$\mu\colon \bar{G} \times^{\bar{B}} \mathfrak{u} \to \mathcal{N},\ [(g,x)] \mapsto \mathrm{Ad}(g)x.$$

The Springer sheaf $\underline{\mathsf{Spr}}$ is the pushforward

$$\underline{\mathsf{Spr}} = \mu_* \overline{\mathbb{Q}}_\ell[\dim \mathcal{N}]$$

of the constant sheaf, which is $\bar{G}$-equivariant perverse sheaf on $\overline{\mathbb{Q}}_\ell$. Moreover, it is known that $\underline{\mathsf{Spr}}$ has a canonical W_G-equivariant structure with respect to the trivial W_G-action on $\mathcal{N}$. Thus we get a functor

$$\mathbb{S}_G\colon \mathrm{Perv}_{\bar{G}}(\mathcal{N}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Rep}(W_G, \overline{\mathbb{Q}}_\ell), A \mapsto \mathrm{Hom}(\underline{\mathsf{Spr}}, A)$$

Here the category $\mathrm{Perv}_{\bar{G}}(\mathcal{N}, \overline{\mathbb{Q}}_\ell)$ is the category of $\bar{G}$-equivariant étale perverse sheaves on $\mathcal{N}$. The category $\mathrm{Rep}(W_G, \overline{\mathbb{Q}}_\ell)$ is the category of W_G-representations on fintie dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces.

Example 4.1. *If $G = \mathrm{GL}_n$, then the $\bar{G}$-orbits of $\mathcal{N}$ correspond to the partitions of n which their Jordan normal forms make. Let IC_ξ be the intersection cohomology sheaf on the $\bar{G}$-orbit of $\mathcal{N}$ corresponding to ξ. Then $\mathrm{Perv}_{\bar{G}}(\mathcal{N}, \overline{\mathbb{Q}}_\ell)$ is a semisimple category with simple objects IC_ξ's, and*

$$\mathbb{S}_G(\mathrm{IC}_\xi)$$

is an irreducible $W_G(= \mathfrak{S}_n)$-representation corresponding to (the transpose of)ξ.

5 The statement of the main theorem

Let $\check{\Phi}$ be the set of roots of $\check{G}$.

Definition 5.1. *We say that $\mu \in X_*(T)$ is small if $\mu \in \mathbb{Z}\check{\Phi}$ and the convex hull of $W_G \cdot \mu$ does not contain twice a root.*

Example 5.2. *If $G = \mathrm{GL}_n$, $\mu = (m_1, \dots, m_n) \in X_*^+(T)$ is small if and only if $m_1 + \cdots + m_n = 0$ and either $m_1 \leq 1$ or $m_n \geq -1$.*

We define a closed subscheme $\mathrm{Gr}_G^{\mathrm{sm}}$ with reduced structure in Gr_G by

$$\mathrm{Gr}^{\mathrm{sm}} := \mathrm{Gr}_G^{\mathrm{sm}} := \bigcup_{\substack{\mu \in X_*^+(T) \\ \mu:\text{small}}} \mathrm{Gr}_\mu \subset \mathrm{Gr}_G.$$

This is finite dimensional as there are only finite small cocharacters.

The full subcategory $\mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}} \subset \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)$ is defined by

$$\mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}} = \{V \in \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell) \mid \text{all the } \check{T}\text{-weights of } V \text{ are small}\}.$$

Then the geometric Satake equivalence $\mathscr{S}_G$ restricts to an equivalence

$$\mathscr{S}_G^{\mathrm{sm}} \colon \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G^{\mathrm{sm}}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}}.$$

Also we define a functor $\Phi_{\check{G}} \colon \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Rep}(W_G, \overline{\mathbb{Q}}_\ell)$ by

$$\Phi_{\check{G}}(V) = V^{\check{T}} \otimes \varepsilon$$

where $V^{\check{T}}$ is a zero-weight space of V, which is a $W_G = N_G(\check{T})/\check{T}$-module, and $\varepsilon \colon W_G \to \{\pm 1\} \subset \overline{\mathbb{Q}}_\ell^\times$ is a sign character of the Coxeter group W_G. We also write $\Phi_{\check{G}}$ for its restriction to $\mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}}$.

The main theorem is the following:

Theorem 5.3. *There is a constant $C_{\bar{G}}$ depending only on $\bar{G}$ such that the following holds:*

Assume $p \neq 2$ and $v_F(p) \geq C_{\bar{G}}$. Then there is an open subscheme $j \colon \mathcal{M}_G \hookrightarrow \mathrm{Gr}_G^{\mathrm{sm}}$ and a morphism $\pi \colon \mathcal{M}_G \to \mathcal{N}_G^{p^{-\infty}}$ (where $(-)^{p^{-\infty}}$ means the perfection of $(-)$) such that

- *The morphism π can be written as a perfection of a finite morphism.*
- *The functor*

$$\Psi_G := j^* \circ \pi_* \colon \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G^{\mathrm{sm}}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}_{\bar{G}}(\mathcal{N}_G^{p^{-\infty}}, \overline{\mathbb{Q}}_\ell) = \mathrm{Perv}_{\bar{G}}(\mathcal{N}_G, \overline{\mathbb{Q}}_\ell)$$

  is well-defined and makes the following diagram naturally commute:

$$\begin{array}{ccc} \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G^{\mathrm{sm}}, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\mathscr{S}_G^{\mathrm{sm}}} & \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}} \\ \downarrow{\scriptstyle \Psi_G} & & \downarrow{\scriptstyle \Phi_{\check{G}}} \\ \mathrm{Perv}_{L^+G}(\mathcal{N}_G, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\mathbb{S}_G} & \mathrm{Rep}(W_G, \overline{\mathbb{Q}}_\ell) \end{array} \tag{1}$$

  In other words, there is a natural isomorphism

$$\Phi_{\check{G}} \circ \mathscr{S}_G^{\mathrm{sm}} \cong \mathbb{S}_G \circ \Psi_G.$$

If $F = \mathbb{C}((t))$, the result was proved by Achar–Rider–Henderson [AHR15].

6 About proof of main theorem

6.1 Rough sketch of proof

Let $L \subset G$ be a Levi subgroup. As in [AHR15], we define restriction functors from G to L for each of the four categories in the diagram 1):

$$\begin{aligned}
\mathfrak{R}_L^G &\colon \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G^{\mathrm{sm}}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}_{L^+L}(\mathrm{Gr}_L^{\mathrm{sm}}, \overline{\mathbb{Q}}_\ell)\\
\mathrm{R}_{\check{L}}^{\check{G}} &\colon \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}} \to \mathrm{Rep}(\check{L}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}},\\
\mathcal{R}_L^G &\colon \mathrm{Perv}_{\bar{G}}(\mathcal{N}_G, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}_{\bar{L}}(\mathcal{N}_L, \overline{\mathbb{Q}}_\ell),\\
\mathsf{R}_{W_L}^{W_G} &\colon \mathrm{Rep}(W_G, \overline{\mathbb{Q}}_\ell) \to \mathrm{Rep}(W_L, \overline{\mathbb{Q}}_\ell).
\end{aligned}$$

And we will define natural isomorphisms called *transitivity isomorphisms*

$$\begin{aligned}
\mathfrak{R}_T^G &\Longleftrightarrow \mathfrak{R}_T^L \circ \mathfrak{R}_L^G,\\
\mathrm{R}_{\check{T}}^{\check{G}} &\Longleftrightarrow \mathrm{R}_{\check{T}}^{\check{L}} \circ \mathrm{R}_{\check{L}}^{\check{G}},\\
\mathcal{R}_T^G &\Longleftrightarrow \mathcal{R}_T^L \circ \mathcal{R}_L^G,\\
\mathsf{R}_{W_T}^{W_G} &\Longleftrightarrow \mathsf{R}_{W_T}^{W_L} \circ \mathsf{R}_{W_L}^{W_G}
\end{aligned}$$

and *intertwining isomorphisms*

$$\begin{aligned}
\mathsf{R}_{W_L}^{W_G} \circ \Phi_{\check{G}} &\Longleftrightarrow \Phi_{\check{L}} \circ \mathrm{R}_{\check{L}}^{\check{G}},\\
\mathcal{R}_L^G \circ \Psi_G &\Longleftrightarrow \Psi_L \circ \mathfrak{R}_L^G,\\
\mathrm{R}_{\check{L}}^{\check{G}} \circ \mathscr{S}_G^{\mathrm{sm}} &\Longleftrightarrow \mathscr{S}_L^{\mathrm{sm}} \circ \mathfrak{R}_L^G,\\
\mathsf{R}_{W_L}^{W_G} \circ \mathbb{S}_G &\Longleftrightarrow \mathbb{S}_L \circ \mathcal{R}_L^G.
\end{aligned}$$

By proving the compatibilities of these isomorphisms, the main theorem reduces to the case where G is semisimple of rank 1 or a torus. In these cases, the results follow by a direct calculation.

The sketch above is the same as the equal characteristic case in [AHR15]. However, some parts are different in mixed characteristic. We will explain that in the next two subsections.

6.2 Partial isomorphism between equal and mixed characteristic affine Grassmannian

In the case of equal characteristic, $\mathcal{M}_G$ is defined as follows:
Let Gr_0^- be a $G(k[t^{-1}])$-orbit of the image of $1 \in G(k((t)))$ in Gr_G. Put

$$\mathcal{M}_G = \mathrm{Gr}_0^- \cap \mathrm{Gr}_G^{\mathrm{sm}}.$$

However, in the case of mixed characteristic, this definition does not work. Thus we show the following theorem:

Theorem 6.1. *There exists a constant $C_{\bar{G}} > 0$ depending only on $\bar{G}$ such that if $v_F(p) > C_{\bar{G}}$, then there is a natural isomorphism*

$$\mathrm{Gr}_G^{\mathrm{sm}} \cong \mathrm{Gr}_G^{\mathrm{sm},\flat}$$

where $\mathrm{Gr}_G^{\mathrm{sm},\flat}$ is the version of $\mathrm{Gr}_G^{\mathrm{sm}}$ for $F = k((t))$ and $G = \bar{G} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[[t]]$.

By this theorem, if $v_F(p) > C_{\bar{G}}$, then we can define $\mathcal{M}_G$ for F as the pullback of $\mathcal{M}_G$ for $k((t))$. This theorem holds not only for Gr_G^{sm} but also for any finite union of Schubert varieties in Gr_G.

Sketch of proof. If $v_F(p) \geq N$, then there is a natural isomorphism

$$(W(R) \otimes_{W(k)} \mathcal{O}_F)/\varpi^N \cong k[[t]]/t^N$$

mapping ϖ to t, where $\varpi \in F$ is a uniformizer. By considering a resolution of a Schubert variety in Gr_G called the Demazure resolution, and using the above isomorphism, we can prove the result. □

6.3 Monoidal structure on restriction functor

To construct an intertwining isomorphism in the diagram

$$\begin{array}{ccc} \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G^{\mathrm{sm}}, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\mathscr{S}_G^{\mathrm{sm}}} & \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}} \\ \downarrow{\scriptstyle \mathfrak{R}_L^G} & & \downarrow{\scriptstyle \mathrm{R}_{\check{L}}^{\check{G}}} \\ \mathrm{Perv}_{L^+L}(\mathrm{Gr}_L^{\mathrm{sm}}, \overline{\mathbb{Q}}_\ell) & \xrightarrow[\mathscr{S}_L^{\mathrm{sm}}]{} & \mathrm{Rep}(\check{L}, \overline{\mathbb{Q}}_\ell)_{\mathrm{sm}}, \end{array}$$

it is enough to construct an intertwining isomorphism in the diagram

$$\begin{array}{ccc} \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\mathscr{S}_G} & \mathrm{Rep}(\check{G}, \overline{\mathbb{Q}}_\ell) \\ \downarrow{\scriptstyle \overline{\mathfrak{R}}_L^G} & & \downarrow{\scriptstyle \mathrm{R}_{\check{L}}^{\check{G}}} \\ \mathrm{Perv}_{L^+L}(\mathrm{Gr}_L, \overline{\mathbb{Q}}_\ell) & \xrightarrow[\mathscr{S}_L]{} & \mathrm{Rep}(\check{L}, \overline{\mathbb{Q}}_\ell) \end{array}$$

where $\overline{\mathfrak{R}}_L^G$ is a restriction functor of $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell)$ defined by using what is called a hyperbolic localization. To get an intertwining isomorphism, we have to construct a monoidal structure on the restriction functor

$$\overline{\mathfrak{R}}_L^G \colon \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}_{L^+L}(\mathrm{Gr}_L, \overline{\mathbb{Q}}_\ell).$$

In the case of equal characteristic, this is constructed by using what is called the fusion product. However, in the case of mixed characteristic, the fusion product can not directly be defined in this situation. Thus we constructed this by using some fundamental sheaf theoretic argument without the fusion product.

7 Future prospects

We believe that the method used in §6.2 can be also used for a partial isomorphism between an object in equal characeristic and its mixed characeristic version other than affine Grassmannians.

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