Linear relations among algebraic points on Drinfeld modules

YEN-TSUNG CHEN

Abstract. This is a survey article for a current development about linear relations among algebraic points on a Drinfeld module defined over a global function field. This result can be regarded as an analogue of Masser's theorem for linear relations among algebraic points on an elliptic curve defined over an algebraic number field.

1. Introduction

1.1. **A theorem of Masser.** Let E be an elliptic curve defined over an algebraic number field K. We embed E in $\mathbb{P}^2$ by choosing a Weierstrass model

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3.$$

Thus, the invariants g_2, g_3 lie in the number field K. Let $\hat{h}$ be the Neron-Tate height on $E(K)$. We set $\eta_1 := \inf_{P\in E(K)\setminus E(K)_{\mathrm{tor}}}\{\hat{h}(P)\} \neq 0$ and $\omega := |E(K)_{\mathrm{tor}}|$. For $\mathbf{x} := (x_1, \ldots, x_\ell) \in \mathbb{Z}^\ell$, we define $|\mathbf{x}| := \max_{i=1}^{\ell}\{|x_i|\}$. Then Masser's theorem is stated as follows.

Theorem 1.1.1 ([Mas88, Thm. E])**.** *Let $K/\mathbb{Q}$ be a finite extension. Let E be an elliptic curve defined over K and $P_1, \ldots, P_\ell \in E(K)$ be distinct non-zero points with Neron-Tate heights at most $\eta_2 \geq \eta_1$. If we set*

$$G := \{(a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell \mid \sum_{i=1}^{\ell} a_iP_i = 0\} \subset \mathbb{Z}^\ell,$$

then there exists a set $\{\mathbf{m}_1, \ldots, \mathbf{m}_\nu\} \subset G$ such that $G = \mathrm{Span}_{\mathbb{Z}}\{\mathbf{m}_1, \ldots, \mathbf{m}_\nu\}$ with bounded size

$$|\mathbf{m}_i| \leq \ell^{\ell-1}\omega(\eta_2/\eta_1)^{(\ell-1)/2}$$

for each $1 \leq i \leq \nu$.

In fact, his result includes all connected commutative algebraic groups defined over an algebraic number field K. This survey article aims to present an analogue of Masser's result for Drinfeld modules.

1.2. **Drinfeld modules.** Let $\mathbb{F}_q$ be a fixed finite field with q elements, for q a power of a prime number p. Let $\mathbb{P}^1_{/\mathbb{F}_q}$ be the projective line defined over $\mathbb{F}_q$ with a fixed point at infinity ∞. Let A be the ring of rational functions regular away from ∞ and k be its fraction field. Let k_∞ be the completion of k at ∞ and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of k_∞. Let $\bar{k}$ be the algebraic closure of k in $\mathbb{C}_\infty$. Let θ be a variable. We identify A with the polynomial ring $\mathbb{F}_q[\theta]$ and k with the rational function field $\mathbb{F}_q(\theta)$.

Let R be an $\mathbb{F}_q$-algebra and $\tau := (x \mapsto x^q) : R \to R$ be the Frobenius q-th power operator. We set $R[\tau]$ be the twisted polynomial ring in τ over R subject to the relation

Date: March 24, 2022.
2010 *Mathematics Subject Classification.* Primary 11G09, 11R58.

$\tau\alpha = \alpha^q\tau$ for $\alpha \in R$. Let $L \subset \bar{k}$ be a finite extension of k. Drinfeld modules defined over L are 1-dimenstional non-trivial t-modules defined over L (see Sec. 2.1 for the definition of t-modules). More precisely, a Drinfeld module is a pair $E = (\mathbb{G}_a, \rho)$ with

$$\rho_t = \theta + \kappa_1\tau + \cdots + \kappa_r\tau^r \in L[\tau],\ \kappa_i \in L$$

so that $\kappa_r \neq 0$ and $r \geq 1$. We define r to be the rank of the Drinfeld module E.

Given finitely many points $P_1, \ldots, P_\ell \in E(L)$, we consider the *relation module over* $\mathbb{F}_q[t]$ associated to these points

$$G := \{(a_1, \ldots, a_\ell) \in \mathbb{F}_q[t]^\ell \mid \rho_{a_1}(P_1) + \cdots + \rho_{a_\ell}(P_\ell) = 0\}.$$

The main theme of this survey article is to establish an upper bound of the size of the generators for the relation module G over $\mathbb{F}_q[t]$ in the case of Drinfeld modules.

1.3. **Overview.** In Section 2, we review the theory of Anderson t-modules [And86] and the notion of dual t-motives [ABP04]. Then we focus on the case of Drinfeld modules. In Section 3, we state an analogue of Theorem 1.1.1 for Drinfeld modules and then sketch the main strategy of the proof.

Acknowledgement

The author thanks to the organizers of the conference "Algebraic Number Theory and Related Topics 2021" in RIMS. The author is also grateful to C.-Y. Chang, J. Yu, O. Gezmis, R. Harada, C. Namoijam, M. Papanikolas and F.-T. Wei for many helpful discussions and comments on this project.

2. Reviews of Anderson's theory

2.1. **Anderson t-modules and dual t-motives.** In this section, We recall the notion of Anderson t-modules [And86] and dual t-motives [ABP04, Def. 4.4.1]. For further information of these objects, one can consult [BP20, HJ20, NP21]. We first recall the definition of Anderson t-modules.

Definition 2.1.1. *Let $L \subset \bar{k}$ be an A-field and $d \in \mathbb{Z}_{>0}$. A d-dimensional t-module defined over L is a pair $E = (\mathbb{G}_a^d, \rho)$ where $\mathbb{G}_a^d$ is the d-dimensional additive group scheme over L and ρ is an $\mathbb{F}_q$-algebra homomorphism*

$$\rho : \mathbb{F}_q[t] \to \mathrm{Mat}_d(L[\tau])$$
$$a \mapsto \rho_a$$

such that $\partial\rho_t - \theta I_d$ is a nilpotent matrix. Here, for $a \in \mathbb{F}_q[t]$ we define $\partial\rho_a := \alpha_0$ whenever $\rho_a = \alpha_0 + \sum_i \alpha_i\tau^i$ for $\alpha_i \in \mathrm{Mat}_d(L)$.

Let F be a subfield of $\bar{k}$ containing L. Then we denote by $E(F)$ the F-valued points of the Anderson t-module E defined over L. More precisely, it is a pair $(\mathbb{G}_a^d(F), \rho)$ of the F-valued points of $\mathbb{G}_a^d$ together with an $\mathbb{F}_q[t]$-module structure on $\mathbb{G}_a^d(F)$ via ρ.

For $n \in \mathbb{Z}$, we define the n-fold Frobenius twisting

$$\mathbb{C}_\infty((t)) \to \mathbb{C}_\infty((t))$$
$$f := \sum_i a_i t^i \mapsto \sum_i a_i^{q^n} t^i =: f^{(n)}.$$

We denote by $\overline{k}[t,\sigma]$ the non-commutative $\overline{k}[t]$-algebra generated by σ subject to the following relation:

$$\sigma f = f^{(-1)}\sigma, \quad f \in \overline{k}[t].$$

Note that $\overline{k}[t,\sigma]$ contains $\overline{k}[t]$, $\overline{k}[\sigma]$, and its center is $\mathbb{F}_q[t]$. Now we recall the notion of dual t-motives.

Definition 2.1.2. *A dual t-motive is a left $\overline{k}[t,\sigma]$-module $\mathscr{M}$ satisfying that*

(i) *$\mathscr{M}$ is a free left $\overline{k}[t]$-module of finite rank.*
(ii) *$\mathscr{M}$ is a free left $\overline{k}[\sigma]$-module of finite rank.*
(iii) *$(t-\theta)^n\mathscr{M} \subset \sigma\mathscr{M}$ for any sufficiently large integer n.*

In what follows, we explain how to associate a $\overline{k}[t,\sigma]$-module $\mathscr{M}_E$ for a given t-module $E = (\mathbb{G}_a^d, \rho)$ defined over L. We set $\mathscr{M}_E := \operatorname{Mat}_{1\times d}\overline{k}[\sigma]$. It naturally has a left $\overline{k}[\sigma]$-module structure. The left $\overline{k}[t]$-module of $\mathscr{M}_E$ is given by the following setting: for each $m \in \mathscr{M}_E$, we define

$$tm := m\rho_t^{\star}. \tag{2.1.3}$$

Here we define

$$\rho_t^{\star} := \alpha_0^{\mathrm{tr}} + (\alpha_1^{(-1)})^{\mathrm{tr}}\sigma + \cdots + (\alpha_s^{(-r)})^{\mathrm{tr}}\sigma^s \tag{2.1.4}$$

whenever $\rho_t = \alpha_0 + \sum_{i=0}^{s}\alpha_i\tau^i$ with $\alpha_i \in \operatorname{Mat}_d(L)$. It is clear that $\mathscr{M}_E$ is free of rank d over $\overline{k}[\sigma]$ and it is a straightforward computation that

$$(t-\theta)^d\mathscr{M}_E \subset \sigma\mathscr{M}_E.$$

If $\mathscr{M}_E$ is free of finite rank over $\overline{k}[t]$, namely it defines an Anderson dual t-motive, then the t-module E is called *t-finite*. In this case, we define $r := \operatorname{rank}_{\overline{k}[t]}\mathscr{M}_E$ to be the rank of the dual t-motive $\mathscr{M}_E$.

Now we are going to explain how to recover a t-finite t-module $E = (\mathbb{G}_a^d, \rho)$ from its associated Anderson dual t-motive $\mathscr{M}_E = \operatorname{Mat}_{1\times d}(\overline{k}[\sigma])$. Let $m = \sum_{i=0}^{n}\mathbf{a}_i\sigma^i \in \mathscr{M}_E$ with $\mathbf{a}_i \in \operatorname{Mat}_{1\times d}(\overline{k})$. Then we define

$$\epsilon_0(m) := \mathbf{a}_0^{\mathrm{tr}} \in \overline{k}^d,\ \epsilon_1(m) := \left(\sum_{i=0}^{n}\mathbf{a}_i^{(i)}\right)^{\mathrm{tr}} \in \overline{k}^d$$

Note that $\epsilon_0 : \mathscr{M}_E \to \overline{k}^d$ is a $\overline{k}$-linear map and $\epsilon_1 : \mathscr{M}_E \to \overline{k}^d$ is an $\mathbb{F}_q$-linear map. We have the following lemma due to Anderson.

Lemma 2.1.5 (Anderson, [HJ20, Prop. 2.5.8], [NP21, Lem. 3.1.2])**.** *For any $a \in \mathbb{F}_q[t]$, we have the following commutative diagrams with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{M}_E & \xrightarrow{\sigma(\cdot)} & \mathscr{M}_E & \xrightarrow{\epsilon_0} & \overline{k}^d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a(\cdot)} & & \downarrow{\scriptstyle a(\cdot)} & & \downarrow{\scriptstyle \partial\rho_a(\cdot)} & & \\
0 & \longrightarrow & \mathscr{M}_E & \xrightarrow{\sigma(\cdot)} & \mathscr{M}_E & \xrightarrow{\epsilon_0} & \overline{k}^d & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{M}_E & \xrightarrow{(\sigma-1)(\cdot)} & \mathscr{M}_E & \xrightarrow{\epsilon_1} & \overline{k}^d & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a(\cdot)} & & \downarrow{\scriptstyle a(\cdot)} & & \downarrow{\scriptstyle \rho_a(\cdot)} & & \\ 0 & \longrightarrow & \mathscr{M}_E & \xrightarrow{(\sigma-1)(\cdot)} & \mathscr{M}_E & \xrightarrow{\epsilon_1} & \overline{k}^d & \longrightarrow & 0. \end{array}$$

In particular, ϵ_0 and ϵ_1 induce isomorphisms:

$$\epsilon_0 : \mathscr{M}_E/\sigma\mathscr{M}_E \cong \mathrm{Lie}(E)(\overline{k}),\ \epsilon_1 : \mathscr{M}_E/(\sigma-1)\mathscr{M}_E \cong E(\overline{k})$$

where ϵ_0 is $\overline{k}[t]$-linear and ϵ_1 is $\mathbb{F}_q[t]$-linear.

Now we recall the notion of t-frame. Let $\{m_1, \ldots, m_r\}$ be a $\overline{k}[t]$-basis of $\mathscr{M}_E$. Then there is an unique matrix $\Phi_E \in \mathrm{Mat}_r(\overline{k}[t])$ such that

$$\sigma(m_1, \ldots, m_r)^{\mathrm{tr}} = \Phi_E(m_1, \ldots, m_r)^{\mathrm{tr}}.$$

Now we define a map

$$\begin{aligned} \iota : \mathrm{Mat}_{1\times r}(\overline{k}[t]) &\to \mathscr{M}_E \\ (a_1, \ldots, a_r) &\mapsto a_1 m_1 + \cdots + a_r m_r. \end{aligned}$$

We call the pair (ι, Φ_E) a t-frame of E. Note that for any $(a_1, \ldots, a_r) \in \mathrm{Mat}_{1\times r}(\overline{k}[t])$, we have

$$\begin{aligned} \sigma\iota(a_1, \ldots, a_r) &= \sigma(a_1, \ldots, a_r)(m_1, \ldots, m_r)^{\mathrm{tr}} \\ &= (a_1^{(-1)}, \ldots, a_r^{(-1)})\sigma(m_1, \ldots, m_r)^{\mathrm{tr}} \\ &= (a_1^{(-1)}, \ldots, a_r^{(-1)})\Phi_E(m_1, \ldots, m_r)^{\mathrm{tr}} \\ &= \iota((a_1^{(-1)}, \ldots, a_r^{(-1)})\Phi_E). \end{aligned}$$

2.2. **Dual t-motives of Drinfeld modules.** In what follows, let $E = (\mathbb{G}_a, \rho)$ be a Drinfeld module with

$$\rho_t = \theta + \kappa_1\tau + \cdots + \kappa_r\tau^r \in L[\tau],\ \kappa_i \in L.$$

Let $\mathscr{M}_E = \overline{k}[\sigma]$ be the associated $\overline{k}[t,\sigma]$-module of E constructed in Section 2.1, whose t-action on the element $m \in \mathscr{M}_E$ is given by

$$tm = m\rho_t^\star = m\left(\theta + \kappa_1^{(-1)}\sigma + \cdots + \kappa_r^{(-r)}\sigma^r\right).$$

Note that $\mathscr{M}_E$ is free of rank r over $\overline{k}[t]$ and $\{1, \sigma, \ldots, \sigma^{r-1}\}$ forms a $\overline{k}[t]$-basis of $\mathscr{M}_E$. Then we have the associated t-frame (ι, Φ_E) where $\Phi_E \in \mathrm{Mat}_r(\overline{k}[t])$ is given by

$$\Phi_E = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \frac{t-\theta}{\kappa_r^{(-r)}} & \frac{-\kappa_1^{(-1)}}{\kappa_r^{(-r)}} & \cdots & \frac{-\kappa_{r-1}^{(-r+1)}}{\kappa_r^{(-r)}} \end{pmatrix} \in \mathrm{Mat}_r(\overline{k}[t]). \tag{2.2.1}$$

Let $P \in E(L)$ be an element in L. Then

$$\epsilon_1^{-1}(P) = \iota(P, 0, \ldots, 0) + (\sigma-1)\mathscr{M}_E \in \mathscr{M}_E/(\sigma-1)\mathscr{M}_E$$

where ϵ_1 is the induced isomorphism of $\mathbb{F}_q[t]$-modules given in Lemma 2.1.5 between $E(\overline{k})$ and $\mathscr{M}_E/(\sigma-1)\mathscr{M}_E$. The crucial point of our strategy is to identify algebraic points on Drinfeld modules to elements in the quotient of its associated dual t-motives via ϵ_1 and then studying the corresponding question in the t-motivic language.

3. Linear equations on Drinfeld modules

3.1. **The main result.** Let $L \subset \overline{k}$ be a finite extension of k. For a divisor D of L, we set $\mathscr{L}(D)$ to be the $\mathbb{F}_q$-vector space of meromorphic functions f over the curve over $\mathbb{F}_q$, whose closed points are the places of L, and such that $\mathrm{div}(f)+D$ is an effective divisor. In other words,

$$\mathscr{L}(D)=\{f\in L^\times \mid \mathrm{div}(f)+D\geq 0\}\cup\{0\}. \tag{3.1.1}$$

For $\mathbf{m}:=(m_1,\ldots,m_\ell)\in\mathbb{F}_q[t]^\ell$, we define $\deg_t(\mathbf{m}):=\max_{i=1}^{\ell}\{\deg_t(m_i)\}$.

The main result of this survey article is the following theorem, which can be viewed as an analogue of Masser's theorem [Mas88] for Drinfeld modules.

Theorem 3.1.2 ([Che20, Thm. 1.2.5]). *Let $L\subset\overline{k}$ be a finite extension of k. Let $E=(\mathbb{G}_a,\rho)$ be a Drinfeld module defined over L and $P_1,\ldots,P_\ell\in E(L)$ be distinct non-zero elements in L. If we set*

$$G:=\{(a_1,\ldots,a_\ell)\in\mathbb{F}_q[t]^\ell \mid \sum_{i=1}^{\ell}\rho_{a_i}(P_i)=0\}\subset\mathbb{F}_q[t]^\ell,$$

then there exist an explicitly constructed divisor D of L and a set $\{\mathbf{m}_1,\ldots,\mathbf{m}_\nu\}\subset G$ such that $G=\mathrm{Span}_{\mathbb{F}_q[t]}\{\mathbf{m}_1,\ldots,\mathbf{m}_\nu\}$ with bounded degree $\deg_t(\mathbf{m}_i)<\dim_{\mathbb{F}_q}\mathscr{L}(D)+\ell$.

3.2. **Key ingredients of the proof.** In what follows, we present the key ideas of our strategy. We shall mention that our approach is different from Masser's. The first step of our method is stated as follows which reduces our task to solving Frobenius difference equations.

Lemma 3.2.1 ([Che20, Lem. 3.1.1]). *Let $L\subset\overline{k}$ be a finite extension of k. Let $E=(\mathbb{G}_a,\rho)$ be a Drinfeld $\mathbb{F}_q[t]$-module defined over L with $\phi_t=\theta+\kappa_1\tau+\cdots+\kappa_r\tau^r\in L[\tau]$ and $\kappa_r\neq 0$. Let $P_1,\ldots,P_\ell\in E(L)$ be distinct non-zero elements in L and $a_1,\ldots,a_\ell\in\mathbb{F}_q[t]$ not all zero. Then the following assertions are equivalent.*

(1) $\rho_{a_1}(P_1)+\cdots+\rho_{a_\ell}(P_\ell)=0$ *in* $E(L)$.
(2) $\iota(a_1P_1+\cdots+a_\ell P_\ell,0,\ldots,0)=0$ *in* $\mathscr{M}_E/(\sigma-1)\mathscr{M}_E$.
(3) *There exist $\delta_1,\ldots,\delta_r\in\overline{k}[t]$ such that*

$$(a_1P_1+\cdots+a_\ell P_\ell,0,\ldots,0)=(-\delta_1^{(-1)},\ldots,-\delta_r^{(-1)})\Phi_E-(-\delta_1,\ldots,-\delta_r).$$

(4) *There exist $\delta_1,\ldots,\delta_r\in\overline{k}[t]$ such that*

$$\begin{cases}\delta_1 &=\delta_r^{(-1)}\left(\frac{t-\theta}{\kappa_r^{(-r)}}\right)+a_1P_1+\cdots+a_\ell P_\ell\\ \delta_2^{(1)} &=\delta_1+\delta_r\left(\frac{-\kappa_1}{\kappa_r^{(-r+1)}}\right)\\ &\vdots\\ \delta_r^{(r-1)} &=\delta_{r-1}^{(r-2)}+\delta_r^{(r-2)}\left(\frac{-\kappa_{r-1}}{\kappa_r^{(-1)}}\right)\end{cases}.$$

(5) *There exists $g \in L[t]$ such that*

$$\kappa_r g^{(r)} + \cdots + \kappa_1 g^{(1)} - (t-\theta) g = a_1 P_1 + \cdots + a_\ell P_\ell.$$

The second step of our method is to reduce the task from solving Frobenius difference equations to solving a specific homogeneous linear system over $\mathbb{F}_q[t]$. More precisely, we have the following result.

Theorem 3.2.2 ([Che20, Thm. 3.2.1]). *There exists an explicitly constructed divisor D of L with $d := \dim_{\mathbb{F}_q}(\mathscr{L}(D))$ and a matrix $B \in \mathrm{Mat}_{m\times n}(\mathbb{F}_q[t])$ with $0 < m := \mathrm{rank}(B) < n := d + \ell$ and $\deg_t(B) \leq 1$ such that the canonical projection*

$$\pi : \Gamma := \{\mathbf{x} \in \mathbb{F}_q[t]^{(d+\ell)} \mid B\mathbf{x}^{\mathrm{tr}} = 0\} \twoheadrightarrow G = \{(a_1, \ldots, a_\ell) \in \mathbb{F}_q[t]^\ell \mid \sum_{i=1}^{\ell} \rho_{a_i}(P_i) = 0\}$$

$$(g_1, \ldots, g_d, a_1, \ldots, a_\ell) \mapsto (a_1, \ldots, a_\ell)$$

is a well-defined surjective $\mathbb{F}_q[t]$-module homomorphism.

Once we have Theorem 3.2.2 in hand, we can apply tools from Diophantine geometry to analyze the linear system occurs in question. More precisely, we prove an analogue of [Cas59, Lem. 8 (p.135)] for lattice over $\mathbb{F}_q[t]$ in [Che20, Lem. 2.4.1]. Then we apply a function field analogue of Bombieri-Vaaler's theorem [BV83] which was established by Thunder in [Thu95]. We consult readers to [Che20] for the related details.

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Department of Mathematics, National Tsing Hua University, Hsinchu City 30042, Taiwan R.O.C.
Email address: ytchen.math@gmail.com