Prismatic and q-crystalline sites of higher level : an announcement

Kimihiko Li
Graduate School of Mathematical Sciences,
The University of Tokyo

1 Introduction

This is an announcement of a recent work of the author on the higher level analogs of the prismatic site and the q-crystalline site. The goals of this talk at RIMS were to introduce the construction of the higher level versions of the prismatic and q-crystalline sites, the Frobenius functors between the categories of crystals on these sites, and the relationship between the category of crystals on the m-prismatic site and the $(m-1)$-q-crystalline site. We also related our results to the usual Frobenius descent.

For an integer $m \geq 0$, Berthelot defined the notion of crystalline cohomology of level m as the cohomology of the m-crystalline site with respect to the crystal on this site. One of the motivations of introducing m-crystalline cohomology would be to develop a p-adic cohomology theory over a ramified base. He proved that the category of crystals on the m-crystalline site is equivalent to that on the usual crystalline site on its m-th Frobenius pullback. This equivalence is called the Frobenius descent. These notions are also related to p-adic analysis. On the other hand, Bhatt and Scholze defined two new p-adic cohomology theories generalizing crystalline cohomology, called prismatic cohomology and q-crystalline cohomology, which were defined as the cohomology of the corresponding sites called the prismatic site and the q-crystalline site respectively. Prismatic cohomology is a flexible tool and recovers the other integral p-adic cohomology theories. q-crystalline cohomology is a canonical q-deformation of crystalline cohomology, which is related to the theory of prismatic cohomology. In this work, we consider the higher level analogs of the prismatic and q-crystalline sites, which are expected to be the basis of the theory of the prismatic and q-crystalline sites over the ramified base, and the theory of p-adic analysis which is compatible with prismatic cohomology.

The outline of this article is as follows. In Section 2, we briefly review the theories of the prismatic site and the q-crystalline site, and the theory of the higher level crystalline site. In Section 3, we first construct the level m prismatic and q-crystalline site. After that, we state our main results.

2 Overview of some p-adic cohomology theories

2.1 Prismatic and q-crystalline cohomology

We first briefly recall prismatic cohomology. Fix a prime p. First we define the notion of prism:

Definition 2.1 ([BS19]) A *prism* is a pair (A, I) where

- A is a commutative ring with a δ-structure (this defines a Frobenius lift ϕ on A),
- $I \subseteq A$ is an ideal defining a Cartier divisor,

such that:

1. A is derived (p, I)-complete.
2. $p \in I + \phi(I)A$.

(A, I) is called a *bounded prism* if in addition A/I has bounded p^∞-torsion.

The boundedness is a kind of finiteness condition. For a bounded prism (A, I), the derived (p, I)-completeness in Definition 2.1 is equivalent to the classical (p, I)-completeness. The following holds:

Proposition 2.2 (Rigidity property of prisms) *If $(A, I) \to (B, J)$ is a map of prisms, then $J = IB$.*

Prisms are flexible enough to consider the relations to the other p-adic cohomology theories. Here are some examples:

Example 2.3
1. **(Crystalline prism)** Let A be a p-torsion free p-complete ring with a Frobenius lift ϕ. This induces a unique δ-ring structure on A and $(A, (p))$ is a prism.
2. **(Perfect prism)** A prism (A, I) is called perfect if A is perfect, i.e. $\phi : A \to A$ is an isomorphism. Then the following two categories are equivalent:
   - The category of perfectoid rings R.
   - The category of perfect prisms (A, I).

   The functors are $R \mapsto (A_{\mathrm{inf}}(R), \ker(\theta))$ and $(A, I) \mapsto A/I$ respectively.
3. **(q-crystalline prism)** Let $A = \mathbf{Z}_p[\![q-1]\!]$, the $(p, q-1)$-adic completion of $\mathbf{Z}[q]$ with δ-structure given by $\delta(q) = 0$. Let $I = ([p]_q)$, where $[p]_q = \frac{q^p-1}{q-1} = 1 + q + \cdots + q^{p-1}$ is the q-analog of p. Then (A, I) is a prism.

Our next goal is to define the prismatic site and the q-crystalline site.

Definition 2.4 (Prismatic site) Let (A, I) be a bounded prism, and let X be a smooth p-adic formal scheme over A/I. Let $(X/A)_{\Delta}$ be the category of maps $(A, I) \to (B, IB)$ of bounded prisms together with a map $\mathrm{Spf}(B/IB) \to X$ over A/I; the notion of morphism is the obvious one.

$$\begin{array}{ccccc} \mathrm{Spf}(A/I) & \longleftarrow & X & \longleftarrow & \mathrm{Spf}(B/IB) \\ \downarrow & & & & \downarrow \\ \mathrm{Spf}(A) & \longleftarrow & & \longleftarrow & \mathrm{Spf}(B). \end{array}$$

We denote the object by $(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X) \in (X/A)_{\Delta}$. The category $(X/A)_{\Delta}$ with the topology defined by flat covers is called the *prismatic site*.

By using the q-crystalline prism $(A, I) = (\mathbf{Z}_p[\![q-1]\!], ([p]_q))$ in Example 2.3, we can define the q-PD pair. In the following definition, we call a pair consisting of a δ-ring and its ideal a δ-pair.

Definition 2.5 (q-PD pair) A q-PD pair is given by a derived $(p, [p]_q)$-complete δ-pair (D, I) over $(A, (q-1))$ satisfying the following conditions:

1. For any $f \in I$, $\phi(f) - [p]_q\delta(f) \in [p]_q I$.
2. The pair $(D, ([p]_q))$ is a bounded prism over $(A, ([p]_q))$, i.e., D is $[p]_q$-torsion free and $D/([p]_q)$ has bounded p^∞-torsion.
3. The ring $D/(q-1)$ is p-torsion free with finite $(p, [p]_q)$-complete Tor-amplitude over D.
4. D/I is classically p-complete.

Then we can define $\gamma(f) := \frac{\phi(f)}{[p]_q} - \delta(f)$ as the q-analog of the p-th divided power. By condition (1) in Definition 2.5, $\gamma(I) \subseteq I$. The q-crystalline site is defined as follows:

Definition 2.6 (q-crystalline site) Let (D, I) be a q-PD pair, and let R be a p-completely smooth D/I-algebra. Let $(R/D)_{q\text{-crys}}$ be the category of maps $(D, I) \to (E, J)$ of q-PD pairs together with a D/I-algebra map $R \to E/J$; the notion of morphism is the obvious one.

$$\begin{array}{ccccc} \mathrm{Spf}(D/I) & \longleftarrow & \mathrm{Spf}(R) & \longleftarrow & \mathrm{Spf}(E/J) \\ \downarrow & & & & \downarrow \\ \mathrm{Spf}(D) & & \longleftarrow & & \mathrm{Spf}(E). \end{array}$$

We denote the object by $(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J) \to \mathrm{Spf}(R)) \in (R/D)_{q\text{-crys}}$. The category $(R/D)_{q\text{-crys}}$ with the indiscrete topology is called the *q-crystalline site*.

We can naturally define the structure sheaf $\mathcal{O}_{\mathbb{\Delta}}$ on $(X/A)_{\mathbb{\Delta}}$ by the assignment

$$(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X) \mapsto B.$$

Then we can define prismatic cohomology as

$$R\Gamma_{\mathbb{\Delta}}(X/A) := R\Gamma((X/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}).$$

q-crystalline cohomology

$$q\Omega(R/D) := R\Gamma((R/D)_{q\text{-crys}}, \mathcal{O}_{q\text{-crys}})$$

can be defined similarly.

Prismatic cohomology recovers most known integral p-adic cohomology theories. For example:

Theorem 2.7 (Comparison theorem, [BS19]) *Let (A, I) be a bounded prism, and let X be a smooth p-adic formal scheme over A/I.*

- **(Crystalline comparison)** *If $I = (p)$, then*

$$R\Gamma_{\mathrm{crys}}(X/A) \simeq \phi_A^* R\Gamma_{\Delta}(X/A),$$

where ϕ_A^* denotes the pullback by the Frobenius lift ϕ_A of A.

- **(Étale comparison)** *Assume A is perfect. Let X_η be the generic fibre of X over $\mathbf{Q}_p$, as a (pre-)adic space. Then*

$$R\Gamma_{\acute{e}t}(X_\eta, \mathbf{Z}/p^n\mathbf{Z}) \simeq (R\Gamma_{\Delta}(X/A)/p^n[\frac{1}{I}])^{\phi=1}.$$

Let (D, I) be a q-PD pair, and let R be a p-completely smooth D/I-algebra. Let R' be the Frobenius pullback $R \mathop{\widehat{\times}}_{D/I,\phi^*} (D/[p]_q D)$. Then:

- **(q-crystalline comparison)** $R\Gamma_{\Delta}(X'/D) \simeq q\Omega(X/D)$.

On the other hand, q-crystalline cohomology recovers the usual crystalline cohomology when $q = 1$:

Theorem 2.8 (q-crystalline and usual crystalline cohomology, [BS19]) *If (D, I) is a q-PD pair and R is a p-completely smooth D/I-algebra, then there is a canonical identification*

$$q\Omega(R/D)\widehat{\otimes}^L_D D/(q-1) \simeq R\Gamma_{\mathrm{crys}}(R/(D/(q-1))).$$

2.2 m-crystalline site

Next we review the theory of the higher level crystalline site. Let m be a non-negative integer. We consider the following base ring:

Definition 2.9 (m-PD ring, [Ber90]) Let D be a $\mathbf{Z}_{(p)}$-algebra and let J be an ideal of D. An *m-PD structure* on J is a PD ideal (I, γ) of D which satisfies the following two conditions:

1. $J^{(p^m)} + pJ \subseteq I \subseteq J$;
2. The PD structure γ is compatible with the unique one on $p\mathbf{Z}_{(p)}$.

Here, $J^{(p^m)}$ denotes the ideal of D generated by x^{p^m} for all elements x of J. We call (J, I, γ) an *m-PD ideal* of D, and (D, J, I, γ) an *m-PD ring*.

Then we can define the m-crystalline site. When $m = 0$, it coincides with the usual crystalline site.

Definition 2.10 (m-crystalline site) Let (D, J, I, γ) be an m-PD ring on which p is nilpotent with $p \in I$. Let X be a scheme smooth and seperated over D/J. Let $(X/D)_{m\text{-crys}}$ be the category of maps $(D, J, I, \gamma) \to (E, J_E, I_E, \gamma_E)$ of m-PD rings together with a map $\mathrm{Spec}(E/J_E) \to X$ over

D/J such that there exists some $n \geq 0$ such that $p^n E = 0$. The notion of morphism is the obvious one.

$$\begin{array}{ccccc} \mathrm{Spec}(D/J) & \longleftarrow & X & \longleftarrow & \mathrm{Spec}(E/J_E) \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(D) & & \longleftarrow & & \mathrm{Spec}(E). \end{array}$$

The category $(X/D)_{m\text{-crys}}$ with the topology defined by flat covers is called the *m-crystalline site.*

We can consider the category of quasi-coherent crystals $\mathscr{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}})$ on the m-crystalline site. The crystal $\mathscr{F}$ is a sheaf of modules with the following property:

- for any morphism $(E, I_E) \to (E_1, I_{E_1})$ in $(X/D)_{m\text{-crys}}$, the canonical map

$$\mathscr{F}(E, I_E) \otimes_E E_1 \to \mathscr{F}(E_1, I_{E_1})$$

  is an isomorphism of E_1-modules.

We usually use the crystals to define m-crystalline cohomology, so it is important to compare the categories of crystals. Let $X' := X \underset{\mathrm{Spec}(D/J),(\phi^m)^*}{\times} \mathrm{Spec}(D/I)$. Berthelot constructed the functor of sites $\sigma : (X/D)_{m\text{-crys}} \to (X'/D)_{\mathrm{crys}}$. He also proved that σ induces the equivalence of categories of crystals, which we call the Frobenius descent functor:

Theorem 2.11 (Frobenius descent, [Ber00]) *There exists an equivalence of categories of crystals:*

$$\sigma^* : \mathscr{C}^{\mathrm{qcoh}}((X'/D)_{crys}) \xrightarrow{\simeq} \mathscr{C}^{\mathrm{qcoh}}((X/D)_{m\text{-}crys}).$$

3 m-prismatic and m-q-crystalline sites

In this section, we first construct the m-prismatic site and the m-q-crystalline site. Then we show the prismatic and q-crystalline versions of the Frobenius descent (Theorem 2.11). After that, we show a categorical analog of the q-crystalline comparison in Theorem 2.7 by comparing the categories of crystals on the m-prismatic site and the $(m-1)$-q-crystalline site. Finally, we show that when $q = 1$, the higher level categorical analog of Theorem 2.8 holds true, and that the q-crystalline version of the Frobenius descent we constructed is compatible with the usual Frobenius descent by this identification.

3.1 Prismatic and q-analogs of the Frobenius descent

The definition of the m-prismatic site is the following:

Definition 3.1 (m-prismatic site) Let $J = (\phi^m)^{-1}(I)$, and let X be a (p, I)-adic formal scheme smooth and separated over A/J. We define the *m-prismatic site* $(X/A)_{m-\Delta}$ of X over A as follows.

- Objects: Maps $(A, I) \to (E, I_E)$ of bounded prisms with $\mathrm{Spf}(E/J_E) \to X$ over A/J (where $J_E = (\phi^m)^{-1}(I_E)$) satisfying:

(*) $\mathrm{Spf}(E/J_E) \to X$ factors over some open affine $\mathrm{Spf}(R) \subseteq X$.

- Morphisms: Obvious ones.
- Covers: Faithfully flat maps $(E, I_E) \to (E', I_{E'})$ of prisms, i.e., E' is (p, I_E)-completely faithfully flat over E.

We shall often denote an object of $(X/A)_{m-\Delta}$ by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X).$$

Since q-PD pairs are constructed by using the prisms, the m-q-crystalline site can be constructed in much the same way as the m-prismatic site:

Definition 3.2 (m-q-crystalline site) Let $J = (\phi^m)^{-1}(I)$ and let X be a p-adic formal scheme smooth and separated over D/J. We define the *m-q-crystalline site* $(X/D)_{m\text{-}q\text{-crys}}$ of X over D as follows.

- Objects: Maps $(D, I) \to (E, I_E)$ of q-PD pairs with $\mathrm{Spf}(E/J_E) \to X$ over D/J (where $J_E = (\phi^m)^{-1}(I_E)$) satisfying:

  (*) $\mathrm{Spf}(E/J_E) \to X$ factors over some open affine $\mathrm{Spf}(R) \subseteq X$.

- Morphisms: Obvious ones.
- Covers: $(p, [p]_q)$-completely faithfully flat maps $(E, I_E) \to (E', I_{E'})$ such that

  $$\widehat{I_E E'} = I_{E'},$$

  where the completion is the classical $(p, [p]_q)$-completion.

We shall often denote an object of $(X/D)_{m\text{-}q\text{-crys}}$ by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X).$$

Next we construct the functors from the level m site to the level 0 site, which induce the prismatic and q-crystalline version of the Frobenius descent. We can give the proofs of the results for the m-q-crystalline site parallel to that for the m-prismatic site, so it is enough to focus mainly on the m-prismatic site.

Construction 3.3 Let X' be $X \mathbin{\widehat{\times}}_{\mathrm{Spf}(A/J),(\phi^m)^*} \mathrm{Spf}(A/I)$, where the completion is the classical (p, I)-completion. Define the functor

$$\rho : (X/A)_{m-\Delta} \to (X'/A)_{\Delta}$$

by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_E) \to X) \mapsto (\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/I_E) \xrightarrow{(1)} X'),$$

where (1) is:

$$\mathrm{Spf}(E/I_E) \xrightarrow{(2)} \mathrm{Spf}(E/J_E) \underset{\mathrm{Spf}(A/J),(\phi^m)^*}{\widehat{\times}} \mathrm{Spf}(A/I)$$
$$\longrightarrow X \underset{\mathrm{Spf}(A/J),(\phi^m)^*}{\widehat{\times}} \mathrm{Spf}(A/I) = X',$$

and (2) is induced by a map of rings

$$E/J_E \underset{A/J,\phi^m}{\widehat{\otimes}} A/I \to E/I_E; \qquad e \otimes a \mapsto \phi^m(e)a.$$

For a q-PD pair (D, I), let X' be $X \underset{\mathrm{Spf}(D/J),(\phi^m)^*}{\widehat{\times}} \mathrm{Spf}(D/I)$, where the completion is the classical p-completion. Then we can construct the q-crystalline version of the functor of sites:

$$\rho : (X/D)_{m\text{-}q\text{-crys}} \to (X'/D)_{q\text{-crys}}.$$

To prove that the pullbacks of these functors induce the equivalences of categories of crystals, we apply the site-theoretic proof of Oyama [Oya17] and Xu [Xu19]. So we need the following proposition about the site-theoretic properties of ρ.

Proposition 3.4 (L.)

1. *The functor ρ is fully faithful, continuous and cocontinuous.*
2. *Let $(E', I_{E'})$ be an object in $(X'/A)_{\Delta}$. Then there exists an object (E, I_E) in $(X/A)_{m-\Delta}$ and a cover of the form*
$$(E', I_{E'}) \to \rho(E, I_E).$$

The proof of Proposition 3.4 is given by applying the ring theoretic argument in the proof of [Xu19] to the prismatic case. The cover in Proposition 3.4.2 is constructed by using the property of the prismatic (and the q-PD) envelope.

By Proposition 3.4.1, we obtain a morphism of topoi

$$\widehat{\rho} : \widetilde{(X/A)}_{m-\Delta} \to \widetilde{(X'/A)}_{\Delta},$$
$$\widehat{\rho} : \widetilde{(X/D)}_{m\text{-}q\text{-crys}} \to \widetilde{(X'/D)}_{q\text{-crys}}.$$

By Proposition 3.4.2 and abstract theory on sites, we can show that $\widehat{\rho}$'s above are equivalences of topoi.

To make sure that $\widehat{\rho}$'s induce the Frobenius descent (the equivalences of categories of crystals), we need to define a suitable category of crystals with some technical conditions so that our argument works.

Definition 3.5 (Category of crystals on m-prismatic site) Let $\mathscr{C}_{\Delta}((X/A)_{m-\Delta})$ be the category of presheaves $\mathscr{F}$ on $(X/A)_{m-\Delta}$ such that:

1. For any object (E, I_E) in $(X/A)_{m-\Delta}$, $\mathscr{F}(E) \in \mathcal{M}_{\Delta}(E)$.

2. For any morphism $(E, I_E) \to (E', I_{E'})$, the induced map $\mathscr{F}(E)\widehat{\otimes}_E E' \xrightarrow{\simeq} \mathscr{F}(E')$ is an isomorphism of E'-modules.

where $\mathcal{M}_{\Delta}(E)$ is the category of E-modules M with suitable "sheaf" property and descent property:

Definition 3.6 (Category $\mathcal{M}_{\Delta}(E)$)

1. **(Sheaf property)** Let $\widetilde{\mathcal{M}}_{\Delta}(E, I)$ be the category of E-modules M such that, for any map $(E, I) \to (E_0, I_0)$ of bounded prisms and for any faithfully flat map $(E_0, I_0) \to (E'_0, I'_0)$ of bounded prisms,
$$0 \to M\widehat{\otimes}_E E_0 \to M\widehat{\otimes}_E E'_0 \to M\widehat{\otimes}_E (E'_0\widehat{\otimes}_{E_0} E'_0)$$
is exact.

2. **(Descent property)** Let $\{\mathcal{M}_{\Delta}(E, I) \subseteq \widetilde{\mathcal{M}}_{\Delta}(E, I)\}_{(E,I)}$ be the largest family of full subcategories such that, for any $M \in \mathcal{M}_{\Delta}(E, I)$, any $(E, I) \to (E'_0, I'_0)$ of bounded prisms and any faithfully flat map $(E_0, I_0) \to (E'_0, I'_0)$ of bounded prisms, any descent data ϵ on $M\widehat{\otimes}_E E'_0$ descends uniquely to $M_0 \in \mathcal{M}_{\Delta}(E_0, I_0)$.

We simply denote $\mathcal{M}_{\Delta}(E, I)$ by $\mathcal{M}_{\Delta}(E)$.

Remark 3.7 1. The category $\mathcal{M}_{\Delta}(E)$ with the complicated properties defined above is large enough. To see this, define $\mathcal{M}^{\text{fp}}(E)$ to be the category of finite projective E-modules, and $\mathcal{M}^{\text{tors}}(E)$ to be the category of all (p, I)-power torsion E-modules. Then we can show $\mathcal{M}^{\text{fp}}(E) \subseteq \mathcal{M}_{\Delta}(E)$ by the Proposition A.12 of [AB19] and $\mathcal{M}^{\text{tors}}(E) \subseteq \mathcal{M}_{\Delta}(E)$ by the usual descent argument.

2. Presheaves $\mathscr{F}$ in $\mathscr{C}_{\Delta}((X/A)_{m-\Delta})$ in Definition 3.5 are automatically sheaves by the sheaf property of $\mathcal{M}_{\Delta}(E)$ in Definition 3.6.1.

One of our main results is the following:

Theorem 3.8 (The Frobenius descent on the m-prismatic site, L.) *The morphisms $\widehat{\rho}$ induce equivalences of categories*

$$\mathscr{C}_{\Delta}((X'/A)_{\Delta}) \xrightarrow{\simeq} \mathscr{C}_{\Delta}((X/A)_{m-\Delta}),$$
$$\mathscr{C}_{\Delta}((X'/D)_{q\text{-}crys}) \xrightarrow{\simeq} \mathscr{C}_{\Delta}((X/D)_{m\text{-}q\text{-}crys}).$$

The proof of Theorem 3.8 is given by following the proof of [Xu19] and using the descent property of crystals in Definition 3.6.2.

3.2 q-crystalline comparison of higher level

By the comparison theorems we have seen, it would be natural to regard prismatic cohomology as a kind of 'level (-1) q-crystalline cohomology',

$$R\Gamma((X''/D)_{\Delta}, \mathscr{F}) \xrightarrow[\text{Thm 2.7}]{\simeq} R\Gamma((X/D)_{q\text{-crys}}, \mathscr{F}) \xrightarrow[\text{Thm 3.8}]{\simeq} R\Gamma((X'/D)_{(-1)\text{-}q\text{-crys}}, \mathscr{F})$$

(Here the negative level of the q-crystalline site is not defined, but can be formally used to observe the above comparison). Based on this observation, it would be natural to compare the m-prismatic site with the $(m-1)$-q-crystalline site.

Construction 3.9 Let (D, I) be a q-PD pair, let $J_q = (\phi^{m-1})^{-1}(I)$ and let $J_{\Delta} = (\phi^m)^{-1}([p]_q D)$. Note that $I \subseteq \phi^{-1}([p]_q D)$ by Corollary 16.8 of [BS19]. In particular, $J_q \subseteq J_{\Delta}$.

Let X^* be $X \widehat{\times}_{\mathrm{Spf}(D/J_q)} \mathrm{Spf}(D/J_{\Delta})$, where the completion is the classical $(p, [p]_q)$-completion. Define the functor

$$\alpha : (X/(D, I))_{(m-1)\text{-}q\text{-crys}} \to (X^*/(D, [p]_q D))_{m-\Delta}$$

by

$$(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_{E,q}) \to X) \mapsto (\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/J_{E,\Delta}) \xrightarrow{(1)} X^*),$$

where (1) is:

$$\mathrm{Spf}(E/J_{E,\Delta}) \xrightarrow{(2)} \mathrm{Spf}(E/J_{E,q}) \widehat{\times}_{\mathrm{Spf}(D/J_q)} \mathrm{Spf}(D/J_{\Delta}) \longrightarrow X \widehat{\times}_{\mathrm{Spf}(D/J_q)} \mathrm{Spf}(D/J_{\Delta}) = X^*,$$

and (2) is induced by the natural surjection on the underlying rings. One can check that α is continuous and thus induces the morphism of topoi

$$\widehat{\alpha} : \widetilde{(X^*/D)}_{m-\Delta} \to \widetilde{(X/D)}_{(m-1)\text{-}q\text{-crys}}.$$

Theorem 3.10 (q-crystalline comparison of higher level, L.) *$\widehat{\alpha}$ induces an equivalence of categories of crystals*

$$\widehat{\alpha}_* : \mathscr{C}_{\Delta}((X^*/D)_{m-\Delta}) \xrightarrow{\simeq} \mathscr{C}_{\Delta}((X/D)_{(m-1)\text{-}q\text{-}crys}).$$

The proof of Theorem 3.10 is given by comparing the categories of crystals with the categories of certain stratifications. The stratification is a kind of algebraic structure that appears naturally in the theory of crystals. For example, the Frobenius descent was proved by identifying the crystals with the stratifications. We can construct certain stratifications in the m-q-crystalline case, and consider the relations to the categories of crystals on the sites appeared in Theorem 3.10.

3.3 Relation to the usual Frobenius descent

As we mentioned above, the Frobenius descent was proved by identifying the crystals with the stratifications. It did not follow from a certain equivalence of topoi: In Definition 2.10, the objects of the m-crystalline site are p-nilpotent and J_E is not uniquely determined by (E, I_E). But in Definition 3.2, the objects of the m-q-crystalline site are p-torsion free when $q = 1$, and J_E is uniquely determined by (E, I_E). So the m-crystalline site was not suitable enough to apply the site-theoretic argument.

For the rest of this section, we first give an alternative, site-theoretic proof of the Frobenius descent in a certain setting. Our strategy is to suitably modify the definition of the m-crystalline site without changing the category of crystals. Then we can apply the site-theoretic argument we have used to the modified site. Next, we assume that (D, I) is a q-PD pair with $q = 1$ in D, and that $p \in I$. We can use the modified m-crystalline site to prove that, the equivalence between the category of crystals on the m-q-crystalline site and that on the usual q-crystalline site in Theorem 3.8 is compatible with the Frobenius descent in Theorem 2.11.

The definition of the variant of the m-crystalline site is the following:

Definition 3.11 (Maximal m-PD ring) Let (D, I, γ) be a PD ring with $p \in I$. We define

$$J_{\max} := \mathrm{Ker}(D \twoheadrightarrow D/I \xrightarrow{\phi^m} D/I).$$

Then $(J_{\max}, I, \gamma)$ is an m-PD ideal. For any m-PD ideal of the form (J, I, γ), we have $J \subseteq J_{\max}$. A *maximal m-PD ring* is an m-PD ring (D, J, I, γ) satisfying $J = J_{\max}$.

Definition 3.12 (New m-crystalline site) Let (D, J, I, γ) be a p-torsion free p-complete maximal m-PD ring with $p \in I$. Let X be a scheme smooth and seperated over D/J. Then the *new m-crystalline site* $(X/D)_{m\text{-crys,new}}$ is defined as follows. Objects are maps $(D, J, I, \gamma) \to (E, J_E, I_E, \gamma_E)$ of p-torsion free p-complete maximal m-PD rings with $\mathrm{Spec}(E/J_E) \to X$ over D/J. Covers are p-completely faithfully flat maps $(E, J_E, I_E, \gamma_E) \to (E', J_{E'}, I_{E'}, \gamma_{E'})$ such that $I_{E'} = \widehat{I_E E'}$.

We can define the functor of sites

$$\sigma_{\mathrm{new}} : (X/D)_{m\text{-crys,new}} \to (X'/D)_{\mathrm{crys,new}}$$

and apply the site-theoretic argument as before to get an equivalence of topoi and an equivalence of categories of certain crystals:

$$\sigma_{\mathrm{new}} : \widetilde{(X/D)}_{m\text{-crys,new}} \xrightarrow{\simeq} \widetilde{(X'/D)}_{\mathrm{crys,new}},$$
$$\sigma^*_{\mathrm{new}} : \mathscr{C}((X'/D)_{\mathrm{crys,new}}) \xrightarrow{\simeq} \mathscr{C}((X/D)_{m\text{-crys,new}}).$$

We can also construct the continuous functor of sites

$$\nu_n : (X/D)_{m\text{-crys,new}} \to (X/D)_{m\text{-crys}}$$

which sends $(\mathrm{Spf}(E) \leftarrow \mathrm{Spec}(E/J_E) \to X)$ to

$$(\mathrm{Spec}(E/p^n E) \leftarrow \mathrm{Spec}((E/p^n E)/(J_E/p^n E)) \to X).$$

Then we have the following diagram:

$$\begin{array}{ccc} \mathscr{C}^{p^n\text{-tors}}((X'/D)_{\mathrm{crys}}) & \xrightarrow{\sigma^*} & \mathscr{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}) \\ \simeq\big\downarrow \widehat{\nu}_{n,*} & & \simeq\big\downarrow \widehat{\nu}_{n,*} \\ \mathscr{C}^{p^n\text{-tors}}((X'/D)_{\mathrm{crys,new}}) & \xrightarrow[\simeq]{\sigma^*_{\mathrm{new}}} & \mathscr{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}}). \end{array}$$

Here σ^* is the Frobenius descent and $\mathscr{C}^{p^n\text{-tors}}$ is the category of p^n-torsion crystals. As before, we can show that $\widehat{\nu}_{n,*}$'s are equivalences by constructing the category of certain stratifications and comparing the categories of crystals with it. After taking projective limits, this gives the site-theoretic proof of the equivalence of the Frobenius descent

$$\sigma^* : \mathscr{C}^{\mathrm{qcoh}}((X'/D)_{\mathrm{crys}}) \to \mathscr{C}^{\mathrm{qcoh}}((X/D)_{m\text{-crys}}),$$

where $\mathscr{C}^{\mathrm{qcoh}}$ denotes the category of quasi-coherent crystals. We can compare σ^* with the q-crystalline version of the Frobenius descent when $q = 1$.

Let (D, I) be a q-PD pair with $q = 1$ in D. By Remark 16.3 of [BS19], there exists a canonical PD-structure on I. We suppose further that $p \in I$, and let $J = (\phi^m)^{-1}(I)$. We can show that (D, J, I, γ) is an m-PD ring. This defines a continuous functor of sites

$$\tau : (X/D)_{m\text{-}q\text{-crys}} \to (X/D)_{m\text{-crys,new}}.$$

By comparing with the category of certain stratifications, we can show that the functor τ induces equivalence of categories of certain crystals

$$\widehat{\tau}_* : \mathscr{C}((X/D)_{m\text{-crys,new}}) \xrightarrow{\simeq} \mathscr{C}((X/D)_{m\text{-}q\text{-crys}}).$$

This is a categorical analog of Theorem 2.8.

The functors we have constructed fit into the following commutative diagram:

$$\begin{array}{ccc} \mathscr{C}^{p^n\text{-tors}}((X'/D)_{\mathrm{crys}}) & \xrightarrow{\sigma^*} & \mathscr{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys}}) \\ \simeq\big\downarrow \widehat{\nu}_{n,*} & & \simeq\big\downarrow \widehat{\nu}_{n,*} \\ \mathscr{C}^{p^n\text{-tors}}((X'/D)_{\mathrm{crys,new}}) & \xrightarrow[\simeq]{\sigma^*_{\mathrm{new}}} & \mathscr{C}^{p^n\text{-tors}}((X/D)_{m\text{-crys,new}}) \\ \simeq\big\downarrow \widehat{\tau}_* & & \simeq\big\downarrow \widehat{\tau}_* \\ \mathscr{C}^{p^n\text{-tors}}((X'/D)_{q\text{-crys}}) & \xrightarrow[\simeq]{\widehat{\rho}^*} & \mathscr{C}^{p^n\text{-tors}}((X/D)_{m\text{-}q\text{-crys}}). \end{array}$$

In this sense, when $q = 1$, the q-crystalline version of the Frobenius descent is compatible with the usual Frobenius descent.

3.4 Future goals

In this subsection, we briefly explain what remains to be done.

The original motivation of introducing m-crystalline cohomology would be to develop a p-adic cohomology theory over a ramified base. For example, if the base is a complete discrete valuation ring V of mixed characteristic $(0,p)$ in which p is not a uniformizer, then we can consider V as the base m-PD ring of the m-crystalline site for a suitable m. However, there exists no δ-ring structure on the ring V above. So our definition of the m-prismatic site and the m-q-crystalline site is not enough for this purpose. We hope to generalize our results to the case of a possibly ramified base in the future.

One way to allow the ramified base in the prismatic site is to consider the δ_π-structure for a uniformizer π in V. By Remark 2.4 of [BS19], specifying a δ-structure on R is equivalent to specifying a section $w : R \to W_2(R)$. If we could use the theory of ramified Witt vectors of Hazewinkel, then a section of truncated ramified Witt vector will correspond to a δ_π-structure on V. Then it could be possible to modify our m-prismatic site to allow the ramified base ring.

In [BS19], Bhatt and Scholze constructed the q-de Rham complex and showed that q-crystalline cohomology can be computed by using the q-de Rham complex. By the q-crystalline comparison in Theorem 2.7, the q-de Rham complex gives an explicit complex computing prismatic cohomology. So it is expected that there exists the notion of m-q-de Rham complex which computes m-q-crystalline cohomology. If we follow the proof of Bhatt and Scholze, we first need to construct the higher level analog of the q-derivation $\nabla_{q,s}$ in Construction 16.19 of [BS19]. One of the candidates is the map $\partial_q^{[p^m]}$ appeared in the Taylor map in Section 5 of [GLSQ20]. To prove that the m-q-de Rham complex computes m-q-crystalline cohomology, we need to show the analog of Lemma 16.21 in [BS19].

As it would be natural to consider the higher level structure for log schemes to treat the case where the schemes in consideration are open or degenerate, we should consider the m-prismatic site and the m-q-crystalline site for log schemes and generalize our results to the case of log schemes.

Rigid cohomology is an important tool used in the p-adic cohomology theory, which was introduced by Berthelot. Overconvergent isocrystals can be considered as the coefficients of rigid cohomology. An overconvergent isocrystal is a kind of system of p-adic differential equations. When the scheme in consideration is proper, it is obtained from a convergent isocrystal and the category of convergent isocrystals can be considered as the intersection of the categories of isocrystals on the crystalline sites of all higher levels. Based on our result, it could be possible to construct the q-analog of convergent isocrystal. We could also represent it by p-adic q-difference equation, and consider the corresponding q-de Rham complex which computes the q-analog of convergent cohomology in proper case. Ultimately, we could consider the q-analog of overconvergent isocrystals, and develop the theory of q-analog of rigid cohomology which is compatible with prismatic cohomology.

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