Retract $(-i)$ rationality and its necessary conditions expressed by unramified presheaves

- Noether's problem of a finite group G as an example

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Abstract

This is another short introduction to the author's study of the rationality problem, which centers the hierarchies of the form: lower rationality = higher ruledness. A particular emphasis is given for Noether's problem of a finite group G, where a technical difficult emerges because the relevant geometric object BG is not approximated by smooth proper varieties. The author's novelty here is a construction of the stable birational subsheaf for any unramified presheaf in the sense of Morel. This gives us a very strong necessary condition for retract $(-i)$ rationality of smooth, not necessary proper, varieties over a perfect field.

1 Introduction

In this paper, all the schemes are defined over a perfect base field k.

Recently, I have been working on the hierarchies of the hierarchy

$$\begin{aligned} &\text{rational} \Longrightarrow \text{stable rational} \Longrightarrow \text{retract rational} \\ &\Longrightarrow \text{separably unirational} \Longrightarrow \text{separably rationally connected} \end{aligned} \tag{1}$$

in the framework of

$$\textit{Lower rationality} = \textit{Higher ruledness}\ , \tag{2}$$

which, for the first three of (1), take the following forms:

Definition 1.1. *For a n-dimensional k-variety* [1] *X, let us say:*

(i)

$$X \text{ is } \underline{\underline{(-i)\textbf{-rational}}} \text{ or } \underline{\underline{(n-i)\textbf{-ruled}}} \qquad (0 \leq i \leq n)$$

if there exist an i-dimensional smooth proper k-variety Z^i and a birational map

$$\mathbb{A}^{n-i} \times Z^i - - > X.$$

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[1] By a variety, we mean an integral k-scheme, which is separated and of finite type as in [sp18, Tag 020D].

(ii) X *is* **stable $(-i)$-rational** *or* **stable $(n-i)$-ruled** $(0 \leq i \leq n)$

if there exist $N \in \mathbb{Z}_{\geq n}$, an i-dimensional smooth proper k-variety Z^i and a birational map

$$\mathbb{A}^{N-n} \times \mathbb{A}^{n-i} \times Z^i - - > \mathbb{A}^{N-n} \times X.$$

(iii) X *is* **retract $(-i)$-rational** *or* **retract $(n-i)$-ruled** $(0 \leq i \leq n)$

if there exist $N \in \mathbb{Z}_{\geq n}$, an i-dimensional smooth projective k-variety $Z^i$and rational maps

$$f : X - - > \mathbb{A}^{N-i} \times Z^i, \quad g : \mathbb{A}^{N-i} \times Z^i - - > X$$

such that the composition

$$g \circ f : X - - > X$$

is defined,[2] *yielding an identity on a dense open subset of X.*

We now list up some basic properties of these hierarchies soon in Proposition 1.3, where we shall state (ii) and (iii) are invariant with respect to the following standard equivalence relation (see e.g. [CTS07, §1]):

Definition 1.2. *Two varieties of possibly different dimensions X and Y are said to be* **stable birational equivalent** *if for some natural numbers r, s, $X \times \mathbb{A}^r$ and $Y \times \mathbb{A}^s$ are birationally equivalent.*

Proposition 1.3. (i) *When $i = 0$, those concepts presented in Definition 1.1 reduce to the usual classical concepts (mentioned in the first line of* (1)*).*

0-rational = rational; stable 0-rational = stable rational;
retract 0-rational = retract rational.

(ii) *Each concept in Definition 1.1 is a hierarchy; i.e. for any $0 \leq i \leq j \leq n$,*

$(-i)$-rational $\Longrightarrow$ $(-j)$-rational;
stable $(-i)$-rational $\Longrightarrow$ stable $(-j)$-rational;
retract $(-i)$-rational $\Longrightarrow$ retract $(-j)$-rational.

(iii) *Concepts in Definition 1.1 define a hierarchy of hierarchies stated in above (ii); i.e. for any $0 \leq i \leq n$,*

$(-i)$-rational $\Longrightarrow$ stable $(-i)$-rational $\Longrightarrow$ retract $(-i)$-rational

(iv) *For any $0 \leq i \leq n$, stable $(-i)$-rationality and retract $(-i)$-rationality are stable birational invariants in the sense of Definition 1.2.*

Then we have the following problem as our technical motivation:

Fundamental Problem

Extend the following implications of the above hierarchies to some hierarchy expressed by some (sheaf cohomology theoretical) $\underline{\text{stably birational invariant } \boxed{?} \ :}$

$$\{(-i)\text{-rational}\}_{i \in \mathbb{Z}_{\geq 0}} \Longrightarrow \{\text{stable } (-i)\text{-rational}\}_{i \in \mathbb{Z}_{\geq 0}} \Longrightarrow \{\text{retract } (-i)\text{-rational}\}_{i \in \mathbb{Z}_{\geq 0}} \Longrightarrow \boxed{?}$$

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[2] This means f and g are *composable* in the sense of [KS15, p.285, 1.4] [KM13, p.198, Crollary RC.11].

When we restrict our attention to smooth and PROPER varieties, I presented one answer [M21] to the above problem by considering a hierarchical version of a recent nice paper of Kai-Otabe-Yamazaki [KOY21]. Amongst of all, I gave hierarchical interpretations of the famous hypersurface non rationality theorems of Totaro [T16] and Schreieder [S19][S21a] in [M21].

However, [M21] is NOT APPLICABLE for some number theoretical applications like Noether's problem or the rationality problem of algebraic tori, because in these cases the relevant smooth varieties are NOT PROPER.

Now, I shall report an alternative answer to the above problem, which is (at least theoretically) applicable even to such number theoretical problems because this alternative approach DOES NOT REQUIRE PROPERNESS. The key to this approach is Morel's theory of unramified sheaves which we shall review in the next section.

2 Stable birational presheaf and unramified sheaf

In search of an answer for our problem, we first note that, if a smooth variety X is retract $(-i)$-rational, then there is a commutative diagram of the following form:

$$\begin{tikzcd}
U \arrow[r, "f"'] \arrow[d, hook] \arrow[rr, bend left, "id_U"] & g^{-1}(U) \arrow[r, "g"'] \arrow[d, hook] & U \arrow[d, hook] \\
X \arrow[r, dashed, "f"] \arrow[rr, bend right, "id_X"'] & \mathbb{A}^{N-i} \times Z^i \arrow[r, dashed, "g"] & X
\end{tikzcd}, \tag{3}$$

where vertical arrows are smooth dense open inclusions.

Then because of the stably birational invariant property which we shall state soon in Remark 2.3, the following concept of Asok-Morel [AM11] is clearly of fundamental importance for our purpose:

Definition 2.1. ([AM11, Definition 6.1.1]) *Let Sm_k be the category of smooth finite type k-schemes. Then a presheaf*

$$M : Sm_k \ \to \ \mathcal{C}(= Sets, Groups, ...),$$

is called birational if the following properties are satisfied:

(B0) *For any $X \in Sm_k$ with irreducible components X_α's, $\alpha \in X^{(0)}$, the obvious map*

$$S(X) \to \prod_{\alpha \in X^{(0)}} S(X_\alpha)$$

is a bijection;

(B1) *For any $X \in Sm_k$ and any everywhere dense open subscheme $U \subset X$ the restriction map*

$$S(X) \to S(U)$$

is a bijection.

In fact, by applying a birational presheaf to (3), we immediately obtain the following easy, but very important, conclusion:

Theorem 2.2. *If X is $(-i)$-retract rational, then, for any birational presheaf S, $S(X)$ is a direct summand of $S(Z^i)$ for some smooth Z^i of dimension i.* ☐

A birational presheaf is actually a stable birational Nisnevich sheaf from the following remark:

Remark 2.3. (i) ([AM11, Lemm.6.1.2] [MV99, §3, Proposition 1.4]) Any birational presheaf is automatically a Nisnevich sheaf.
(ii) ([KS15, p.330, Appendix A by Jean-Louis Colliot-Thélène]) Any birational presheaf S is automatically stably birational invariant, thanks to a beautiful observation of Colliot-Thélène.

Thus, we are urged to search after birational presheaves. For this purpose, we shall turn our attention to Morel's unramified presheaves, for which, Morel realized their essence lies in their values at the function fields $k(X)$ and local rings $\mathcal{O}_{X,x}$ for $x \in X \in Sm_k$. However, $\operatorname{Spec} k(X)$ and $\operatorname{Spec} \mathcal{O}_{X,x}$ do not belong to Sm_k, because they are not finite type k-schemes in general, Morel enlarged the domain of the definition of his presheaf S from Sm_k to the category of its pro-objects $Pr(Sm_k)$ [3] (which contains $\operatorname{Spec} k(X)$ and $\mathcal{O}_{X,x}$) as follows:

Definition 2.4. ([M12, Definition 2.1, Remark 2.2]) *An* **unramified** *presheaf S of $\mathcal{C}$ ($\mathcal{C}$ = sets. groups, or abelian groups) on Sm_k (resp. on Sm_k^{Sm}) is a presheaf of S of $\mathcal{C}$ ($\mathcal{C}$ = sets. groups, or abelian groups) on Sm_k (resp. on Sm_k^{Sm}), which we uniquely extend to a presheaf on $Pr(Sm_k)$ (resp. on $Pr\left(Sm_k^{Sm}\right)$)* [4] *by the left Kan extension*

$$S : Pr(Sm_k)^{op} \to \mathcal{C} \quad \big(resp.\ S : Pr(Sm_k^{Sm})^{op} \to \mathcal{C}\big)$$
$$\varprojlim_{\alpha} X_\alpha \mapsto \varinjlim_{\alpha} S(X_\alpha),$$

such that the following three conditions hold (Here, for an affine scheme $\operatorname{Spec} A \in Pr(Sm_k)$, we have abbreviated $S(\operatorname{Spec} A)$ simply as $S(A)$.):

(U0) *For any $X \in Sm_k$, the obvious map*

$$S(X) \to \prod_{\alpha \in X^{(0)}} S(X_\alpha)$$

is a bijection;

(U1) *For any $X \in Sm_k$ and any everywhere dense open subscheme $U \subset X$ the restriction map*

$$S(X) \to S(U)$$

is injective;

(U2) *With* **(U0)** *at hand, let us suppose $X \in Sm_k$ is irreducible. Then the injective map*

$$S(X) \to \cap_{x \in X^{(1)}} S\left(\mathcal{O}_{X,x}\right)$$

is a bijection, where $\cap_{x \in X^{(1)}} S\left(\mathcal{O}_{X,x}\right)$ is computed in $S(k(X))$.

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[3] Strictly speaking, Morel [M12, p.vi] considered the category Sm_k' of essential smooth k-schemes rather than $Pr(Sm_k)$.

[4] This extension of S allows us to define $S(k(X))$ and $S(\mathcal{O}_{X,x})$ ($x \in X$) for $X \in Sm_k$, as desired. Furthermore, we can also define $S(R_v)$ for a divisorial valuation v, which we shall define in the next section and which plays a very important role in this paper, and, more generally, $S(R_v)$ for a valuation v with separable residue field $\kappa(v)$ (in fact, this assumption is always satisfied under our perfect base field assumption by [M89, Theorem 26.3] [sp18, 030Y,030Z]) amongst of all.

It should be noted that Morel's unramified presheaf itself is not (stably) birational yet, but there are a plenty of its examples as follows:

Example 2.5. (i) ([M12, Remark 6.10] [K21a, Corollary 2.8]) Any strongly $\mathbb{A}^1$ invariant Nisnevich sheaf of group G, i.e. $U \mapsto H^n_{Nis}(U, G)$ is $\mathbb{A}^1$ invariant for $n = 0, 1$, is unramified.

(ii) Consequently, familiar $\mathbb{A}^1$ invariant Nisnevich sheaves with transfer are unramified. In fact, these are special cases of (i), because, by [MVW06, Theorem 13.8], any $\mathbb{A}^1$ invariant Nisnevich sheaf of abelian groups with transfer F is strictly $\mathbb{A}^1$ invariant, i.e. $U \mapsto H^n_{Nis}(U, F)$ is $\mathbb{A}^1$ invariant for any $n \in \mathbb{Z}_{\geq 0}$.

Déglise [D06, Proposition 6.9] [D11, Théorème 3.7] showed an $\mathbb{A}^1$ invariant Nisnevich sheaves of abelian groups with transfer F is essentially equivalent to Rost's cycle module [R96]. Rost's cycle module is a data of the following form:

$$\begin{aligned} M = \Big(& M_* : \mathcal{F}_k \to \mathcal{A}b_*, \{\phi^* : M_*(F) \to M_*(F) \mid \text{finite extension } E \subset F \text{ in } \mathcal{F}_k\}, \\ & \{\partial_v : M_*(F) \to M_*(\kappa(v)) \mid v, \text{ a geometric discrete valuation on } F|k \in \mathcal{F}_k\} \Big), \end{aligned}$$

and the desired $\mathbb{A}^1$ invariant Nisnevich sheaf of abelian groups with transfer is given by the associated Rost Chow groups with coefficients in the given cycle module M:

$$U \mapsto A^0(U, M_*) := \mathrm{Ker}\left(\oplus_{x \in U^{(0)}} M_*(\kappa(x)) \xrightarrow{\oplus_{x \in U^{(0)}} \left(\oplus_{y \in U^{(1)}} \partial^x_y\right)} \oplus_{y \in U^{(1)}} M_{*-1}(\kappa(y)) \right), \tag{4}$$

where $U^{(c)}$ is the set of codimension c schematic points of U. While we only consider the degree 0 part in (4), the higher degree parts also emerges as the Bloch-Ogus [BO74] type theorem for Rost's cycle modules [R96, Theorem (6.1), Corollary (6.5)], which gives us a conceptual transparent description of $A^p(X, M_*)$ when X is a smooth k-variety:

$$A^p(X, M_*) \cong H^p_{Zar}(X, \mathcal{M}_*), \tag{5}$$

where $\mathcal{M}_*$ is the Zariski sheaf given by (4).

For instance, starting with the Galois cohomology as Rost's cycle module $(H^i_{Gal}(-, \mu), \{\phi^*\}, \{\partial_v\})$, where μ is either $\mu_n^{\otimes(i-1)}$ or $\mathbb{Q}/\mathbb{Z}(i-1) = \varinjlim \mu_n^{\otimes(i-1)}$ with $(n, \mathrm{char} k) = 1$ [R96, p.335, Remark (1.11)] [AB17, 3.2], we obtain (the geometric version of) the unramified cohomology as follows:

$$H^i_{nr}(X, \mu) = A^0\left(X, (H^i(-, \mu), \{\phi^*\}, \{\partial_v\})\right) \cong H^0_{Zar}\left(X, \mathcal{H}^i_{\acute{e}t}(\mu)\right).$$

Again, see [R96] [M05a, 2.2] [AB17, 3] for more details including the notation.

(iii) ([F20][F21a][F21b]) The concept of Rost's cycle module was generalized by Feld [F20] to his Milnor-Witt cycle modules. Whereas Rost's cycle module admits a graded action of the Milnor K-theory $\mathbf{K}^M_*$ [M70], Feld's Milnor-Witt cycle module admits a graded action of the Morel's Milnor-Witt K-theory $\mathbf{K}^{MW}_*$ [M12, Definition 3.1], so that the Rost cycle module is a special case of the Milnor-Witt cycle module with the action of $\eta \in \mathbf{K}^{MW}_{-1}$ equal to 0. (Recall $\mathbf{K}^{MW}_*/(\eta) \cong \mathbf{K}^M_*$ [M12, Remark 3.2].)

For any Milnor-Witt cycle module M and any smooth k-scheme X, just like the case of Rost's cycle module, a $\mathbb{Z}$-graded pair $\left\{F^M_n, \omega_n : F^M_{n-1} \to (F^M_n)_{-1}\right\}_{n \in \mathbb{Z}}$ are produced so that the $\mathbb{Z}$-graded Zariski sheaves of abelian groups $\left\{F^M_n\right\}_{n \in \mathbb{Z}}$ are of the form

$$\begin{cases} U \mapsto F^M_n(U) := A^0\left(U, M, -\Omega_{U/k} + \langle n \rangle\right) \cong H^0_{Zar}\left(U, \mathcal{M}_{U,n}\right) \\ A^p\left(X, M, -\Omega_{X/k} + \langle n \rangle\right) \;\cong\; H^p_{Zar}\left(X, \mathcal{M}_{X,n}\right) \end{cases},$$

making $\{F_n^M, \omega_n : F_{n-1}^M \to (F_n^M)_{-1}\}_{n\in\mathbb{Z}}$ into a homotopy module [F20, Corollary 8.5] [F21a, Theorem 4.1.7], i.e. a pair $\{M_*, \omega_*\}_{n\in\mathbb{Z}}$ of a $\mathbb{Z}$-graded strictly $\mathbb{A}^1$-invariant Nisnevich sheaf of abelian groups M_* on Sm_k and the desuspension map $\omega_n : M_{n-1} \to (M_n)_{-1}$. See [F20][F21a][F21b] for more details including the notation.

Consequently, each F_n^M is unramified by (i).

(iv) ([S20, Theorem 0.2] [K21b, Corollary 1.17]) So far, all of our unramified sheaves have been $\mathbb{A}^1$-invariant. However, Saito [S20, Theorem 0.2] and Koizumi [K21b, Corollary 1.17] showed reciprocity sheaves in the sense of [KSY16], which are not necessarily $\mathbb{A}^1$-invariant, are also unramified. Examples of reciprocity sheaves include $\mathbb{A}^1$-invariant Nisnevich sheaves with transfer treated in (i), smooth commutative algebraic groups over k, e.g. the additive group $\mathbb{G}_a$, the modules of absolute Kähler differentials Ω^i, and de Rham-Witt differentials $W_n\Omega^i$, as was observed by Rülling [KSY16, Appendix].

3 Geometric valuation and Stable birational Nisnevich sheaf

In order to convert such rich unramified presheaves into stable birational Nisnevich sheaves, let us review some basic of the higher rank valuation theory, because many number theorists might be used only to those valuations whose valuation rings are rank 1 discrete valuation rings.

- Given a valuation $v : K \to \Gamma \cup \{+\infty\}$ on a field K to a totally ordered commutative group Γ (and the extra greatest element $+\infty$), we obtain the following familiar algebraic objects (see e.g. [ZS60b, p.34, VI, §8] [B72, VI, §3, Definition 1, Proposition 2] [M89, §10]):

  - The valuation ring $R_v := \{x \in K \mid v(x) \geq 0\}$, which is a local ring with the maximal ideal $\mathfrak{m}_v := \{x \in K \mid v(x) > 0\}$ and the residue field $\kappa(v) := R_v/\mathfrak{m}_v$;
  - The value group $\Gamma_v := v(K^*) \cong K^*/(R_v)^*$, which is a totally ordered commutative group.

  From these, we have the following numerical invariants of a valuation ν :

  - ([ZS60b, p.50, VI, §10] [V06, p.485, Definition]) The rational rank of v is defined by $rat.rank(v) := \dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q})$;
  - ([ZS60b, p.39, VI, §10; p.9, VI, §3, Definition 1; p.40, VI, §10, Theorem 15] [V06, p.482, Definition, Corollary; p.483, Proposition 1.8; p.484, lines 3-5]) The rank of v, denoted by $rank(v)$, is defined by the maximal length $r \in \mathbb{Z}_{\geq 0}$ of either one of the following three chains:
    - $0 = \Gamma_r \subsetneq \Gamma_{r-1} \subsetneq \cdots \subsetneq \Gamma_1 \subsetneq \Gamma_0 = \Gamma_v$, a chain of isolated subgroups of the totally ordered commutative group Γ_v. [5)]
    - $R_v = R_r \subsetneq R_{r-1} \subsetneq \cdots \subsetneq R_1 \subsetneq R_0 = K$, a chain of local rings, which are actually valuation rings over $R_v = R_r$ is the valuation ring of v, of K.
    - $\mathfrak{m}_v = \mathfrak{p}_r \supsetneq \mathfrak{p}_{r-1} \supsetneq \cdots \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_0 = 0$, a chain of prime ideals of R_v.

    In particular, $rank(v)$ is the Krull dimension of R_v :

$$rank(v) = \dim R_v.$$

[5)] ([ZS60b, p.40,VI, §10] [V06, p.481-482, Definition]) A subgroup H of a totally ordered commutative group A is called isolated if any $a \in A$ with $0 \leq a \leq h \in H$ is contained in $H : a \in H$.

  - Between the above numerical invariants, the following inequality holds ([B72, p.438, VI, §10.2, Corollary, Proposition 4]):
$$rank(v) \leq rat.rank(v). \tag{6}$$
When Γ_v is finitely generated, if the equality $rank(v) = rat.rank(v)$ holds in (6), then Γ_v is discrete in the sense of [ZS60b, p.48, -3rd line] [V06, p.484, Definition], i.e. $\Gamma_v \cong \mathbb{Z}^{rat.rank(v)}$ as totally ordered commutative groups, where $\mathbb{Z}^{rat.rank(v)}$ is given the lexicographical order.

- Suppose K is a field extension of its subfield k, then a valuation $v : K \to \Gamma \cup \{+\infty\}$ is called a valuation of K/k, if $v(k^*) = 0$ [V06, p.481, 1.1].
  - When R is a subring of K and $\mathfrak{m}_v \cap R = \mathfrak{p} \subset R$, we say v has a center on R at $\mathfrak{p}$.
When a local ring $(S, \mathfrak{m}_S)$ with $k \subset S \subset Frac(S) = K$ has $\mathfrak{m}_S$ as the center of v, its local uniformization problem [Z40] [V06] [NS16] asks us to find a local blowing up [NS16, §2] of $(S, \mathfrak{m}_S)$ with respect to v
$$(S, \mathfrak{m}_S) \to (R, \mathfrak{m}_R) \ \ (\to (R_v, \mathfrak{m}_v)) \tag{7}$$
such that $(R, \mathfrak{m}_R)$ is a regular local ring. Local uniformization may be regarded as a local version of the resolution of singularities, and was proved affirmatively when the base field k is characteristic 0 by Zariski [Z40]. However, it is still unsolved for the positive characteristic case $char\ k > 0$ just as the resolution of singularities problem.
  - Given a valuation $v : K \to \Gamma \cup \{+\infty\}$ of K/k, we have a couple of field extensions:
$$K/k, \qquad \kappa(v)/k,$$
for which let us consider their transcendental degrees:
$$tr.deg_k K, \qquad tr.deg_k \kappa(v). \tag{8}$$
The latter is called the dimension of v ([ZS60b, p.34, VI, §8] [V06, p.494]), and denoted by
$$\dim(v) := tr.deg_k \kappa(v). \tag{9}$$

- ([A56a][ZS60b, p.331, Appendix 2, Proposition 2; p.335, Appendix 2, Proposition 3] [B72, p.439, VI, §10.3, Corollary 1]) For any valuation v of K/k, the following Abhyankar's inequality holds:
$$rat.rank(v) + \dim(v) \ \leq \ tr.deg_k K \tag{10}$$
When K/k is a finitely generated field extension, if the equality $rat.rank(v) + \dim(v) \ = \ tr.deg_k K$ holds in Abhyankar's inequality (10), then v is called an Abhyankar valuation on K/k.

  Generalizing and building upon earlier works of Zariski [Z40] and Abhyankar [A56b], Knaf-Kuhlmann [KK05, p.835, Theorem 1.1] (see also Temkin [T13, p.114, Theorem 5.5.2] and Cutkosky [C21, Theorem 1.3] for improvements) proved a local uniformization theorem (7) for an Abhyankar valuation v whose residue field $\kappa(v)$ is separable. (6)

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(6) This assumption is automatically satisfied in our perfect base field k case [M89, Theorem 26.3] [sp18, 030Y,030Z].

- For a valuation v on K/k, we have the following inequality from (6) and Abhyankar's inequality (10):

$$rank(v) + \dim(v) \ \leq \ tr.deg_k K \tag{11}$$

When K/k is a finitely generated field extension, if the equality $rank(v) + \dim(v) \ = \ tr.deg_k K$ holds in (11), then v is called a geometric valuation on K/k in some motivic literature (e.g. [R96, p.328, 3rd and 4th paragraphs] for the rank 1 case and [M08, p.53, 2nd paragraph] for arbitrary rank cases). If v is a geometric valuation of K/k, then v is of course Abhyankar. Furthermore, Γ_v is not only finitely generated, but also discrete: $\Gamma_v \cong \mathbb{Z}^{rat.rank(v)}$ as totally ordered commutative groups, where $\mathbb{Z}^{rat.rank(v)}$ is given the lexicographical order (for a direct proof of this fact, see [ZS60b, p.90, VI §14, Corollary]).

- A rank 1 geometric valuation v is nothing but a rational rank 1 Abyankar valuation, and was called a prime divisor [ZS60b, p.88 VI, §14; p.89, VI, §14, Theorem 31], but is also called a divisorial valuation in many recent literature. For the rest, I shall mostly use this terminology "divisorial valuation" for the sake of brevity.

Now, we are ready to state my answer to the fundamental problem stated in §1:

My answer - Stable birational Nisnevich subsheaf S_{sb} of a birational sheaf S

Theorem and Definition 3.1. (i) Given an unramified sheaf S on Sm_k, let us set

$$S_{sb}\,(K/k) := \cap_{\substack{v,\ \text{divisorial} \\ \text{valuation of } K/k}} S(R_v) \tag{12}$$

for any finitely generated field extension K/k. Then the correspondence

$$U \ \mapsto \ S_{sb}(U) := S_{sb}(k(U)/k) = \cap_{\substack{v,\ \text{divisorial} \\ \text{valuation of } k(U)/k}} S(Rv)$$

$$\left(\subseteq \ S(U) := \cap_{\substack{v,\ \text{divisorial valuation of } k(U)/k \\ \text{s.t. } R_v = \mathcal{O}_{U,x} \text{ for some } x \in U^{(1)}}} S(R_v) = \cap_{x \in U^{(1)}} S(\mathcal{O}_{U,x}) \ \subset \ S(k(U)) \right)$$

defines a birational subsheaf S_{sb} of an unramified sheaf S on Sm_k which we call the stable birational subsheaf S_{sb} of an unramfied sheaf S because S_{ab} is by definition a birational sheaf (which immediately implies S_{ab} is actually staby birational invariant by [KS15, p.330, Appendix A by Jean-Louis Colliot-Thélène] as was recalled in Remark 2.3 (ii)).
(ii) For any smooth proper k-scheme X,

$$S_{sb}(X) = S(X).$$

Consequently, any unramified sheaf S is stably birational invariant among smooth proper k-schemes.

From Theorem 2.2 and the above Theorem and Definition 3.1, we immediately obtain the following;

Corollary 3.2. *If X is $(-i)$-retract rational, then $S_{sb}(X)$ is a direct summand of $S_{sb}(Z^i)$ for some smooth Z^i of dimension i.* ☐

Since [7)]

$$\left(H^j_{nr}\left(k(Z^i)/k,\mu_{p^m}^{\otimes k}\right)\right)_{sb} \subset H^j_{Gal}\left(k(Z^i)/k,\mu_{p^m}^{\otimes k}\right) \qquad (1\leq m\leq +\infty),$$

the classical Tate's thoerem (formally a conjecture of Grothendieck) [S72, p.119, §4, Th. 28] (see also [AGV73, Exposé xiv 3]) about the cohomological dimension and Corollary 3.2 imply:

Corollary 3.3. *When* $(char\ k,p)=1$, *if*

$$\left(H^j_{nr}\left(k(X)/k,\mu_{p^m}^{\otimes k}\right)\right)_{sb} \neq 0 \qquad (1\leq \exists m\leq +\infty)$$

for some

$$j>i+cd_p(k),$$

then X *is not retract* $(-i)$*-rational.*

Corollary 3.4. *When* $k=\mathbb{C}$, *if*

$$\left(H^j_{nr}\left(k(X)/k,\mu_{p^m}^{\otimes k}\right)\right)_{sb} \neq 0 \qquad (1\leq \exists m\leq +\infty)$$

for some

$$j>i,$$

then X *is not retract* $(-i)$*-rational.*

Remark 3.5. (i) I believe the above Theorem and Definition 3.1, which is applicable to arbitrary smooth, not necessary proper, k-scheme is the most conceptually transparent result along the line of [CT95] (whose philosophy might go back to [G68] which deals with Brauer Groups).
(ii) Notice that we do not impose any properness assumption on X in our Corollary 3.2, Corollary 3.3 and Corollary 3.4 . In fact, with such properness assumption on X, they were already obtained in [M21] by considering a hierarchical version of a recent nice paper of Kai-Otabe-Yamazaki [KOY21].
(iii) When the unramified sheaf S is provided by a Rost cycle module M, i.e. when $S(X)=A^0(X,M)$, S_{sb} was already defined by Merkurjev [M08, p.56, 2.2] and an intimately related claim [M08, p.61, Proposition 2.15] was already stated [M08, p.56, 2.2; p.61, Proposition 2.15]. This was also considered by Kahn-Ngan [KN14, Definition 6.1], but be aware that different notations are used for this case:

$$\underbrace{A^0_{sb}(-,M)}_{\text{our notation}} \quad=\quad \underbrace{M(-)_{nr}}_{\text{Merkurjev's notation}} \quad=\quad \underbrace{A^0_{nr}(-,M)}_{\text{Kahn-Ngan's notation}}$$

(iv) Stably birational invariance of an unramified sheaf among smooth proper k-schemes, claimed in Theorem and Definition 2.2 (ii), was proved for Rost's cycle modules mentioned in Example 2.5 (ii) by Rost [R96, Corollary (12.10)], and for Feld's Milnor-Witt cycle modules mentioned in Example 2.5 (iii) by Feld [F21a, Theorem 5.3.1]. However, from these approaches which stick only to smooth and proper k-schemes, we can never deduce Corollary 3.2 which is valid for arbitrary smooth k-schemes.

[7)] We warn the readers that our $\left(H^j_{nr}\left(k(Z^i)/k,\mu_{p^m}^{\otimes k}\right)\right)_{sb}$ is simply denoted by $H^j_{nr}\left(k(Z^i)/k,\mu_{p^m}^{\otimes k}\right)$ in some literature like [S21b]. Also, in some literature, including the original literature [CTO89], all the rank 1 discrete valuations are used to define the unramified cohomology. Of course, the unramified cohomology so defined is a possibly strict subgroup of the unramified cohomology of our sense (see [S21b, Remark 4.4] for this point), but, our Corollary 3.3 and Corollary 3.4 are also applicable to such unramified cohomologies, as a matter of course.

In fact, when a presheaf of some general class of presheaves, including unramified presheaves, is given, then its stably birational invariance for smooth proper varieties is already known to Colliot-Thélène [CT95, Proposition 2.1.8(e)] [KS15, p.330, Appendix A by Jean-Louis Colliot-Thélène].

Main ingredients of the proof of Theorem and Definition 3.1:.
The following two ingredients are the essence to my proof:

- Morel's argument which he used to prove [M12, Theorem 2.11, Lemma 2.12] that an unramified sheaf is determined by its values on the function fields and the valuation rings of divisorial valuations. [8] enjoying certain axioms, which Morel calls unramified datum.
- My own local uniformization theorem for general geometric valuations, which is, unlike the general theorems of Knaf-Kuhlmann [KK05, p.835, Theorem 1.1], Temkin [T13, p.114, Theorem 5.5.2] and Cutkosky [C21, Theorem 1.3] for Abhyankar valuations, only valid for geometric valuations, but produces a better smooth model needed for our proof (actually, only the rank 2 case of my local uniformization theorem for geometric valuations is needed for this purpose).

□

4 Applications

Since I [M21] knew Corollary 3.2, Corollary 3.3 and Corollary 3.4 when the PROPER assumption is imposed, by considering a hierarchical version of a recent nice paper of Kai-Otabe-Yamazaki [KOY21], I gave some applications of Corollary 3.4 to the case of hypersurfaces, which are smooth and PROPER, in [M21] to give hierarchical interpretations of the famous hypersurface non rationality theorems of Totaro [T16] and Schreieder [S19][S21a].

However, our Corollary 3.2, Corollary 3.3 and Corollary 3.4 do not impose any such proper assumption. Thus, we may apply them to number theoretical problems like Noether's problem and the rationality problem of algebraic tori.

For applications to the Noether's problem for $k = \mathbb{C}$, consult various examples of Noether's problem summarized nicely in Hoshi's RIMS Kôkyûroku surveys [H14][H20] (see also [HKY20]).

In these references, whenever you encounter an example of X such that [9]

$$H^i_{ur}(\mathbb{C}(X)/\mathbb{C}, A) \neq 0,\ \ i \in \mathbb{N},$$

you can instantaneously upgrade the conclusion from merely "non retract rational" to "non retract $(-i)$-rational", and you are prompted to the next level question:

$$\boxed{\textit{Is } X\ (-(n+1))\textit{-rational?}}$$

So, don't leave the scene immediately simply because X is found to be non retract rational!

We shall come back to the rationality problem of algebraic tori elsewhere.

The details will be put in ArXiv soon.

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[8] Although Morel [M12, §2] stated general (rank 1) discrete valuation rings in his formulation, his argument indicates that it is enough to consider only divisorial valuations.

[9] In these literature, like the original literature [CTO89], all the rank 1 discrete valuations are used to define the unramified cohomology. Of course, if the unramified cohomology so defined is non zero, then the unramified cohomology of our sense must be non zero too.

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