FUNDAMENTAL GROUPS AND SPECIALIZATION IN RIGID GEOMETRY

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1. Introduction

In most areas adjacent to arithmetic geometry the role of 'covering space' has historically been assumed by the notion of a finite étale covering. This is for good reason, as if one is concerned with well-behaved (e.g. geometrically unibranch) schemes, then (disjoint unions of) finite étale coverings account for essentially all notions of 'covering space' one is likely to define (see Example 4.4).

That said, there are two notable examples of profitable theories of infinite degree covering spaces arising from arithmetic geometry:

- de Jong's theory of covering spaces for rigid spaces over non-archimedean fields,
- Bhatt–Scholze's theory of geometric coverings for locally topologically Noetherian schemes.

In this expository note I discuss recent work of myself and my coauthors showing that these two ostensibly disparate notions are intimiately connected via the idea of *specialization* and how this points to a more all-encompassing theory of 'covering spaces' in rigid geometry.

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Notation and Conventions. This article is written in an informal manner. I encourage the reader to consult [1], [3], and [2] for precise statements, references, definitions, and conventions.

2. Preliminary ideas

We briefly recall some background material needed for the rest of the article.

2.1. Rigid spaces.

As de Jong's theory of covering spaces concerns rigid geometry, we now briefly recall the various incarnations of 'rigid spaces' and the relationships between them.

Fix $(K, |\cdot|)$ to be a non-archimedean field (so K is complete and non-discrete), ϖ an element of K with $0 < |\varpi| < 1$, $\mathcal{O}$ the valuation ring of K, $\mathfrak{m}$ the valuation ideal of $\mathcal{O}$, and k the residue field of $\mathcal{O}$. In this article, a *rigid K-space* means an adic space locally of finite type over $\mathrm{Spa}(K)$. Denote by $\mathbf{Rig}_K$ (resp. $\mathbf{Rig}_K^{\mathrm{qs}}$, resp. $\mathbf{Rig}_K^{\mathrm{qcqs}}$) the category of rigid K-spaces (resp. quasi-separated rigid K-spaces, resp. quasi-compact and quasi-separated rigid K-spaces).

The universal separated quotient. The underlying topological space of a rigid K-space X is *valuative* in the sense of [12, Chapter 0, Definition 2.3.1]. This means that while the topology of X is locally spectral, and thus is scheme-theoretic in nature, the generizations of any point of X form a totally ordered set. In particular, unlike the case of schemes locally of finite type over a field, rigid K-spaces admit non-trivial continuous maps to separated (i.e. T_1) topological spaces.

Definition 2.1 ([12, Chapter 0, §2.3.(c)])**.** The *universal separated quotient of X*, denoted $[X]$, is the quotient topological space $X/\sim$ where $x \sim y$ if x and y are related by generization/specialization.

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The quotient map $X \to [X]$ is denoted by sep_X. As the name suggests, the space $[X]$ is separated and the map sep_X is initial amongst maps from X to separated spaces. The association of $[X]$ to X is functorial. The fibers of the map sep_X are Riemann–Zariski-esque spaces.

If X is a so-called *taut* rigid K-space (see [16, Definition 5.1.2]), then $[X]$ is in fact a locally compact Hausdorff space and $[X]$ agrees with the underlying topological space of the Berkovich space X^{Berk} associated to X (cf. [17, §8.3]).[1]

The map sep_X allows us to endow X with a coarser topology. Call an open subset of X *overconvergent* if it is of the form $\mathrm{sep}_X^{-1}(U)$ for an open subset $U \subseteq [X]$. One might also call these, in light of the above mentioned relation to Berkovich spaces, Berkovich open subsets. If $X = \mathrm{Spa}(A)$ then a basis for the toplogy of X are the open subsets of the form $\{x \in X : |f(x)| \leqslant 1\}$ for $f \in A$. In contrast, a basis for the overconvergent topology on X are those open subsets of the form

$$\{x \in X : |f(x)| < 1\}^\circ = \bigcup_{0<\varepsilon<1} \{x \in X : |f(x)| \leqslant \varepsilon\}.$$

Example 2.2. Assume that K is algebraically closed and let $\mathbf{B}_K = \mathrm{Spa}(K\langle T\rangle)$ be the closed unit disk over K. The structure of $[\mathbf{B}_K]$ is explained in great detail in [5, Chapter 1]. Recall (see [20, Example 2.20]) that the points x of $\mathbf{B}_K$ are classified into five types, and $\mathrm{sep}_X^{-1}(\mathrm{sep}_X(x))$ is $\{x\}$ unless x is a point of Type 2, in which case $\mathrm{sep}_X^{-1}(\mathrm{sep}_X(x))$ may be identified with $\mathbf{P}^1_k$ except when x is the Gauss point η_{Gauss} of $\mathbf{B}_K$ in which case it may be identified with $\mathbf{A}^1_k$.[2]

Generic fibers of formal schemes. Denote the category of formal schemes locally formally of finite type (resp. locally of finite type, resp. of finite type, resp. finite type and flat) over $\mathcal{O}$ by $\mathbf{FSch}^{\mathrm{lfft}}_{\mathcal{O}_K}$ (resp. $\mathbf{FSch}^{\mathrm{lft}}_{\mathcal{O}_K}$, resp. $\mathbf{FSch}^{\mathrm{ft}}_{\mathcal{O}_K}$, resp. $\mathbf{FSch}^{\mathrm{adm}}_{\mathcal{O}_K}$).

Let A be a topologically finite type $\mathcal{O}_K$-algebra, so $A_K = A[\frac{1}{\varpi}]$ is then topologically of finite type over K. The subring $A_K^\circ \subseteq A_K$ of powerbounded elements coincides with the integral closure of (the image of) A in A_K. There exists a unique functor

$$(-)_\eta \colon \mathbf{FSch}^{\mathrm{ft}}_{\mathcal{O}_K} \to \mathbf{Rig}^{\mathrm{qcqs}}_K$$

such that $\mathrm{Spf}(A)_\eta = \mathrm{Spa}(A_K)$ for every topologically finite type $\mathcal{O}_K$-algebra A, and which respects open immersions and open covers. This functor naturally extends to a functor

$$(-)_\eta \colon \mathbf{FSch}^{\mathrm{lft}}_{\mathcal{O}_K} \to \mathbf{Rig}^{\mathrm{qs}}_K,$$

and for $\mathfrak{X}$ locally of finite type over $\mathcal{O}_K$, the rigid K-space $\mathfrak{X}_\eta$ is called the *rigid generic fiber* of $\mathfrak{X}$. Furthermore, $(-)_\eta$ sends the class W of admissible blowups (see [12, Chapter II, §1.1]) to isomorphisms and induces equivalences of categories

$$\mathbf{FSch}^{\mathrm{adm}}_{\mathcal{O}_K}[W^{-1}] \xrightarrow[\mathrm{incl}]{\sim} \mathbf{FSch}^{\mathrm{ft}}_{\mathcal{O}_K}[W^{-1}] \xrightarrow[(-)_\eta]{\sim} \mathbf{Rig}^{\mathrm{qcqs}}_K.$$

Here $(-)[W^{-1}]$ denotes the localization with respect to W. By a *formal model* of a rigid K-space X we shall mean a formal scheme $\mathfrak{X}$ such that $\mathfrak{X}_\eta \simeq X$. The notion of the generic fiber of a formal scheme can be extended to $\mathbf{FSch}^{\mathrm{lfft}}_{\mathcal{O}}$ by a gluing construction (cf. [12, Chapter II, §9.6.(a)]).

If X belongs to $\mathbf{Rig}^{\mathrm{qcqs}}_K$, then the construction of the rigid generic fiber allows one to identify the locally topologically ringed space $(X, \mathcal{O}_X^+)$ as $\varprojlim(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ where $\mathfrak{X}$ runs over admissible formal models of X. In particular, for any model $\mathfrak{X}$ of X one has a map of topological spaces

$$\mathrm{sp}_{\mathfrak{X}} \colon |X| \to |\mathfrak{X}| = |\mathfrak{X}_k|$$

[1] This tautness assumption is quite mild, and holds true for any quasi-paracompact and quasi-separated rigid K-spaces (e.g. affinoids, analytifications of separated locally of finite type K-schemes, etc.) as well as any quasi-separated rigid K-space of equidimension 1 (see [1, Proposition 3.4.7]).

[2] This discrepancy between Gauss points and other type 2 points is related to the fact that $\mathbf{B}_K$ is not partially proper over $\mathrm{Spa}(K)$ (see [1, Example 3.4.3]).

called the *specialization map*. If $\mathfrak{X} = \mathrm{Spf}(A)$, then for a valuation $\nu\colon A_K \to \Gamma \cup \{0\}$, $\mathrm{sp}_{\mathfrak{X}}(\nu)$ is the open prime ideal $\{x \in A : \nu(x) < 1\}$ in $\mathrm{Spf}(A)$. The specialization map is continuous, quasi-compact, closed, and (for $\mathfrak{X}$ in $\mathbf{FSch}_{\mathcal{O}}^{\mathrm{adm}}$) surjective.

For an object $\mathfrak{X}$ of $\mathbf{FSch}_{\mathcal{O}}^{\mathrm{ft}}$, and a closed subset $Z \subseteq |\mathfrak{X}|$, we define the *tube open subset* of $\mathfrak{X}_\eta$ associated to Z, denoted $T(\mathfrak{X}|Z)$, to be the open subset $\mathrm{sp}_{\mathfrak{X}}^{-1}(Z)^\circ$. Tube open subsets are overconvergent opens, have the property that a closed cover $\{Z_i\}$ gives an overconvergent open cover $\{T(\mathfrak{X}|Z_i)\}$ of $\mathfrak{X}_\eta$, and the natural map $\widehat{\mathfrak{X}}_Z \to \mathfrak{X}$, where $\widehat{\mathfrak{X}}_Z$ is the completion of $\mathfrak{X}$ along Z, has generic fiber which is an open embedding with image $T(\mathfrak{X}|Z)$.

Example 2.3 (cf. [17, Proposition 1.9.6])**.** Suppose that $X \to \mathrm{Spec}(\mathcal{O})$ is a separated and locally of finite type morphism of schemes. Denote by $\mathfrak{X}$ the ϖ-adic completion of X. Then there is a functorial open immersion $\mathfrak{X}_\eta \to X_\eta^{\mathrm{an}}$ which is an isomorphism if $X \to \mathrm{Spec}(\mathcal{O})$ is proper.

Example 2.4. Let $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}\langle T\rangle)$ be the ϖ-adic completion of $\mathbf{A}^1_{\mathcal{O}}$ so then $\mathfrak{X}_\eta = \mathbf{B}_K$. For a point $\alpha \in \mathcal{O} = \mathbf{B}_K(K)$ one has $\mathrm{sp}_{\mathfrak{X}}(\alpha) = \overline{\alpha} \in k = \mathbf{A}^1_k(k)$, where $\overline{\alpha}$ is the image of α in k. In particular $\mathrm{sp}_{\mathfrak{X}}$ collapses every point in $\mathfrak{m} \subseteq \mathbf{B}_K(K)$ to the closed point $0 \in \mathbf{A}^1_k(k)$. In contrast, if ξ is the generic point of $\mathbf{A}^1_k$ then $\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi) = \{\eta_{\mathrm{Gauss}}\}$. Finally, if $Z = 0 \in \mathbf{A}^1_k(k)$, then

$$T(\mathfrak{X}|Z) = \mathbf{D}_K := \bigcup_{0<\varepsilon<1} \{x \in \mathbf{B}_K : |T(x)| \leqslant \varepsilon\} \subsetneq \mathrm{sp}_{\mathfrak{X}}^{-1}(Z) = \{x \in \mathbf{B}_K : |T(x)| < 1\}$$

where this containment is strict as (for example) the Type 5 point with $x = 0$ and $r = 1$, and $? =<$ (in the notation of [20, Example 2.20]) is in the right-hand side, but not the left. Observe that here we can visibly see that the generic fiber of the completion $\widehat{\mathfrak{X}}_Z = \mathrm{Spf}(\mathcal{O}[\![T]\!])$ of $\mathfrak{X}$ along Z agrees with $T(\mathfrak{X}|Z) = \mathbf{D}_K$.

Example 2.5. Consider the affinoid

$$A_{1,\varpi} := \mathrm{Spa}\left(K\left\langle T, \frac{\varpi}{T}\right\rangle\right) = \{x \in \mathbf{B}_K : |\varpi| \leqslant |T| \leqslant 1\}.$$

This has a model $\mathfrak{A}_{1,\varpi}$ given by $\mathrm{Spa}(\mathcal{O}\langle T, \frac{\varpi}{T}\rangle)$ whose special fiber is $\mathrm{Spec}(k[T,S]/(TS))$, the (reduced) union of the axes in $\mathbf{A}^2_{2,k}$. One may glue two copies of $A_{1,\varpi}$ together along the (rational open) unit circle

$$C := \mathrm{Spa}\left(K\langle T, T^{-1}\rangle\right) = \{x \in A_{1,\varpi} : |T(x)| = 1\}$$

via the automorphism $T \mapsto T^{-1}$ of C. The result is

$$A_{\varpi^{-1},\varpi} = \{x \in \mathbf{A}^{1,\mathrm{an}}_K : |\varpi| \leqslant |x| \leqslant |\varpi|^{-1}\}$$

This has a model $\mathfrak{A}_{\varpi^{-1},\varpi}$ given by gluing $\mathrm{Spf}(\mathcal{O}\langle T, \frac{\varpi}{T}\rangle)$ to itself along $\mathrm{Spf}(\mathcal{O}\langle T, T^{-1}\rangle)$ via $T \mapsto T^{-1}$. The special fiber of $\mathfrak{A}_{\varpi^{-1},\varpi}$ is two copies of $\mathrm{Spec}(k[T,S]/(TS))$ glued along the non-vanishing locus $D(S) = \mathrm{Spec}(k[T,T^{-1}])$ via the automorphism $T \mapsto T^{-1}$. In other words, the special fiber of $\mathfrak{A}_{\varpi^{-1},\varpi}$ looks like $\mathbf{P}^1_k$ with copies of $\mathbf{A}^1_k$ glued to each pole. It is then perhaps not surprising that if $\mathfrak{X}_1$ is the admissible blowup of $\widehat{\mathbf{P}}^1_{\mathcal{O}}$ along the two poles 0 and ∞ then

$$\mathfrak{A}_{\varpi^{-1},\varpi} = \mathfrak{X}_1 - \{0_1, \infty_1\}, \qquad A_{\varpi^{-1},\varpi} = \mathrm{sp}_{\mathfrak{X}_1}^{-1}(\mathfrak{A}_{\varpi^{-1},\varpi}) \subseteq \mathbf{P}^{1,\mathrm{an}}_K,$$

where 0_1 and ∞_1 are the poles of the exceptional divisor of the blowup not intersecting the original copy of $\widehat{\mathbf{P}}^1_{\mathcal{O}}$. Continuing in this way, either by the blowup or gluing procedure, we obtain spaces

$$\mathfrak{A}_{\varpi^{-n},\varpi^n} = \mathfrak{X}_n - \{0_n, \infty_n\}, \qquad A_{\varpi^{-n},\varpi^n} = \{x \in \mathbf{A}^{1,\mathrm{an}}_K : |\varpi|^n \leqslant |x| \leqslant |\varpi|^{-n}\} = \mathrm{sp}_{\mathfrak{X}_n}^{-1}(\mathfrak{A}_{\varpi^{-n},\varpi^n}) \subseteq \mathbf{P}^{1,\mathrm{an}}_K.$$

Iterating infinitely is seen, either by the blowup or gluing procedure, to yield (a model of) $\mathbf{G}^{\mathrm{an}}_{m,K}$.

2.2. **Tame infinite Galois categories.** The technical underpinning for the notion of 'fundamental group', in the generality that we will need it, is the notion of a tame infinite Galois category. In essence, this theory seeks to axiomatize the study of the category $G\text{-}\mathbf{Set}$ of discrete sets with a continuous action of a topological group. It is in analogy with the classical theory of Galois categories, where one studies finite sets with a continuous action of a profinite group.

Definition 2.6 ([7, Definition 7.2.1])**.** Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathbf{Set}$ be a functor. We then call the pair $(\mathcal{C}, F)$ an *infinite Galois category* if the following properties hold:

- **(IGC1)** The category $\mathcal{C}$ is cocomplete and finitely complete.
- **(IGC2)** Each object X of $\mathcal{C}$ is a coproduct of categorically connected objects.[3]
- **(IGC3)** There exists a set S of connected objects of $\mathcal{C}$ which generates $\mathcal{C}$ under colimits.
- **(IGC4)** The functor F is faithful, conservative, cocontinuous, and finitely continuous.

We say that $(\mathcal{C}, F)$ is *tame* if for every categorically connected object X of $\mathcal{C}$ the action of $\pi_1(\mathcal{C}, F)$ on $F(X)$ is transitive. The *fundamental group of* $(\mathcal{C}, F)$, denoted $\pi_1(\mathcal{C}, F)$ is the group $\mathrm{Aut}(F)$ endowed with the compact-open topology.[4]

For a topological group G, we denote by $G\text{-}\mathbf{Set}$ the category of discrete sets with a continuous of G. The upshot of the theory of (tame) infinite Galois categories is the following.

Proposition 2.7 ([7, Example 7.2.2 and Theorem 7.2.5])**.** *Let $(\mathcal{C}, F)$ be an infinite Galois category and G a Noohi group*[5]*. Then, the following statements are true.*

(a) The group $\pi_1(\mathcal{C}, F)$ with its compact-open topology is a Noohi group.
(b) The pair $(G\text{-}\mathbf{Set}, F_G)$, where $F_G\colon G\text{-}\mathbf{Set} \to \mathbf{Set}$ is the forgetful functor, is a tame infinite Galois category with a canonical isomorphism $G \simeq \pi_1(G\text{-}\mathbf{Set}, F_G)$.
(c) The natural map $\mathrm{Hom}((\mathcal{C}, F), (G\text{-}\mathbf{Set}, F_G)) \to \mathrm{Hom}_{\mathrm{cnts}}(G, \pi_1(\mathcal{C}, F))$ is a bijection.
(d) If $(\mathcal{C}, F)$ is tame then F induces an equivalence $F\colon \mathcal{C} \xrightarrow{\sim} \pi_1(\mathcal{C}, F)\text{-}\mathbf{Set}$.

3. The category of de Jong covering spaces

In this section we discuss the theory of covering spaces of rigid spaces developed by de Jong in [8] using more modern language.

Motivation. To properly discuss de Jong's theory of covering spaces, as well as later examples of covering spaces, it is useful to first develop some notation. Fix a site $(\mathcal{S}, \tau)$ and a stack $\mathcal{D} \to \mathcal{S}$ such that $\mathcal{D}_X$ has all coproducts for all objects X of $\mathcal{S}$. For a fibered subcategory $\mathcal{C}$ of $\mathcal{D}$ define

- $L_\tau \mathcal{C}$ to be the τ-stackification of $\mathcal{C}$ (i.e. $(L_\tau \mathcal{C})_X$ consists of those elements Y of $\mathcal{D}_X$ for which there exists a τ-cover $\{U_i \to X\}$ such that Y_{U_i} belongs to $\mathcal{C}_{U_i}$ for all i),
- $\mathbf{U}\mathcal{C}$ (resp. $\mathbf{U}_{\mathrm{fin}}\mathcal{C}$) to be the subfibered category of $\mathcal{D}$ such that $(\mathbf{U}\mathcal{C})_X$ (resp. $(\mathbf{U}_{\mathrm{fin}}\mathcal{C})_X$) consists of all (resp. all finite) coproducts of elements of $\mathcal{C}_X$.

One may observe that many categories of 'covering spaces' occur by starting with a category of 'nice morphisms' (usually '*isomorphisms*') $\mathcal{C}$ and iterating the above operations.

Example 3.1. Let $(\mathcal{S}, \tau) = (\mathbf{Top}, \mathrm{op})$ be the category topological spaces with the usual Grothendieck topology, and let $\mathcal{D}$ be the arrow category of $\mathbf{Top}$. If $\mathbf{Isom}$ denotes the fibered subcategory of $\mathcal{D}$ consisting of isomorphisms then the stack $\mathbf{Cov}^{\mathrm{top}} \to \mathbf{Top}$ consisting of covering spaces is $L_{\mathrm{op}}\mathbf{U}\,\mathbf{Isom}$.

[3] An object Y is *categorically connected* if every monomorphism $Y' \to Y$ with Y' non-initial is an isomorphism.

[4] More precisely, for each s in S, where S is as in **(IGC3)**, we endow $\mathrm{Aut}(s)$ with the compact-open topology. We then endow $\mathrm{Aut}(F)$ with the subspace topology inherited from the natural map $\mathrm{Aut}(F) \to \prod_{s \in S} \mathrm{Aut}(s)$.

[5] A *Noohi group* is a Hausdorff topological group which has a neighborhood basis of 1 given by open (not necessarily normal) subgroups, and which is Raĭkov complete

Example 3.2. Let $(\mathcal{S}, \tau) = (\mathbf{Sch}, \text{ét})$ be the category **Sch** of schemes with the big étale topology, and let $\mathcal{D}$ be the arrow category of **Sch**. Then the stack $\mathbf{F\acute{E}t} \to \mathbf{Sch}$ of finite étale morphisms is precisely $L_{\text{ét}}\mathbf{U}_{\text{fin}}\,\mathbf{Isom}$.

Definition 3.3 (Berkovich, de Jong)**.** Let X be a rigid K-space. Then, the category of *de Jong covering spaces*, denoted $\mathbf{Cov}_X^{\text{oc}}$, consists of all morphisms $Y \to X$ of rigid K-spaces such that there exists an overconvergent open cover $\{U_i\}$ of X such that Y_{U_i} belongs to $\mathbf{UF\acute{E}t}_{U_i}$ for all i.

If oc denotes the Grothendieck topology consisting of overconvergent open covers we see that

$$\mathbf{Cov}^{\text{oc}} = L_{\text{oc}}\,\mathbf{UF\acute{E}t} = L_{\text{oc}}\mathbf{U}L_{\text{ét}}\mathbf{U}_{\text{fin}}\,\mathbf{Isom}\,.$$

Intuitively we may think of $\mathbf{Cov}_X^{\text{oc}}$ as being the synthesis of the notions of topological covering space of X^{Berk} and finite étale covering space of X^{Berk}.

The original motivation of de Jong in his paper [8] to work with de Jong covering spaces is that examples of such covering spaces were abundant in nature

Example 3.4. Let q be an element of K satisfying $0 < |q| < 1$. Then, $q^{\mathbf{Z}}$ acts on $\mathbf{G}_{m,K}^{\text{an}}$ by translation, and this action is properly discontinuous for the overconvergent topology. Thus, one gets a quotient space $\mathbf{G}_{m,K}^{\text{an}}/q^{\mathbf{Z}}$ and the mapping $\mathbf{G}_{m,K}^{\text{an}} \to \mathbf{G}_{m,K}^{\text{an}}/q^{\mathbf{Z}}$ is a topological $\mathbf{Z}$-covering on the level of Berkovich spaces and so is a de Jong covering space.

Example 3.5 (see [4, Chapter III, Example 1.2.6])**.** The logarithm map

$$\log\colon \mathbf{D}_K \to \mathbf{A}_K^{1,\text{an}}, \qquad x \mapsto \log(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

is a de Jong covering space.

Example 3.6 (Yu, see [8, Proposition 7.2])**.** The Gross–Hopkins period mapping

$$\pi_{\text{GH}}\colon \mathbf{D}_{\mathbf{C}_p} \to \mathbf{P}_{\mathbf{C}_p}^{1,\text{an}}$$

is a de Jong covering space.

The de Jong fundamental group. While the definition of a de Jong covering space is useful for capturing the natural examples mentioned above, the notion would not be a true instace of 'covering space' if the category $\mathbf{Cov}_X^{\text{oc}}$ could not be studied by fundamental group theoretic means. The main theorem of [8] addresses this question. Indeed, if we use the notation

$$F_{\overline{x}}\colon\ \mathbf{\acute{E}t}_X \to \mathbf{Set}, \qquad (Y \to X) \mapsto \operatorname{Hom}_X(\overline{x}, Y).$$

then the main theorem op. cit. may be interpreted (in modern language) as follows.[6]

Theorem 3.7 (de Jong)**.** *Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then the pair $(\mathbf{UCov}_X^{\text{oc}}, F_{\overline{x}})$ is a tame infinite Galois category.*

Before we sketch the proof of this result, let us first make some notational preparations. For simplicity we assume that X is tame and write $\mathscr{X} := X^{\text{Berk}}$. For an open subset $\mathscr{U}$ of $\mathscr{X}$ we write U for the overconvergent open subset $\text{sep}_X^{-1}(\mathscr{U})$. Finally, for two geometric points $\overline{x}$ and $\overline{y}$ and a subcategory $\mathcal{C}$ of $\mathbf{\acute{E}t}_X$, denote by $\operatorname{Isom}_{\mathcal{C}}(F_{\overline{x}}, F_{\overline{y}})$ the set of isomorphisms $(F_{\overline{x}})|_{\mathcal{C}} \to (F_{\overline{y}})_{\mathcal{C}}$.

[6]As the theorem suggests, the notion of a de Jong covering space is not closed under disjoint unions and so cannot be a tame infinite Galois category. For an example of this type of phenomenon see [2, Remark 3.4]. Intuitively the issue is that unlike the case of complex manifolds, rigid K-spaces are not locally contractible for the étale topology (e.g. the closed unit disk has non-trivial finite étale covers). One can similarly create examples to show that the composition of two de Jong covering spaces needn't be a de Jong covering space.

Idea of proof. The only difficult condition to verify is tameness. It suffices to show that for any two geometric points $\overline{x}$ and $\overline{y}$ that $\mathrm{Isom}_{\mathbf{Cov}_X^{\mathrm{oc}}}(F_{\overline{x}}, F_{\overline{y}})$ is non-empty (cf. [1, Proof of Proposition 5.4.9]). To prove this we use the fact (see [6, Theorem 3.2.1 and Corollary 4.3.3]) that $\mathscr{X}$ is arc connected, i.e. any two points a and b are the endpoints of a subspace $\ell \subseteq \mathscr{X}$ homeomorphic to $[0,1]$. Let ℓ be such an arc with endpoints x and y. For any 'nice, linearly ordered' finite open cover $\mathfrak{U} = \{\mathscr{U}_1, \ldots, \mathscr{U}_n\}$ of ℓ (see [8, Proof of Theorem 2.9]) we have the category $\mathbf{Cov}_{\mathfrak{U}}$ of morphisms $Y \to X$ such that Y_{U_i} is in $\mathbf{UF\acute{E}t}_{U_i}$ for all i. As ℓ is compact,

$$\mathbf{Cov}_X^{\mathrm{oc}} = \varinjlim_{\mathfrak{U}} \mathbf{Cov}_{\mathfrak{U}}, \qquad \mathrm{Isom}_{\mathbf{Cov}_X^{\mathrm{oc}}}(F_{\overline{x}}, F_{\overline{y}}) = \varprojlim \mathrm{Isom}_{\mathbf{Cov}_{\mathfrak{U}}}(F_{\overline{x}}, F_{\overline{y}}).$$

For each $\mathfrak{U}$ let $K_{\mathfrak{U}}$ be the image of the composition map

$$\mathrm{Isom}_{\mathbf{UF\acute{E}t}_{U_1}}(F_{\overline{x}}, F_{\overline{x}_1}) \times \cdots \times \mathrm{Isom}_{\mathbf{UF\acute{E}t}_{U_n}}(F_{\overline{x}_{n-1}}, F_{\overline{y}}) \to \mathrm{Isom}_{\mathbf{Cov}_{\mathfrak{U}}}(F_{\overline{x}}, F_{\overline{y}}).$$

Here each $\overline{x}_i$ is a geometric point anchored in $\mathscr{U}_i \cap \mathscr{U}_{i+1}$, and note this image $K_{\mathfrak{U}}$ may be shown to be independent of such choices. Each set $\mathrm{Isom}_{\mathbf{UF\acute{E}t}_{U_i}}(F_{\overline{x}_{i-1}}, F_{\overline{x}_i})$ is a pseudo-torsor under the profinite group $\pi_1^{\mathrm{alg}}(U_i, \overline{x}_i)$ and, as $\mathbf{F\acute{E}t}_{U_i}$ is a Galois category, is in fact a torsor (cf. [24, Tag 0BN5]). This endows each $K_{\mathfrak{U}}$ with the structure of a compact Hausdorff space. The transition maps $K_{\mathfrak{U}'} \to K_{\mathfrak{U}}$ are continuous, and thus $\varprojlim K_{\mathfrak{U}}$ is a projective limit of non-empty compact spaces and so non-empty. As $\varprojlim K_{\mathfrak{U}}$ admits a map to $\varprojlim \mathrm{Isom}_{\mathbf{Cov}_{\mathfrak{U}}}(F_{\overline{x}}, F_{\overline{y}})$ we're done. □

Denote by $\pi_1^{\mathrm{dJ}}(X, \overline{x})$ the Noohi group $\pi_1(\mathbf{UCov}_X^{\mathrm{oc}}, F_{\overline{x}})$ and call it the *de Jong fundamental group*. As this group does not depend on the chosen base point we shall often omit it from the notation.

It is important to note that the de Jong fundamental group can be quite complex even in relatively simple situations. In particular, the following example shows that even for a space as simple as $\mathbf{P}^{1,\mathrm{an}}_{\mathbf{C}_p}$ the de Jong fundamental group need not be pro-discrete (i.e. is not, in the category of topological groups, an inverse limit of discrete groups).[7]

Example 3.8 (de Jong, [8, Propsition 7.4])**.** The Gross–Hopkins period map π_{GH} gives rise to a continuous surjection $\pi_1^{\mathrm{dJ}}(\mathbf{P}^{1,\mathrm{an}}_{\mathbf{C}_p}) \to \mathrm{SL}_2(\mathbf{Q}_p)$.

4. The category of geometric coverings of schemes

In this section we talk about the category of geometric coverings of schemes discussed by Bhatt–Scholze in [7].

Motivation. Given the situation with de Jong covering spaces, it's natural to ask what happens for schemes. For instance, why does the category $L_{\text{ét}}\mathbf{U}L_{\text{ét}}\mathbf{U}_{\text{fin}}\mathbf{Isom}$ not show up in the classical study of covering spaces of schemes? As it turns out, the reason is (at least partially) because one doesn't obtain anything new if the scheme is reasonablly well-behaved.

Proposition 4.1. *Let X be a topologically Noetherian geometrically unibranch scheme. Then,* $L_{\mathrm{fpqc}}\,\mathbf{UF\acute{E}t}_X = \mathbf{UF\acute{E}t}_X$.

We delay proving this proposition until later, as it will fall out of more general machinery. Until then, we note that this proposition patently fails for even relatively benign varieties which are not geometrically unibranch.

Example 4.2. Let k be a field and let

$$X = V(y^2z - x^3 - x^2z) \subseteq \mathbf{P}^2_k$$

[7] There is a minor error in [8] where it is claimed that $\pi_1^{\mathrm{dJ}}(X)$ is always pro-discrete, see [7, Remark 7.4.11].

be the projective nodal cubic curve. Note X is the result of pinching (see [11]) together 0 and ∞ on $\mathbf{P}^1_k$. Let d be a non-negative integer and set $Y'_d = \bigsqcup_{m \in \mathbf{Z}/d\mathbf{Z}} X_m$ where each X_m is a copy of $\mathbf{P}^1_k$ and let Y_d be the result of pinching together each ∞ in X_m with 0 in X_{m+1}. The tautological map $Y'_d \to \mathbf{P}^1_k$ gives rise to a map $Y_d \to X$ via the universal property of pinching. Away from the pinched points the map $Y_d \to X$ is a disjoint union of isomorphisms, and a pinched point y of Y maps to the pinched point x of X. As the induced map $\widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{Y,y}$ is an isomorphism (see [15, Example V.6.3]), the maps $Y_d \to X$ are étale. One checks that Y_d is connected, $\mathrm{Aut}(Y_d/X) = \mathbf{Z}/d\mathbf{Z}$, and $Y_d \times_X Y_0 \cong \bigsqcup_{m \in \mathbf{Z}/d\mathbf{Z}} Y_0$ as Y_0-schemes. In particular, Y_0 belongs to $L_{\text{ét}}\,\mathbf{UF\acute{E}t}_X$ but not to $\mathbf{UF\acute{E}t}_X$.[8]

This example, and others like it, are handled by the following definition of Bhatt–Scholze.

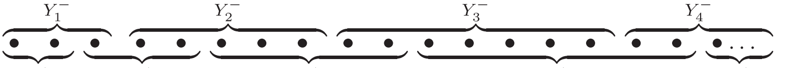
Definition 4.3 ([7, 7.3.1(3)]). Let X be a locally topologically Noetherian scheme. A morphism of schemes $Y \to X$ is a *geometric covering* if it is étale and partially proper.[9] We denote by $\mathbf{Cov}_X$ the category of geometric coverings of X.

We now give some further examples of geometric coverings.

Example 4.4 (See [7, Lemma 7.4.10] and its proof). Let X be a locally topologically Noetherian scheme. If X is geometrically unibranch, then $\mathbf{Cov}_X = \mathbf{UF\acute{E}t}_X$.

It is worth noting that geometric coverings is strictly larger than $L_{\text{ét}}\,\mathbf{UF\acute{E}t}_X$ in some cases.

Example 4.5 ([2, Remark 3.9]). Let k be an algebraically closed field, and let X be the curve obtained by pinching two copies, call them X^+ and X^-, of $\mathbf{G}_{m,k}$ together at a closed point x. For $n \geqslant 0$, let $Y_n^\pm \to X^\pm$ be the connected cyclic covering of degree equal to the n-th prime number invertible in k. Let $Y^+ = \coprod_{n \geqslant 0} Y_n^+$ and $Y^- = \coprod_{n > 0} Y_n^-$, and let $Y \to X$ be the geometric covering of X with $Y|_{X^\pm} \simeq Y^\pm$ obtained by identifying the fibers of $Y^\pm$ at x as in the picture below.



Then $Y \to X$ is a geometric covering but is not the disjoint union of finite étale coverings in any étale neighborhood of x.

The pro-étale fundamental group. Again, for the category of geometric coverings to be useful, it is highly desirable that there is an associated theory of fundamental groups. To prove this it is useful to first recall the context that Bhatt and Scholze first considered geometric coverings in.

Definition 4.6. The *pro-étale site* of X, denoted $X_{\text{proét}}$, has objects consisting of weakly étale morphisms $Y \to X$ (see [24, Tag 094N]) and whose covering families consist of fpqc covers. We denote by $\mathbf{Loc}(X_{\text{proét}})$ the category of locally constant sheaves of sets on $X_{\text{proét}}$.

Using the pro-étale topology, Bhatt–Scholze are able to give an alternative description of $\mathbf{Cov}_X$.

Proposition 4.7 (cf. [7, Lemma 7.3.9]). *For a locally topologically Noetherian scheme X,*

$$\mathbf{Cov}_X = L_{\text{fpqc}}\,\mathbf{Isom} \cong \mathbf{Loc}(X_{\text{proét}}).$$

Using this, one can show that $\mathbf{Cov}_X$ is a tame infinite Galois category.

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[8] In fact, one can see that $Y_0 \to X$ belongs to $L_{\text{ét}}\mathbf{UIsom}$. Such morphisms, called *SGA3 covering spaces*, were already considered in [9] where they are used to classify tori over non-normal bases.

[9] A morphism is called *partially proper* if it is quasi-separated, locally of finite type, and satisfies the valuative criterion for properness (see [24, Tag 03IX]).

Theorem 4.8 ([7, Lemma 7.4.1])**.** *Let X be a connected locally topologically Noetherian scheme and $\overline{x}$ a geometric point of X. Then, the pair $(\mathbf{Cov}_X, F_{\overline{x}})$ is a tame infinite Galois category.*

Idea of proof. By Proposition 4.7 it suffices to show that $\mathbf{Loc}(X_{\text{proét}})$ with the stalk functor is a tame infinite Galois category. As in the proof of Theorem 3.7 it suffices to show that for any two geometric points $\overline{x}$ and $\overline{y}$ that the set $\mathrm{Isom}_{\mathbf{Loc}(X_{\text{proét}})}(F_{\overline{x}}, F_{\overline{y}})$ is non-empty. As X is topologically Noetherian one may connect the underlying points of $\overline{x}$ and $\overline{y}$ by finitely many specialization and generization relations. Thus, one may assume without loss of generality that the underlying point of $\overline{x}$ generizes that of $\overline{y}$. So there exists (cf. [24, Tag 02JQ]) a valuation R with separably closed fraction field and a morphism $\mathrm{Spec}(R) \to X$ such that $\overline{x}$ and $\overline{y}$ lift to $\mathrm{Spec}(R)$. As we have a map

$$\mathrm{Isom}_{\mathbf{Loc}(\mathrm{Spec}(R)_{\text{proét}})}(F_{\overline{x}}, F_{\overline{y}}) \to \mathrm{Isom}_{\mathbf{Loc}(X_{\text{proét}})}(F_{\overline{x}}, F_{\overline{y}})$$

it suffices to show that $\mathrm{Isom}_{\mathbf{Loc}(\mathrm{Spec}(R)_{\text{proét}})}(F_{\overline{x}}, F_{\overline{y}})$ is non-empty. But, by Example 4.4 and [24, Tag 09Z9] this is trivial to do. □

We call the Noohi group $\pi_1(\mathbf{Cov}_X, F_{\overline{x}})$ the *pro-étale fundamental group* of the pair $(X, \overline{x})$ and denote it $\pi_1^{\text{proét}}(X, \overline{x})$. As this group is independent of the choice of $\overline{x}$ we shall often supress it from the notation.

One may notice the similarities between the proof of Theorem 4.8 and that of Theorem 3.7, with the role of γ in the latter being replaced by $\mathrm{Spec}(R)$ in the former. We return to this point later on. We also note that by combining Proposition 4.7 and Example 4.4 we immediately obtain a proof of Proposition 4.1.

We end this section by giving some examples of the pro-étale fundamental group.

Example 4.9. Let X be the projective nodal cubic curve from Example 4.2. then, the map $Y_0 \to X$ is a Galois geometric covering and realizes $\pi_1^{\text{proét}}(X)$ as the discrete group $\mathbf{Z}$.

Example 4.10 (Deligne, [7, Example 7.4.9])**.** Let X be a curve of genus at least 1 over an algebraically closed field k. Let Y be the result of pinching two distinct points of X together. Then, there exists a representation $\pi_1^{\text{proét}}(Y) \to \mathrm{GL}_2(\mathbf{Q}_p)$ with non-(pro-discrete) image.

Example 4.11. Let X and Y be as in Example 4.5. The Noohi group $\pi_1^{\text{proét}}(X)$ is not pro-discrete. Indeed, if it were then by [7, Lemma 7.4.6] (and its proof) one would have that $Y \to X$ is in $L_{\text{ét}}\,\mathbf{Isom}$, but this is false.

5. The specialization morphism

In this section we discuss the result of [3] showing the existence of a specialization map between the de Jong fundamental group and the pro-étale fundamental group.

Motivation. The idea of specialization (for fundamental groups) finds its conceptual roots (as do many things in arithmetic geometry) in complex geometry. Let Δ denote the open unit disk in $\mathbf{C}$.

Theorem 5.1 ([19, Proposition C.11])**.** *Let $X \to \Delta$ be a flat proper morphism of complex analytic spaces with X connected. Then, there exists an open subdisk $0 \in \Delta' \subseteq \Delta$ such that the inclusion $X_0 \hookrightarrow X_{\Delta'}$ is a homotopy equivalence.*

For simplicity let us assume that $\Delta' = \Delta$. From this we deduce the existence of *specialization homomorphisms*

$$\mathrm{sp}\colon \pi_1^{\text{top}}(X_{\Delta^*}) \to \pi_1^{\text{top}}(X_0), \qquad \mathrm{sp}\colon \pi_1^{\text{top}}(X_t) \to \pi_1^{\text{top}}(X_0)$$

where $\Delta^* = \Delta - \{0\}$ and t is any point of Δ. Indeed, this second map is obtained as the composition of the morphism $\pi_1^{\text{top}}(X_t) \to \pi_1^{\text{top}}(X_\Delta)$ with the inverse of the isomorphism $\pi_1^{\text{top}}(X_0) \xrightarrow{\sim} \pi_1^{\text{top}}(X_\Delta)$, and similarly for the first.

Proposition 5.2. *If X is normal, then the specialization homomorphim* $\mathrm{sp}\colon \pi_1^{\mathrm{top}}(X_{\Delta^*}) \to \pi_1^{\mathrm{top}}(X_0)$ *is surjective.*

Proof. As these groups are discrete it suffices to show that if $Y \to X$ is a connected covering space, then Y_{Δ^*} is connected (cf. [18, Proposition 2.37.(2)]). Observe that Y may be given the unique structure of a complex analytic space so that $Y \to X$ is holomorphic. As $Y \to X$ is a local biholomorphism, the fact that X is normal implies that Y is normal. So, Y is locally irreducible (see [13, Chapter 6, §4.2]) and as Y is connected it is thus irreducible. Therefore, as $Y \to \Delta$ is surjective, we know that Y_0 is a proper closed analytic subset of Y and so thin by [13, Chapter 9, §1.2, Theorem]. Thus, Y_{Δ^*} is connected by [13, Chapter 7, §4.2, Criterion of Connectedness]. □

Therefore, we see that if $X \to \Delta$ is proper and flat, and X is normal, one has a sort of *semi-continuity* result which says that the fundamental group shrinks under specialization from a 'general point of Δ' (repsented by Δ^*) to a specific point.

Example 5.3. Let
$$X = V(y^2z + x^3 + x^2z - tz^3) \subseteq \mathbf{P}^2_\Delta,$$
where x, y, z are the parameters of $\mathbf{P}^2$ and t the parameter of Δ, and so X is normal. In this case X_0 is the nodal cubic curve and the generic fiber X_t is an elliptic curve. One can intuitively imagine that X_t is diffeomorphic to $S^1 \times S^1$ but where the second copy of S^1 has radius t which shrinks to 0. One may then intuitively see the specialization map
$$\mathbf{Z}^2 \cong \pi_1^{\mathrm{top}}(X_t) \xrightarrow{\mathrm{sp}} \pi_1^{\mathrm{top}}(X_0) \cong \mathbf{Z}$$
as the surjective map collapsing this second copy of S^1 to 0.

Grothendieck specialization. In [14] one finds an algebraic analogue of specialization in complex geometry. For our purposes we restrict to a special case more directly related to that of the complex situation. Namely, let us fix a non-archimedean field K. Here $\mathrm{Spec}(\mathcal{O})$ acts as a 'contractible object of dimension 1' much like the disk Δ.

Fix $X \to \mathrm{Spec}(\mathcal{O})$ to be a flat proper map of schemes and write $i\colon X_k \to X$ to be the inclusion.

Theorem 5.4 (Grothendieck)**.** *The pullback morphism* $i^*\colon \mathbf{F\acute{E}t}_X \to \mathbf{F\acute{E}t}_{X_k}$ *is an equivalence.*

Proof. Let $\mathfrak{X}$ be the ϖ-adic completion of X. Observe that the inclusion $X_k \to X$ factorizes as $X_k \to \mathfrak{X} \to X$. Now, $X_k \to \mathfrak{X}$ is a universal homeomorphism of formal schemes, so by the topological invariance of the étale site (see [2, Proposition 3.5]) it induces an equivalence of categories $\mathbf{\acute{E}t}_{\mathfrak{X}} \to \mathbf{\acute{E}t}_{X_k}$ and $\mathbf{F\acute{E}t}_{\mathfrak{X}} \to \mathbf{F\acute{E}t}_{X_k}$. On the other hand, as $X \to \mathrm{Spec}(\mathcal{O})$ is proper we know by formal GAGA (see [12, Chapter I, Theorem 10.1.2]) that the completion functor $\mathbf{F\acute{E}t}_X \to \mathbf{F\acute{E}t}_{\mathfrak{X}}$ is an equivalence. We are done as our functor is the composition of these equivalences. □

I emphasize the role of properness in the above proof. At first look it appears as though it is the algebraizability of the unique lift $\mathfrak{Y} \to \mathfrak{X}$ of a finite étale map $Y_k \to X_k$ for which formal GAGA, and thus properness of $X \to \mathrm{Spec}(\mathcal{O})$, is being used. That said, algebraizability of this unique lift happens in much more general situations than the case when $X \to \mathrm{Spec}(\mathcal{O})$ is proper.

For instance, if $X = \mathrm{Spec}(A)$ is the spectrum of a finite type flat $\mathcal{O}$-algebra, then any finite étale cover $Y_k \to X_k$ can be lifted to an algebraic map $Y \to X$. Indeed, choose a presentation $Y_k = \mathrm{Spec}(A_k[x_1, \ldots, x_m]/(f_1, \ldots, f_m))$. Then, if $\mathfrak{Y} = \mathrm{Spf}(A\langle x_1, \ldots, x_m\rangle(\widetilde{f}_1, \ldots, \widetilde{f}_m))$, one has that $\mathfrak{Y}$ is the completion of $Y = \mathrm{Spec}(A[x_1, \ldots, x_m]/(g_1, \ldots, g_m))$ where g_i in $A[x_1, \ldots, x_m]$ are such that $g_i = \widetilde{f}_i \mod \varpi^N$ for $N \gg 0$ (cf. [10, Théorème 7]). But, as $X \to \mathrm{Spec}(\mathcal{O})$ is not proper the map $Y \to X$ need not be étale. Intuitively the issue is contained in the observation made in Example 2.3, that if $X \to \mathrm{Spec}(\mathcal{O})$ is not proper, then X_η is often strictly larger than $\mathfrak{X}_\eta$.

Example 5.5. Let $X = \mathbf{A}^1_{\mathcal{O}} = \operatorname{Spec}(\mathcal{O}[T])$ and consider the finite étale Artin–Schreier cover $Y_k = \operatorname{Spec}(k[S,T]/(S - T^p - T - 1))$. The unique finite étale lift of Y_k to a finite étale morphism $\mathfrak{Y} \to \mathfrak{X}$ is given by $\mathfrak{Y} = \operatorname{Spf}(\mathcal{O}\langle S,T\rangle/(S - T^p - T - 1))$. This is the completion of the finite X-scheme $\operatorname{Spec}(\mathcal{O}[S,T]/(S - T^p - T - 1))$, but $Y \to X$ fails to be étale. Essentially the reason is that $Y_\eta \to X_\eta$ is not etale at any point of X_η corresponding to a p^{th}-root of $-p^{-1}$. This point belongs to X_η but not to $\mathfrak{X}_\eta$.

Returning to the case of proper $\mathcal{O}$-schemes, from Theorem 5.4 one is able to create a functor

$$\operatorname{sp}^*\colon \mathbf{F\acute{E}t}_{X_k} \to \mathbf{F\acute{E}t}_{X_\eta}$$

sending a finite étale morphism $Y_k \to X_k$ to the finite étale morphism $Y_\eta \to X_\eta$ where $Y \to X$ is the unique finite étale deformation of $Y_k \to X_k$ to X. From Proposition 2.7 this gives us a *continuous specialization homomorphism*

$$\operatorname{sp}\colon \pi_1^{\text{ét}}(X_\eta) \to \pi_1^{\text{ét}}(X_k).$$

As in the complex setting, one gets a semi-continuity statement for the étale fundamental group if one assumes that X is normal.

Proposition 5.6 (cf. [24, Tag 0BQM])**.** *Let $X \to \operatorname{Spec}(\mathcal{O})$ be a flat proper morphism with X normal. Then, the specialization homomorphism $\operatorname{sp}\colon \pi_1^{\text{ét}}(X_\eta) \to \pi_1^{\text{ét}}(X_k)$ is surjective.*

The proof of this result is similar to the proof of Proposition 5.2, but largely simpler in this algebraic situation. The only added difficulty, and the main part of the proof, is to show that the normality of X is inherited by finite étale covers of X.

Example 5.7. Let $K = \mathbf{C}_p$, and let X be the curve

$$X = V(y^2z - x^3 - x^2z - pz^3) \subseteq \mathbf{P}^2_{\mathcal{O}}.$$

Let us then note that X_k is the projective nodal cubic curve from Example 4.2 and X_η is an elliptic curve over K with split multiplicative reduction. From Example 4.9 we know that the covers $Y_d \to X_k$ for $d \geqslant 1$ are cofinal in the category of finite étale covers of X_k, and so $\pi_1^{\text{ét}}(X_k) \cong \widehat{\mathbf{Z}}$. On the other hand we know that $\pi_1^{\text{ét}}(X_K) = \widehat{\mathbf{Z}}^2$. As X is regular, we get from the above theory a surjective specialization homomorphism $\pi_1^{\text{ét}}(X_K) \to \pi_1^{\text{ét}}(X_k)$. We may describe this more concretely using Tate's uniformization theorem (e.g. see [23, Chapter V, §3]). Namely, choosing q in K with $0 < |q| < 1$ such that X_K^{an} is isomorphic to $\mathbf{G}^{\text{an}}_{m,K}/q^{\mathbf{Z}}$ we have that $\operatorname{sp}^*(Y_d) \to X_K$ is the morphism whose analytification coincides with the natural quotient map $\mathbf{G}^{\text{an}}_{m,K}/q^{d\mathbf{Z}} \to \mathbf{G}^{\text{an}}_{m,K}/q^{\mathbf{Z}}$.

Specialization for the de Jong fundamental group. As in Example 5.7, even very well-behaved (e.g. smooth) proper schemes X_K over K can have models X over $\mathcal{O}$ whose special fiber X_k is not geometrically unibranch. Thus, it is a natural question to ask whether or not one has a specialization morphism for the pro-etale fundamental group which would allow one to import the richer family of geometric coverings of X_k to interesting covering spaces of X_K.

Question 5.8. *Let $X \to \operatorname{Spec}(\mathcal{O})$ be a flat proper morphism. Does there exist a continuous* specialization morphism $\operatorname{sp}\colon \pi_1^{\text{proét}}(X_K) \to \pi_1^{\text{proét}}(X_k)$ *making the diagram*

$$\begin{array}{ccc} \pi_1^{\text{proét}}(X_K) & \overset{\operatorname{sp}}{\dashrightarrow} & \pi_1^{\text{proét}}(X_k) \\ \downarrow & & \downarrow \\ \pi_1^{\text{ét}}(X_K) & \underset{\operatorname{sp}}{\longrightarrow} & \pi_1^{\text{ét}}(X_k) \end{array}$$

commute?

The answer to this question is a resounding *no*, even for purely group theoretic reasons.

Example 5.9. Let $X \to \mathrm{Spec}(\mathcal{O})$ be as in Example 5.7. On the one hand, from Example 4.4 we know that $\pi_1^{\text{proét}}(X_K) = \pi_1^{\text{ét}}(X_K) = \widehat{\mathbf{Z}}^2$. On the other hand, from Example 4.9 we know that $\pi_1^{\text{proét}}(X_k) = \mathbf{Z}$ and $\pi_1^{\text{ét}}(X_k) = \widehat{\mathbf{Z}}$. Thus, we'd be be looking for an arrow making the following diagram commute

$$\begin{array}{ccc} \widehat{\mathbf{Z}}^2 & \dashrightarrow & \mathbf{Z} \\ \wr\downarrow & & \downarrow \\ \widehat{\mathbf{Z}}^2 & \twoheadrightarrow & \widehat{\mathbf{Z}} \end{array}.$$

But, this is clearly impossible as one of the compositions is surjective and the other cannot be.

Inspecting the proof of Theorem 5.4, $\mathbf{\acute{E}t}_{X_k}$ is still equivalent to $\mathbf{\acute{E}t}_{\mathfrak{X}}$ and so any geometric covering $Y_k \to X_k$ must deform uniquely to an étale morphism $\mathfrak{Y} \to \mathfrak{X}$. The issue is that one can no longer apply formal GAGA to obtain algebraization as the covering $\mathfrak{Y} \to \mathfrak{X}$ is infinite degree. In fact, this precisely underlies the issue highlighted in Example 5.9.

Example 5.10. Let $X \to \mathrm{Spec}(\mathcal{O})$ be as in Example 5.7, and let $Y_0 \to X_k$ to be the $\mathbf{Z}$-cover from Example 4.2. Then, one may uniquely deform this geometric covering to an étale morphism $\mathfrak{Y} \to \mathfrak{X}$. In fact, $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is precisely the non-algebraizable Tate uniformization map $\mathbf{G}_{m,K}^{\text{an}} \to X_K^{\text{an}}$.[10]

Following the hint provided by Example 5.10 we may instead turn our focus away from finding a specialization functor $\mathbf{Cov}_{X_k}$ $\mathbf{Cov}_{X_K}$, which cannot exist, to considering the functor $\mathbf{Cov}_{X_k} \to \mathbf{\acute{E}t}_{X_K^{\text{an}}}$ obtained by sending $Y_k \to X_k$ to the rigid generic fiber of its unique étale formal deformation $\mathfrak{Y} \to \mathfrak{X}$. But, as $\mathbf{\acute{E}t}_{X_K^{\text{an}}}$ is not a tame infinite Galois category this cannot be used as a target to get a specialization homomorphism of fundamental groups. Thus, we need to place the image of this functor in a smaller tame infinite Galois subcategory of $\mathbf{\acute{E}t}_{X_K^{\text{an}}}$. The surprising fact is that one may take the category $\mathbf{Cov}^{\text{oc}}_{X_K^{\text{an}}}$.

More generally, let us fix $\mathfrak{X}$ to be an admissible quasi-paracompact formal scheme over $\mathcal{O}$ (e.g. the ϖ-adic completion of a flat finite type $\mathcal{O}$-scheme). Then we have the following.

Theorem 5.11 ([2, Corollary 3.8])**.** *Let $\mathfrak{Y} \to \mathfrak{X}$ be an étale map. Then, the following are equivalent:*

(a) $\mathfrak{Y}_k \to \mathfrak{X}_k$ is a geometric covering,
(b) $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is a de Jong covering space.

In particular, there exists a continuous *specialization* homomorphism of Noohi groups

$$\mathrm{sp}\colon \pi_1^{\text{dJ}}(\mathfrak{X}_\eta) \to \pi_1^{\text{proét}}(\mathfrak{X}_k).$$

making the diagram

$$\begin{array}{ccc} \pi_1^{\text{dJ}}(\mathfrak{X}_\eta) & \xrightarrow{\mathrm{sp}} & \pi_1^{\text{proét}}(\mathfrak{X}_k) \\ \downarrow & & \downarrow \\ \pi_1^{\text{ét}}(\mathfrak{X}_\eta) & \xrightarrow[\mathrm{sp}]{} & \pi_1^{\text{ét}}(\mathfrak{X}_k) \end{array}$$

commute. Note that if $\mathfrak{X}$ is the ϖ-adic completion of a proper morphism $X \to \mathrm{Spec}(\mathcal{O})$ then the finite étale covers of $\mathfrak{X}_\eta = X_K^{\text{an}}$ agree with those of X by rigid analytic GAGA (e.g. see [6, Corollary

[10] As this is plausible from Example 2.5, which shows that $\mathbf{G}_{m,K}^{\text{an}}$ has a formal model with special fiber Y_0, this claim is not exactly clear. One way to confirm its veracity is to use Example 5.12 to show that $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is a topological $\mathbf{Z}$-cover, all of which are isomorphic to the Tate uniformization morphism.

3.4.13]) and so $\pi_1^{\text{ét}}(\mathfrak{X}_\eta) = \pi_1^{\text{ét}}(X_K)$ thus yielding a commutative diagram of Noohi groups

$$\begin{array}{ccc} \pi_1^{\text{dJ}}(X_K^{\text{an}}) & \xrightarrow{\text{sp}} & \pi_1^{\text{proét}}(X_k) \\ \downarrow & & \downarrow \\ \pi_1^{\text{ét}}(X_\eta) & \xrightarrow[\text{sp}]{} & \pi_1^{\text{ét}}(X_k) \end{array}$$

thus fully addresing the situation suggested by Question 5.8.

Outline of the proof of Theorem 5.11. The reason Theorem 5.11 should be surprising is that while Example 4.5 shows that a geometric covering $\mathfrak{Y}_k \to \mathfrak{X}_k$ need not split into a disjoint union of finite étale covers even étale locally on $\mathfrak{X}_k$, we see that $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ must split into a disjoint union of finite étale covers not just étale locally on $\mathfrak{X}_\eta$, not just admissibe locally on $\mathfrak{X}_\eta$, but *overconvergent open locally* on $\mathfrak{X}_\eta$.

The reason that this result is not entirely implausible is that the topology on $\mathfrak{X}_\eta$ is much more flexible than that of $\mathfrak{X}$, allowing one to replace $\mathfrak{X}$ by an admissible blowup. This motivates the proof which, in some sense, shows that one can split a geometric covering $\mathfrak{Y}_k \to \mathfrak{X}_k$ into a disjoint union of finite étale covers in a particularly simple way using admissible blowups of $\mathfrak{X}$.

We now outline the major steps to the proof of Theorem 5.11. We focus on the most difficult part, showing that if $\mathfrak{Y}_k \to \mathfrak{X}_k$ is a geometric covering, then $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is a de Jong covering space when $\mathfrak{X}$ is quasi-compact.

Outline of proof of Theorem 5.11.

Step 1: Assume that for each irreducible component Z of $\mathfrak{X}_k$ that $\mathfrak{Y} \times_{\mathfrak{X}} Z$ is in $\mathbf{UFÉt}_Z$. Now,

$$\mathfrak{Y} \times_{\mathfrak{X}} Z = (\mathfrak{Y} \times_{\mathfrak{X}} \widehat{\mathfrak{X}}_Z) \times_{\widehat{\mathfrak{X}}_Z} Z \in \mathbf{UFÉt}_Z .$$

As the morphism $Z \to \widehat{\mathfrak{X}}_Z$ is a universal homeomorphism, we know by the topological equivalence of the étale site that $\mathfrak{Y} \times_{\mathfrak{X}} \widehat{\mathfrak{X}}_Z$ is in $\mathbf{UFÉt}_{\widehat{\mathfrak{X}}_Z}$. Thus we see that $\mathfrak{Y}_\eta$ is a disjoint union of finite étale coverings over each $(\widehat{\mathfrak{X}}_Z)_\eta = T(\mathfrak{X}|Z)$. As the tubes $T(\mathfrak{X}|Z)$ as Z varies over the irreducible components of $\mathfrak{X}_k$ form an overconvergent open cover of $\mathfrak{X}_\eta$, we are done.

Step 2: Suppose that there is an admissible blowup $\mathfrak{X}' \to \mathfrak{X}$ such that for each irreducible component Z' in $\mathfrak{X}'_k$ with image Z in $\mathfrak{X}$, the map $Z' \to Z$ factorizes through the normalization $\widetilde{Z} \to Z$. Set $\mathfrak{Y}' = \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ and observe that as

$$\mathfrak{Y}' \times_{\mathfrak{X}'} Z' = (\mathfrak{Y}' \times_{\mathfrak{X}} \widetilde{Z}) \times_{\widetilde{Z}} Z'$$

we have that $\mathfrak{Y}'_{Z'}$ is in $\mathbf{UFÉt}_{Z'}$ by Example 4.4. By Step 1 we know that $\mathfrak{Y}'_\eta \to \mathfrak{X}'_\eta$ is a de Jong covering space. As $\mathfrak{X}' \to \mathfrak{X}$ is an admissible blowup, $\mathfrak{X}'_\eta \to \mathfrak{X}_\eta$ is an isomorphism, so we're done.

Step 3: It remains to show that there is an admissible blowup $\mathfrak{X}' \to \mathfrak{X}$ as in Step 2. To show this one first observes by Raynaud–Gruson that as the normalization $\widetilde{Z}_i \to Z_i$ is proper and birational it may in fact be dominated by the special fiber of an admissible blowup $\mathfrak{X}' \to \mathfrak{X}$. Of course, one must be careful as the admissible blowup $\mathfrak{X}' \to \mathfrak{X}$ may have more irreducible components than those obtained as the strict transform of the Z_i. Thus, one must iterate this procedure and perform a delicate analysis to show that it terminates in finite time. □

The above is illustrated quite well in the situation of Example 5.10.

Example 5.12. Let $\mathfrak{X}' \to \mathfrak{X}$ be the admissible blowup of $\mathfrak{X}$ at the nodal point of $\mathfrak{X}(k)$. Then, $\mathfrak{X}'_k = Y_2$ (in the notation of Example 4.2) and in particular we see that $\mathfrak{X}' \to \mathfrak{X}$ has the desired property from Step 2 of the above proof outline. Let us write the irreducible components of $\mathfrak{X}'_k$ as

Z_1' and Z_2', both of which are isomorphic to $\mathbf{P}^1_k$. As $\pi_1^{\text{ét}}(\mathbf{P}^1_k)$ is trivial we see that Y_0 pulled back to each Z_i' is a disjoint union of isomorphisms. Thus, from the above proof outline we see that $\mathfrak{X}$ has an admissible open cover $\mathfrak{X} = \mathrm{sp}_{\mathfrak{X}'}^{-1}(Z_1') \cup \mathrm{sp}_{\mathfrak{X}'}^{-1}(Z_2')$ over which $\mathfrak{Y}_\eta$ becomes a disjoint union of isomorphisms. In particular, $\mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is a topological covering (compare with Footnote 10).

6. Geometric coverings of rigid spaces and future work

Theorem 5.11 indicates an intimate connection between geometric coverings of schemes and de Jong coverings of rigid K-spaces. But, to say that a fibered subcategory $\mathcal{C}$ of the stack $\mathbf{\acute{E}t} \to \mathbf{Rig}_K$ is the 'correct' analogue of the stack of geometric coverings, one would like:

- for all connected X, $(\mathcal{C}_X, F_{\overline{x}})$ is a tame infinite Galois category
- $L_{\text{ét}}\mathcal{C} = \mathcal{C}$,
- if $Z \to Y$ is in $\mathcal{C}_Y$ and $Y \to X$ is in $\mathcal{C}_X$ then $Z \to Y \to X$ is in $\mathcal{C}_X$,
- for any X there exists a suitable 'pro-étale like site' $X_?$ such that $\mathcal{C}_X = \mathbf{Loc}(X_?)$.

The fibered category $\mathbf{UCov}^{\text{oc}}$ of de Jong covering spaces satisfies only this first property. Most seriously is the fact that being a de Jong covering space is not admissible open local on the target.

Example 6.1 ([3, §2.1])**.** If K is of equal characteristic $p > 0$, then there exists an example of a morphism $Y \to X$ which is not a de Jong covering space, but which is so admissibly locally on X. In short, X is the annulus $A_{\varpi^{-1},\varpi}$ from Example 2.5 covered by the two annuli

$$U^- = \{x \in A_{\varpi^{-1},\varpi} : |\varpi| \leqslant |x| \leqslant 1\} \quad \text{and} \quad U^+ = \{x \in A_{\varpi^{-1},\varpi} : 1 \leqslant |x| \leqslant |\varpi|^{-1}\},$$

intersecting along the unit circle

$$U^+ \cap U^- = C = \{x \in A_{\varpi^{-1},\varpi} : |x| = 1\}.$$

The restriction of $Y \to X$ to $U^\pm$ is a disjoint union of well-chosen Artin–Schreier coverings $Y_n^\pm$ ($n \in \mathbf{Z}$) which are split over C, and $\coprod_n Y_n^-$ and $\coprod_n Y_n^+$ are identified suitably over C.

Given the definition of geometric coverings, it's natural to guess that for $\mathcal{C}_X$ one can take the category of étale and partially proper morphisms $Y \to X$. Unfortunately such a definition is useless.

Example 6.2. For any rigid K-space X the inclusion $U \to X$ of any overconvergent open subset is étale and partially proper. As a concrete example of this, the inclusion $\mathbf{D}_K \to \mathbf{B}_K$ of the open unit disk into the closed unit disk is étale and partially proper.

Ultimately the reason for such examples is that valuative rigid K-spaces have large universal separated quotients $[X]$ in stark contrast to locally topologically Noetherian schemes. Specifically the notion of partially proper is only concerned with liftings specializations in the Riemann–Zariski-esque spaces $\mathrm{sep}_X^{-1}(\mathrm{sep}_X(x))$, and completely ignores specializations that happen in $[X]$. Thus, it intuitively makes sense that to fix this one should add a sort of 'valuative criterion for $[X]$'.

Definition 6.3 ([1, Definition 5.2.2])**.** A morphism of rigid K-spaces $Y \to X$ is called a *geometric covering* if it is étale, partially proper, and satisfies the following valuative criterion: for all smooth and separated rigid L-curves C, where L is a non-archimedean extension of K, and all morphisms $C \to X_L$, any embedding $i\colon [0,1] \to [C]$ and lift of $[0,1) \to [Y_C]$ along $[f_C]$ can uniquely be extended to a lift of i.[11] We denote by $\mathbf{Cov}_X$ the category of geometric coverings of X.

The role of arcs in this definition is perhaps not too surprising considering their large role in the proof of Theorem 3.7. This analogy can be made even stronger by studying an alternative characterization of geometric coverings in terms of the ability to 'uniquely lift geometric arcs' (see [1] for details).

[11] For the reason to introduce the curves C see [1, Remark 5.4.10]

The main results of [1] can be combined to show that the fibered category **Cov** satisfies the first three desirable conditions listed above. It is then a question of considerable interest as to whether the final desirable condition has an affirmative answer for $\mathbf{Cov}_X$.

Question 6.4. *Does there exist a 'pro-étale like topology' $X_?$ on X such that $\mathbf{Cov}_X = \mathbf{Loc}(X_?)$?*

One reasonable guess for such a topology is the pro-étale topology defined in [21]. But, let us define the fibered subcategory $\mathbf{Cov}^{\text{ét}}$ of $\mathbf{\acute{E}t}_X$ as $L_{\text{ét}}\,\mathbf{UF\acute{E}t}_X$. Then, we have the following.

Theorem 6.5 ([3, Theorem 4.4.1])**.** *For any X there is a natural equivalence of categories between $\mathbf{Loc}(X_{\text{proét}})$ and $\mathbf{Cov}_X^{\text{ét}}$.*

We suspect examples like Example 6.1 may be adapted to show that $\mathbf{Cov}_X^{\text{ét}}$ is strictly smaller than $\mathbf{Cov}_X$ in many cases, and thus Theorem 6.5 indicates that the pro-étale topology from [21] is not sufficient. In fact, I suspect a more likely option is that one may take for $X_?$ a modification of the v-topology on the diamond $X^\diamond$ as in [22].

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