Floer K-theory for knots

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1 Introduction and backgrounds

In this report, I will summarize a joint work with Jin Miyazawa and Masaki Taniguchi [17], where we established a version of Floer K-theory for knots. Our construction is based on Seiberg–Witten theory, more precisely Seiberg–Witten Floer stable homotopy type due to Manolescu [23]. First, we shall give a Seiberg–Witten Floer K-theory for 3-manifolds with involution, and finally it yields an interesting constraint on knots and surfaces bounded by them, through double branched coverings along them. Our way to take into account involutions to Seiberg–Witten theory follows Y. Kato's work for closed 4-manifolds [15].

Several authors have already studied effects of group actions in various types of Floer homology and their applications to knots, such as [8, 14, 10, 20, 21, 13, 12, 19, 11, 1, 22, 9, 6, 2, 3, 18, 5]. Compared with these works, a feature of our work is that we provided a 10/8-type inequality involving knots and surfaces bounded by them. Recall that Furuta's original 10/8-inequality [7] was proved by applying (a certain equivariant) K-theory to a finite-dimensional approximation of Seiberg–Witten equations, and the existence of a finite-dimensional approximation is based on the compactness of the moduli space for the Seiberg–Witten equations. Because of this, for now there is no known alternative proof of the 10/8-inequality by other gauge-theoretic methods, such as Yang–Mills theory or Heegaard Floer homology, and this is one of big advantages of Seiberg–Witten theory rather than other gauge theories for now.

There are two generalizations or variants of Furuta's 10/8-inequality relevant to us. The first one is due to Manolescu [24]: he generalized Furuta's 10/8-inequality for spin 4-manifolds with boundary, called the relative 10/8-inequality. The second is due to Y. Kato [15] mentioned above. Kato gave a 10/8-type constraint on smooth involutions on closed spin 4-manifolds, in a different way from usual equivariant theory. Roughly, our framework is a hybrid of Manolescu' and Kato' constructions. We shall give a constraint on smooth involutions on spin 4-manifolds with boundary, which we call the *relative 10/8-inequality for involutions*, and this ends up with a strong constraint on knots in the smooth category.

2 Relative 10/8-inequality for involutions

Now we describe the precise statement of the relative 10/8-inequality for involutions. Let us consider a triple $(Y, \mathfrak{t}, \iota)$, where

- Y is an oriented rational homology 3-sphere,
- $\mathfrak{t}$ is a spin structure on Y, and
- ι is an orientation-preserving smooth involution whose fixed-point set is non-empty and of codimension-2. Suppose that ι preserves the spin structure $\mathfrak{t}$.

In [17], we defined a numerical invariant

$$\kappa(Y, \mathfrak{t}, \iota) \in \frac{1}{16}\mathbb{Z}.$$

This is a variant (taking effects of ι into account) of the K-theoretic Frøyshov-type invariant $\kappa(Y, \mathfrak{t})$ defined by Manolescu [24], which was used in [24] to give his relative version of Furuta's 10/8-inequality [7]. Let $b^+_\iota(W)$ denote the maximal dimension of ι-invariant positive-definite subspaces of $H_2(W; \mathbb{R})$. The following is the first main theorem of [17]: the property (iv) is the main statement, which is our relative 10/8-inequality for involutions.

Theorem 2.1 ([17, Theorem 1.1]) *The invariant $\kappa(Y, \mathfrak{t}, \iota) \in \frac{1}{16}\mathbb{Z}$ satisfies the following properties:*

(i) The mod 2 reduction of $-2\kappa(Y, \mathfrak{t}, \iota)$ coincides with the Rokhlin invariant $\mu(Y, \mathfrak{t})$:

$$-2\kappa(Y, \mathfrak{t}, \iota) = \mu(Y, \mathfrak{t})$$

in $\left(\frac{1}{8}\mathbb{Z}\right)/2\mathbb{Z} \cong \mathbb{Z}/16\mathbb{Z}$.

(ii) The quantity $\kappa(Y, \mathfrak{t}, \iota)$ is invariant under conjugation: for every diffeomorphism f on Y preserving the orientation and the spin structure $\mathfrak{t}$, we have

$$\kappa(Y, \mathfrak{t}, \iota) = \kappa(Y, \mathfrak{t}, f^{-1} \circ \iota \circ f).$$

(iii) For $-Y$, the same manifold with the reversed orientation, we have

$$\kappa(Y, \mathfrak{t}, \iota) + \kappa(-Y, \mathfrak{t}, \iota) \geq 0.$$

(iv) Let $(W, \mathfrak{s}, \iota)$ be a smooth spin cobordism with involution from $(Y_0, \mathfrak{t}_0, \iota_0)$ to $(Y_1, \mathfrak{t}_1, \iota_1)$ with $b_1(W) = 0$. Then we have

$$-\frac{\sigma(W)}{16} + \kappa(Y_0, \mathfrak{t}_0, \iota_0) \leq b^+(W) - b^+_\iota(W) + \kappa(Y_1, \mathfrak{t}_1, \iota_1). \qquad (1)$$

The inequality (1) recovers Kato's 10/8-type inequality for involutions on closed spin 4-manifolds [15].

3 10/8-inequality for knots

From Theorem 2.1, we may extract a constraint on surfaces bounded by knots through double branched coverings, summarized as follows. Throughout we consider oriented knots in S^3. For a given knot K, one can associate an oriented rational homology 3-sphere $\Sigma(K)$ called the branched covering space along K, which is equipped with the covering involution ι_K. Set

$$\kappa(K) := \kappa(\Sigma(K), \mathfrak{t}, \iota_K) \in \frac{1}{16}\mathbb{Z},$$

where $\mathfrak{t}$ is the unique spin structure on $\Sigma(K)$. The following theorem is obtained by translating Theorem 2.1 in terms of knots and surfaces through double branched coverings:

Theorem 3.1 ([17, Theorem 1.4]) *The invariant $\kappa(K)$ satisfies the following properties:*

(i) The invariant $\kappa(K)$ is a knot concordance invariant with $\kappa(U) = 0$, where U is the unknot.

(ii) For every knot K, we have $\kappa(K) = \kappa(-K)$, where $-K$ is the knot with the opposite orientation.

(iii) For every knot K in S^3, we have

$$2\kappa(K) = -\frac{1}{8}\sigma(K)$$

in $\left(\frac{1}{8}\mathbb{Z}\right)/2\mathbb{Z} \cong \mathbb{Z}/16\mathbb{Z}$.

(iv) For every knot K in S^3, we have $\kappa(K) + \kappa(K^*) \geq 0$, where K^* denotes the mirror image of K.

(v) Let K and K' be knots in S^3, X be an oriented smooth compact connected cobordism from S^3 to S^3 with $H_1(X;\mathbb{Z}) = 0$, and S be an oriented compact connected properly and smoothly embedded cobordism in X from K to K' such that the homology class $[S]$ of S is divisible by 2 and $PD(w_2(X)) = [S]/2$ mod 2. Then, we have

$$-\frac{\sigma(X)}{8} + \frac{9}{32}[S]^2 - \frac{9}{16}\sigma(K') + \frac{9}{16}\sigma(K) \leq b^+(X) + g(S) + \kappa(K') - \kappa(K), \quad (2)$$

where $\sigma(K)$ denotes the signature of K and $g(S)$ is the genus of S.

We call the inequality (2) the *10/8-inequality for knots*, which is a consequence of the relative 10/8-inequality for involutions (1).

The invariant $\kappa(K)$ can be computed to some extent: for two-bridge knots and (many of) torus knots and their connected sums, we can compute the κ-invariant [17, Theorem 1.8, Theorem 1.9, Theorem 1.10], and we have a crossing change formula for the κ-invariant [17, Theorem 1.11].

The 10/8-inequality for knots effectively extracts discrepancy of 4-dimensional aspects of knot theory in the smooth and topological categories. We shall see two major applications in that direction below.

4 First application: stabilizing numbers

Now we describe the first application of the 10/8-inequality for knots (Theorem 3.1), which is about a 4-dimensional knot invariant called the *stabilizing number*. We say that a knot K is (smoothly) *H-slice* in a closed smooth oriented 4-manifold X if K bounds a smoothly and properly embedded disk which is null homologous in $X \setminus \mathrm{int}D^4$. It is proven in [16, page 84, Corollary 5.11] and [26] that, for every knot K in S^3 whose Arf invariant $\mathrm{Arf}(K)$ is zero, there is a positive integer N such that K is smoothly H-slice in $\#_N S^2 \times S^2$, the N-fold connected sum of copies of $S^2 \times S^2$. This result enables us to define the following invariants: let $\mathrm{sn}(K)$ be the minimum of natural numbers N for which K is smoothly H-slice in $\#_N S^2 \times S^2$. Similarly, considering locally flat topological embeddings instead of smooth embeddings, we may define a topological version of this quantity, which we denote by $\mathrm{sn}^{\mathrm{Top}}(K)$. These invariants $\mathrm{sn}(K)$ and $\mathrm{sn}^{\mathrm{Top}}(K)$ are called the *smooth and topological stabilizing numbers* of K, respectively.

Recently, Conway and Nagel asked the following question [4, Question 1.4]: whether there exists a knot K with $\mathrm{Arf}(K) = 0$ such that

$$0 < \mathrm{sn}^{\mathrm{Top}}(K) < \mathrm{sn}(K).$$

We give the affirmative answer to this question, and more:

Theorem 4.1 ([17, Theorem 1.13]) *There exists a knot K in S^3 with $\mathrm{Arf}(K) = 0$ that satisfy the following properties:*

- *We have*
$$0 < \mathrm{sn}^{\mathrm{Top}}(K) < \mathrm{sn}(K).$$
- *We have $0 < \mathrm{sn}(\#_n K)$ for all positive integers n, and*
$$\lim_{n\to\infty} \left(\mathrm{sn}(\#_n K) - \mathrm{sn}^{\mathrm{Top}}(\#_n K)\right) = \infty.$$

For example, we can take K in Theorem 4.1 to be $T(3, 11)$. Moreover, in fact, we may detect many examples of K which satisfy the statement of Theorem 4.1.

5 Second application: relative genus bounds

Next application of the 10/8-inequality for knots (Theorem 3.1) is about relative genus bounds. Let X be an oriented smooth closed 4-manifold with a second homology class $x \in H_2(X;\mathbb{Z})$. For a knot K in S^3, let $g_{X,x}(K)$ be the minimum of genera of surfaces S which are properly and smoothly embedded oriented connected compact surfaces in $X \setminus \mathrm{int}D^4$ such that $\partial S = K$ and $[S] = x \in H_2(X;\mathbb{Z})$. Here $[S]$ denotes the relative fundamental class of S, and $H_2(X;\mathbb{Z})$ is naturally identified with $H_2(X \setminus \mathrm{int}D^4;\mathbb{Z})$. This quantity $g_{X,x}(K)$ is called the *smooth (X, x)-genus* of K, and the topological version $g^{\mathrm{Top}}_{X,x}(K)$ is defined by considering locally flat embeddings instead of smooth embeddings.

It is known that the smooth X-genus and topological X-genus have "big" difference: For a definite 4-manifold X and and every $x \in H_2(X;\mathbb{Z})$, using the Ozváth–Szabó τ-invariant [25], one may find a sequence of knots $\{K_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty}\left(g_{X,x}(K_n) - g^{\mathrm{Top}}_{X,x}(K_n)\right) = \infty.$$

The following result detects such difference for 4-manifold without assuming the definiteness, and proves more:

Theorem 5.1 ([17, Theorem 1.14]) *There exists a knot K' that satisfies the following property. Let X be a smooth closed oriented 4-manifold with $H_1(X;\mathbb{Z}) = 0$ and $x \in H_2(X;\mathbb{Z})$ be a second homology class which is divisible by 2 and satisfies $x/2 = PD(w_2(X))$ mod 2. Then, for every knot K in S^3, we have*

$$\lim_{n\to\infty}\left(g_{X,x}(K\#(\#_n K')) - g^{\mathrm{Top}}_{X,x}(K\#(\#_n K'))\right) = \infty.$$

Concretely, we can take K' to be, for example, $K' = T(3,11)$.

6 Floer homotopy type and Floer K-theory

The all results explained until here are derived from the relative 10/8-inequality for involutions (Theorem 2.1). The main ingredients to establish Theorem 2.1 are versions of Floer homotopy type and of Floer K-theory. We shall describe the points below.

Based upon Manolescu's construction of Seiberg–Witten Floer stable homotopy type [23], in [17] we constructed versions of *Seiberg–Witten Floer stable homotopy type for involutions*

$$DSWF_G(Y,\mathfrak{t},\iota)$$

and of *Seiberg–Witten Floer K-theory for involutions*

$$DSWFK_G(Y,\mathfrak{t},\iota),$$

which are defined for a spin rational homology 3-sphere $(Y,\mathfrak{t})$ with an involution ι which preserves $\mathfrak{t}$ and whose fixed-point set is of codimension-2. Here G stands for $G = \mathbb{Z}_4$, which is the subgroup of $\mathrm{Pin}(2)(= S^1 \cup jS^1 \subset \mathbb{H})$ generated by $j \in \mathrm{Pin}(2)$, and D stands for a "doubling" construction, which was made to avoid complications in K-theory.

Via double branched covers, we obtain the *Seiberg–Witten Floer stable homotopy type of a knot K in S^3*

$$DSWF(K)$$

and the *Seiberg–Witten Floer K-theory of a knot K*

$$DSWFK(K).$$

To take into account ι in the construction of $DSWF_G(Y,\mathfrak{t},\iota)$, we used an involutive symmetry on the Seiberg–Witten equations introduced by Kato [15]. Kato's symmetry is quite different from symmetries considered in usual equivariant theory: for example, Kato's involutive symmetry acts on spinors *anti-complex linearly*. A significant effect of this symmetry reflects also in the right-hand side of the relative 10/8 inequality (1): the term $b^+(W) - b^+_\iota(W)$. This does not appear in the usual equivariant theory, and is crucial in the applications to knots explained in Sections 4 and 5

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