Trichotomy of recent progress in Iwasawa theory of knots and links

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Dedicated to the memory of Professor Toshie Takata

1 Introduction

The analogy between low dimensional topology and number theory has played an important role for 200 years. In modern days, the analogy between knots and prime numbers was initially pointed out by B. Mazur in [Maz64] and has been systematically developed by M. Kapranov, A. Reznikov, M. Morishita, M. Kim [Kap95, Rez97, Rez00, Mor02, Mor12, Kim20], and others. Here is a basic part of their dictionary;

Low dimensional topology	Number theory
oriented, connected, closed 3-manifold M	number field k (the ring of integers $\operatorname{Spec}\mathcal{O}_k$)
knot $K : S^1 \hookrightarrow M$ link $L : \sqcup S^1 \hookrightarrow M$	prime ideal $\mathfrak{p} : \operatorname{Spec}\mathbb{F}_\mathfrak{p} \hookrightarrow \operatorname{Spec}\mathcal{O}_k$ family of primes $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$
(branched/unbranched) cover $h : N \to M$ fundamental group $\pi_1(M)$ $\pi_1(M - L)$	(ramified/unramified) extension F/k étale fundamental group $\pi_1^{\text{ét}}(\operatorname{Spec}\mathcal{O}_k)$ $\pi_1^{\text{ét}}(\operatorname{Spec}\mathcal{O}_k - S)$
theory of branched coverings	Hilbert ramification theory
1-cycle group $Z_1(M)$ 1-boundary group $B_1(M)$ $\partial : C_2(M) \to Z_1(M); s \mapsto \partial s$ $H_1(M) = Z_1(M)/B_1(M)$ assumption: $\#H_1(M) < \infty$ (M:$\mathbb{Q}$HS^3)	ideal group I_k principal ideal group P_k $(\ ) : k^\times \to I_k; a \mapsto (a)$ ideal class group $\mathrm{Cl}(k) = I_k/P_k$ fact: $\#\,\mathrm{Cl}(k) < \infty$
Hurewicz isomorphism $\pi_1(M)^{\mathrm{ab}} \cong \mathrm{Gal}(M_{\mathrm{ab}}/M) \cong H_1(M)$	Artin reciprocity law $\pi_1^{\text{ét}}(\overline{\operatorname{Spec}\mathcal{O}_k})^{\mathrm{ab}} \cong \mathrm{Gal}(k_{\mathrm{ab}}^{\mathrm{ur}}/k) \cong \mathrm{Cl}(k)$
Alexander–Fox theory	Iwasawa theory
$\mathbb{Z}$-cover of $S^3 - K$	cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$
Mahler measure of the Alexander polynomial	Iwasawa invariants of the Iwasawa polynomial

Let p be a prime number. The author has been involved in p-adic refinements of the Alexander–Fox theory, say, Iwasawa theory of $\mathbb{Z}_p$-covers, and its connection with the study of profinite rigidity (cf.[Uek17, Uek16, Uek20, Uek18b, Uek21, Uek22, Uek18a], see also [HMM06, KM08, KM13, DR22]).

In this survey article, we briefly overview three new progress in Iwasawa theory of knots and links due to the authors [TU22a, UY22, TU22b], and attach several questions.

2 Twisted Iwasawa invariants of knots [TU22a]

In the number theory side, Iwasawa theory of representations is studied by Greenberg [Gre89] and others. Here we study Iwasawa theory of knot group representations.

Let K be a knot in S^3 with $\pi_K = \pi_1(S^3 - K)$ and let $X_n \to X = S^3 - K$ denote the $\mathbb{Z}/n\mathbb{Z}$-cover for each $n \in \mathbb{Z}_{>0}$. Let p be a prime number and let $m \in \mathbb{Z}$ with $p \nmid m$. Let $\rho : \pi_K \to \mathrm{GL}_N(O_{\mathfrak{p}})$ be a representation over a finite extension $O_{\mathfrak{p}}$ of the p-adic integers $\mathbb{Z}_p$ and let $\Delta_\rho(t)$ denote the twisted Alexander polynomial of ρ. Then, the p-adic Weierstrass preparation theorem assures that there exist unique λ, μ, $\nu \in \mathbb{Z}$ satisfying $\mathrm{Nr}\Delta_\rho(T) \doteq p^\mu(\lambda + p(\text{lower terms}))$ in $\mathbb{Z}_p[[T]]$. Then a standard argument of Iwasawa theory and a generalization of Fox–Weber's formula (cf.[Tan18, Uek22]) yields the following.

Theorem 2.1. *Let (K, p, m, ρ) and $\lambda, \mu, \nu \in \mathbb{Z}$ be as above. Then for any $n \gg 0$, $|H_1(X_{mp^n}, \rho)_{\mathrm{tor}}| = p^{\lambda n + \mu p^n + \nu}$ holds.*

We remark that $\mu = 0$ is often a big theorem or a leading conjecture in number theory. In the knot theory side, the author pointed out that μ may be interpreted as Bowen's p-adic entropy of the Alexander module.

In addition, λ is an analogue of the genus of a Riemann surface in view of the analogy between the Riemann–Hurwitz formula and Kida's formula. Thus it may be a natural question to ask if λ is related to the genus of a knot.

The following is a translation of Friedl–Vidussi's deep results [FV11, FV13, FV15] into twisted Iwasawa invariants.

Theorem 2.2. (1) *For each (K, p, ρ), there exists some m such that $\lambda/[O_{\mathfrak{p}} : \mathbb{Z}_p] = \deg\Delta_\rho(t)$. Hence for each K, there exists some (p, ρ, m) such that λ coincides with the genus of K.*

(2) *For each (p, K, ρ), μ's and λ's determine whether $\Delta_\rho(t)$ is monic in $O_{\mathfrak{p}}[t]$ and whether K is fibered.*

Examples 2.3. (1) The λ's of the lifts $\rho^{\pm}_{\mathrm{hol}} : \pi_K \to \mathrm{SL}_2(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$ of the holonomy representation of the figure eight knot $K = 4_1$; ρ^{+}_{hol} has $\Delta_{\rho+,1}(1+T) = T^2 + 6T + 6$ and hence $\lambda_2/2 = \lambda_3/2 = 2$, $\lambda_p = 0$ $(p \neq 2, 3)$, while ρ^{-}_{hol} has $\Delta_{\rho-,1}(1+T) = T^2 - 2T + 2$ and hence $\lambda_2/2 = 2$, $\lambda_p = 0$ $(p \neq 2)$. Since each lift corresponds to a spin structure, we may say that λ_p's distinguish the spin structures of $K = 4_1$.

(2) For any SL_2-representations of the twist knots $J(2, 2k)$ $(k \in \mathbb{Z})$, we have $\mu = 0$. We may expect that if $k \neq 0, \pm 1$, then there exists some ρ of $J(2, 2k)$ with $\mu > 0$.

Question 2.4. Find a representation ρ of a non-fibered knot group with $\mu > 0$.

Friedl–Vidussi asserts that a knot K is non-fibered if and only if there exists a representation $\overline{\rho} : \pi_K \to \mathrm{GL}_N(\mathbb{F}_p)$ with $\Delta_{\overline{\rho}}(t) = 0$. It seems that if we find such $\overline{\rho}$ and take a generic lift, then we obtain such ρ with $\mu > 0$.

Question 2.5. Study what λ's and μ's of (lifts of) the holonomy representations of hyperbolic knots know (cf.[DFJ12]).

3 Weber's class number problem for knots [UY22]

Weber's class number problem for number fields has been of great importance with quite little solved for 200 years. It is conjectured that the class number of cyclotomic $\mathbb{Z}/p^n\mathbb{Z}$-extension of $\mathbb{Q}$ is always 1, but proved only for $(p,n) = (2, \leq 6), (3, \leq 3), (5, \leq 2), (\geq 7, 1)$. Note that the branch locus of $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is (p). In the knot theory side, we may choose the extension degree p and the branch locus K independently, so we may consider several analogues of Weber's class number problem. One answer is established by Livingston;

Theorem 3.1 (Livingston [Liv02, Theorem 1.2])**.** *Let K be a knot in S^3. Then, the equality $|H_1(X_{p^n})_{\mathrm{tor}}| = 1$ holds for every prime number p and positive integer n if and only if every non-trivial factor of the Alexander polynomial $\Delta_K(t)$ is the m-th cyclotomic polynomial with m being divisible by at least three distinct prime numbers.*

The fourth author Yoshizaki [Yos20] recently gave a new approach to Weber's problem with use of continued fraction expansion and pointed out that the sequence of the class numbers in the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$ converges in the ring of 2-adic integers $\mathbb{Z}_2$. His result extends to any $\mathbb{Z}_p$-extension of any global field. In the knot theory side, we have the following.

Theorem 3.2. *Let K be a knot in S^3 and let $X_n \to X = S^3 - K$ denote the $\mathbb{Z}/n\mathbb{Z}$-cover for each $n \in \mathbb{Z}_{>0}$. Then, the sizes $|H_1(X_{p^n})_{\mathrm{tor}}|$ of the torsion subgroups of $H_1(X_{p^n})$ converges in $\mathbb{Z}_p$.*

This assertion extends to any $\mathbb{Z}_p$-cover over a compact 3-manifold X, namely, a compatible system $\{X_{p^n} \to X\}_n$ of $\mathbb{Z}/p^n\mathbb{Z}$-covers.

Another analogue of Weber's class number problem may be the following.

Question 3.3. For each p and K, study the p-adic limit of $(|H_1(X_{p^n})_{\mathrm{tor}}|)_n$. Find a condition of a prime number p and a knot K for which $\lim_{n\to\infty} |H_1(X_{p^n})| = 1$ holds.

Now let $\overline{\mathbb{Q}}$ be an algebraic closer of $\mathbb{Q}$, let $\mathbb{C}_p$ denote the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, and fixed an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Then the p-adic limit in a $\mathbb{Z}_p$-cover of a knot is given by the roots of unity that are close to the roots of the Alexander polynomial $\Delta_K(t)$ in $\overline{\mathbb{Q}} \subset \mathbb{C}_p$. Recall that for each $n \in \mathbb{Z}_{>0}$, the n-th cyclic resultant of $0 \neq f(t) \in \mathbb{Z}[t]$ is defined by the determinant of the Sylvester matrix, or equivalently, by $\mathrm{Res}(t^n - 1, f(t)) = \prod_{\zeta^n=1} f(\zeta)$, where ζ runs through n-th roots of unity. If $f(t) = a_0 \prod_i (t - \alpha_i)$, then $\mathrm{Res}(t^n - 1, f(t)) = a_0^n \prod (\alpha_i^n - 1)$ holds. The following theorem is purely algebraic.

Theorem 3.4. *Let $0 \neq f(t) \in \mathbb{Z}[t]$. Then,*

(1) *the p-power-th cyclic resultants* $\mathrm{Res}(t^{p^n} - 1, f(t))$ *converge in $\mathbb{Z}_p$.*

(2) *Let $f(t) = a_0 \prod_i (t - \alpha_i)$ in $\overline{\mathbb{Q}}[t]$ with $p \nmid f(1)$. For each i, let $\zeta_i \in \overline{\mathbb{Q}}$ and ξ denote the unique p-prime-th root of unity with $|\alpha_i - \zeta_i|_p < 1$ and $|a_0 - \xi|_p < 1$. Then*

$$\lim_{n\to\infty} \mathrm{Res}(t^{p^n} - 1, f(t)) = \xi \prod_i (\zeta_i - 1)$$

holds in $\mathbb{Z}_p$. If $p | f(1)$, then the limit equals to 0.

The assertion (1) is proved by using the Artin reciprocity law of global class field theory with modulus (cf.[Was97, Appendix §3, Theorem 1(i)]). Since Fox–Weber's formula [Web79] and Livingston's theorem [Liv02, Corollary 3.2] asserts that $|H_1(X_{p^n})_{\mathrm{tor}}| = \mathrm{Res}(t^{p^n}-1, \Delta_K(t))$, our theorem together with Apostol's calculation [Apo70] of resultants of cyclotomic polynomials yields the following.

Examples 3.5. Let (a, b) be a coprime pair of integers with $p \nmid b$ and write $a = p^m a'$ with $m, a' \in \mathbb{Z}$, $p \nmid a'$. If K is the torus knot $T_{a,b}$, then we have $\Delta_K = (1-t)(1-t^{ab})/(1-t^a)(1-t^b)$ and $\lim_{n\to\infty} |H_1(X_{p^n})_{\mathrm{tor}}| = b^{p^m-1}$ in $\mathbb{Z}_p$.

Examples 3.6. If $K = J(2, -2)$ is the figure-eight knot, then we have $\Delta_K(t) = -t^2+3t-1$ and

p	2	3	5	7
lim	-3	-2	-4	$\sqrt{2}-2$

,

where $\sqrt{2}-2 \in \mathbb{Z}_7$ is properly chosen.

If $K = J(2, 2m)$ $(m \in \mathbb{Z})$ is the twist knot with $p \mid m$, then by $\Delta_K(t) = mt^2 + (1-2m)t + m$, we obtain that $\lim_{n\to\infty} |H_1(X_{p^n})_{\mathrm{tor}}| = 1$ in $\mathbb{Z}_p$.

Question 3.7. Study p-adic refinements of [Liv02, Theorem 1.1] and [Kim09] on slice and concordance.

Question 3.8. Study the p-adic limits of other invariants in $\mathbb{Z}_p$-covers.

Modified versions of Theorem 3.4 hold if we replace $t^{p^n}-1$ by $(t^{p^n}-1)/\gcd(t^{p^n}-1, f(t))$. Since we have variants of Fox–Weber's formula for $\mathbb{Z}$-covers of links and for twisted homologies in $\mathbb{Z}$-covers, Theorem 3.4 applies to several other similar situations.

Theorem 3.4 on the p-adic limit of cyclic resultants is applicable to algebraic curves (function fields) as well. It may be interesting to compare covers of knots and extensions of function fields in various situations.

Recently Ozaki [Oza22] generalized Yoshizaki's p-adic convergence theorem to a general extension with a finitely generated pro-p Galois group by developing an analytic method to reveal the relationship amongst several arithmetic invariants; the class numbers, the ratios of p-adic regulators, the square roots of discriminants, and the order of algebraic K_2-groups of the ring of integers. Studing their analogues in the knot theory side would give a new cliff to extend the dictionary of arithmetic topology.

4 Iwasawa invariants of the ${\mathbb{Z}_p}^d$-covers of links [TU22b]

Let p be a prime number and let $d \in \mathbb{Z}_{>0}$. Cuoco–Monsky [CM81] gave variants of the Iwasawa class number formula for ${\mathbb{Z}_p}^d$-extensions of number fields and pointed out the existence of the term $O(1)$. An analogue of [CM81, Theorem I] is the following.

Theorem 4.1. *Let $L = \cup_i K_i$ be a d-component link in a rational homology 3-sphere M such that $[K_i] = 0$ in $H_1(M)$ holds for all i, and let $X_n \to X = M - L$ denote the standard ${\mathbb{Z}/n\mathbb{Z}}^d$-cover. Suppose that the Alexander polynomial $\Delta_L(t_1, \dots, t_d)$ of L satisfies $\Delta_L(\zeta, \zeta, \cdots, \zeta) \neq 0$ for every p-power-th roots ζ of unity. Then, there exist some*

$\lambda, \mu \in \mathbb{Z}_{\geq 0}$ such that for any $n \gg 0$, the size of p-torsion subgroup of $H_1(X_{p^n}, \mathbb{Z})$ is given by

$$|H_1(X_{p^n})_{(p)}| = p^{p^{(d-1)n}(\mu p^n + \lambda n + O(1))},$$

where $O(1)$ is the Bachmann–Landau notation.

If M is an integral homology 3-sphere, then the ${\mathbb{Z}_p}^d$-cover is Greenberg, namely, $O(1)$ is a constant. In addition, for $n \gg 0$, the exponent $p^{(d-1)n}(\mu p^n + \lambda n + O(1))$ is a polynomial with rational coefficients in n and p^n having degree ≤ 1 in n and total degree $\leq d$.

The condition $\Delta_L(\zeta, \zeta, \cdots, \zeta) \neq 0$ may be replaced by a weaker one, namely, $\Delta_L(t)$ has no special prime factor in the sense of Cuoco–Monsky [CM81]. This condition is related to the Betti numbers. By [Sak95, Theorem 7.5], we obtain the following.

Theorem 4.2. *$\Delta_L(t_1, \ldots, t_d)$ has no special prime factors if and only if the maximal degree $D(L)$ of polynomials giving the first Betti numbers satisfies $D(L) \leq d - 2$.*

The assertion of Theorem 4.1 extends to any $\mathbb{Z}_p^d$-cover (i.e., a compatible system of $\mathbb{Z}/p^n\mathbb{Z}$-covers) of a compact 3-manifold X such that the p-torsion subgroups $H_1(X_n)_{(p)}$ form a surjective system.

Sakuma [Sak79, Sak81], Mayberry–Murasugi [MM82], and Porti [Por04] gave variants of Fox–Weber's formula for links. By virtue of them, together with a multivariable analogue of p-adic Weierstrass preparation theorem [Mon81, CM81], we may calculate the invariants μ, λ and sometimes $O(1)$ as well, from the Alexander polynomial.

Examples 4.3. (1) If L is the Solomon's link 4_1^2, then we have $|H_1(X_n)_{(2)}| = 2^{(0 \cdot 2^n + 0 \cdot n + 2) \cdot 2^n}$ and $|H_1(X_n)_{(p)}| = 1$ if $p \neq 2$.

(2) If L is the twisted Whitehead link W_{2k-1} for $k = mp^l \in \mathbb{Z}$ with $p \nmid m$, then $\Delta_L(X, Y) = kXY$ and $\mu = l$.

(3) If $L = 6_1^2$ and $p = 3$, then $\lambda = 2$. If $L = 6_3^3$, then $\lambda = 1$ for any p. If $L = 8_4^2$, then $\lambda = 2$ for any p.

Thus, we have a link with $O(1) \neq 0$, a link with any $\mu \in \mathbb{Z}_{\geq 0}$, and a link with $\lambda \neq 0$.

Question 4.4. Find an example of $\mathbb{Z}_p^d$-cover such that $O(1) \neq$ constant (i.e., "non-Greenberg"). The existence of such $\mathbb{Z}_p^d$-extension is conjectured by Cuoco–Monsky [CM81].

Question 4.5. Study $\mathbb{Z}_p^{d'}$-covers of d-component links with $d' < d$. We may find some subtle phenomenon in this case, as pointed out in [MM82, Section 14].

Question 4.6. Study an analogue of [CM81, Theorem II], or something between [CM81, Theorem I] and [CM81, Theorem II].

5 Further problems

Question 5.1. Find any connection between Iwasawa invariants λ, μ and surgeries via Dijkgraaf–Witten invariants (cf. [Che17]).

Question 5.2. Find any feedback to number theory.

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