Properties of links from the viewpoint of R. Thompson's group F

Akihiro Takano
Graduate School of Mathematical Sciences, The University of Tokyo

1 Introduction

Recently, Jones [4] introduced a method of constructing links from elements of Thompson's group F. Moreover, he showed an analogue of unoriented Alexander's theorem. Namely, for every unoriented link L there exists an element g of F such that the link $\mathcal{L}(g)$ is equivalent to L, where $\mathcal{L}(g)$ is the link obtained from g by Jones' construction. He also proved a slightly weaker version of the theorem for the oriented case, and Aiello [1] completely showed that. In particular, in this case we are able to obtain an element of oriented Thompson's group $\overrightarrow{F}$ which is a subgroup of F for any oriented link.

The purposes of this manuscript are to give a natural sequence of elements of $\overrightarrow{F}$ realizing non-trivial links, and to explain that these links have some geometric properties such as alternating and fibered. Also, we compute their crossing numbers, genera, and braid indices, and we show that it is very easy to obtain minimal genus flat Seifert surfaces of these links from Jones' construction. This is joint work with Yuya Kodama (Tokyo Metropolitan University).

2 Definitions and constructions

2.1 R. Thompson's group F

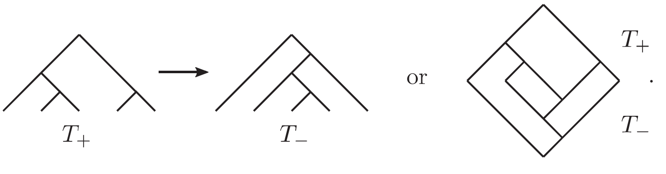
There are several equivalent definitions of Thompson's group F. In this manuscript, we introduce that using binary trees.

Thompson's group F is defined by the following set:

$$F := \left\{ \begin{array}{l} \text{all pairs of rooted, planar, binary trees } (T_+, T_-) \\ \text{with the same number of leaves} \end{array} \right\} \Big/ \sim ,$$

where $\sim$ is the equivalence relation defined below. A pair (T_+, T_-) is called a *tree diagram*, and described as

$(T_+, T_-) =$



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The equivalence relation $\sim$ is defined as follows: two tree diagrams are equivalent in F if and only if there is a finite sequence of additions and reductions of pairs of opposing carets ◇, which deform one to the other (see Figure 1). The tree diagram representing $g \in F$ without pairs of opposing carets is called the *reduced tree diagram* of g. It is known that for any element of F, its reduced tree diagram is unique.

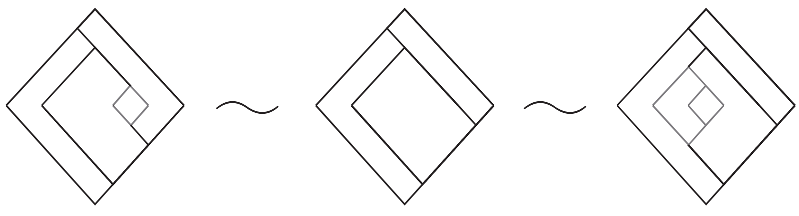


Figure 1: Equivalent tree diagrams in F.

For two tree diagrams (T_{1+}, T_{1-}) and (T_{2+}, T_{2-}), their product $(T_{1+}, T_{1-}) \cdot (T_{2+}, T_{2-})$ is defined as follows: by additions or reductions of pairs of opposing carets, we are able to deform these tree diagrams to equivalent tree diagrams (T'_{1+}, T'_{1-}) and (T'_{2+}, T'_{2-}), respectively, so that the binary trees T'_{1-} and T'_{2+} are the same. Then we define the product $(T_{1+}, T_{1-}) \cdot (T_{2+}, T_{2-})$ as the tree diagram (T'_{1+}, T'_{2-}) (see Figure 2). It is well known that Thompson's group F is finitely presented.

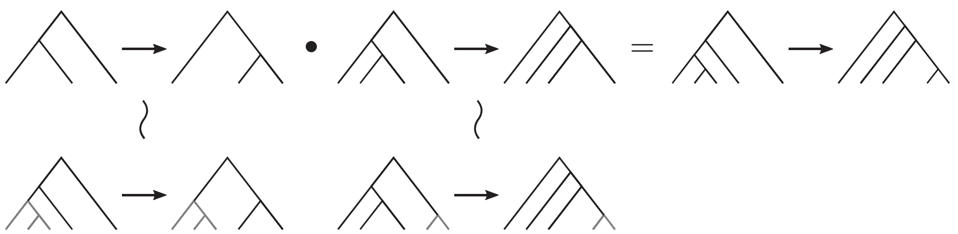


Figure 2: The product of two tree diagrams.

Theorem 2.1 ([5, Theorem 3.1], [2, Section 3])**.** *Thompson's group F admits the following presentations:*

$$\begin{aligned} F &\cong \left\langle x_0, x_1, x_2, \dots \;\middle|\; x_i^{-1} x_j x_i = x_{j+1} \ (i < j) \right\rangle \\ &\cong \left\langle x_0, x_1 \;\middle|\; [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \right\rangle, \end{aligned}$$

where $[x, y]$ is the commutator of x and y, and x_0, x_1 and x_2 correspond to the tree diagrams in Figure 3.

2.2 Jones' construction

In this subsection, we explain the procedure of Jones' construction with an example. We refer to [1, 4]. Let (T_+, T_-) be a reduced tree diagram with $n + 1$ leaves, and place its

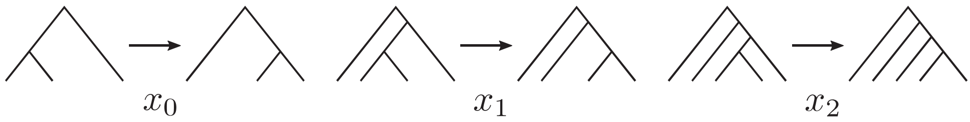


Figure 3: The generators of F.

leaves at $\left(\frac{1}{2},0\right), \left(\frac{3}{2},0\right), \ldots, \left(\frac{2n+1}{2},0\right)$. Note that the tree T_+ is in the upper half-plane, and T_- in the lower half-plane.

Step 1: Construct the planar graph $\Gamma(T_+, T_-)$.

The planar graph $\Gamma(T_+, T_-)$, which is called the Γ-graph of (T_+, T_-), is defined as follows: the vertices of $\Gamma(T_+, T_-)$ are put at $(0,0), (1,0), \ldots, (n,0)$. An edge of $\Gamma(T_+, T_-)$ passes transversely just once an edge $\diagup$ of T_+ (i.e. an edge from top right to bottom left) or an edge $\diagdown$ of T_- (i.e. an edge from top left to bottom right) and does not do the other edges of (T_+, T_-). Figure 4 is an example of this step.

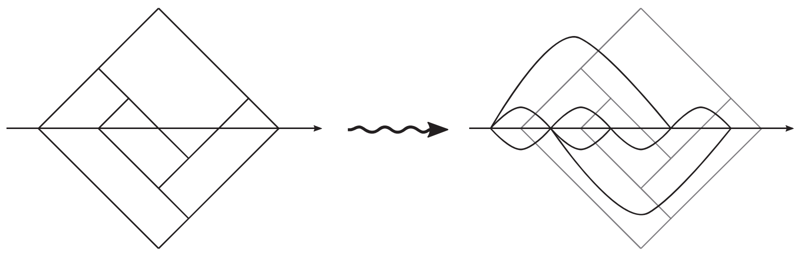


Figure 4: The graph $\Gamma(T_+, T_-)$ obtained from (T_+, T_-).

Step 2: Construct the medial graph $M(\Gamma(T_+, T_-))$.

The medial graph is defined for any connected planar graph. Let G be a connected planar graph. Its *medial graph* $M(G)$ is defined as follows: we put a vertex of $M(G)$ on every edge of G, and join two vertices by an edge if the corresponding edges of G are adjacent on a face of G. Figure 5 is an example of this step.

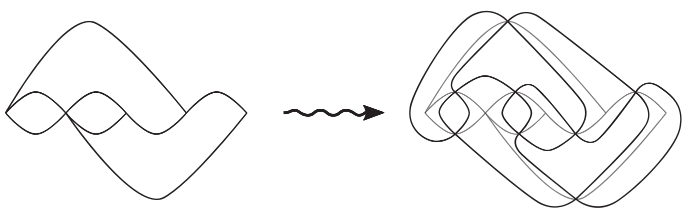


Figure 5: The medial graph $M(\Gamma(T_+, T_-))$ of $\Gamma(T_+, T_-)$.

Step 3: Construct the link diagram $\mathcal{L}(T_+, T_-)$.

In general, since the medial graph is 4-valent graph, we are able to obtain a link diagram $\mathcal{L}(T_+, T_-)$ from turning all vertices of $M(\Gamma(T_+, T_-))$ into crossings. For vertices in the upper half-plane, we use the crossing  and for the other vertices, we use . Figure 6 is an example of this step.

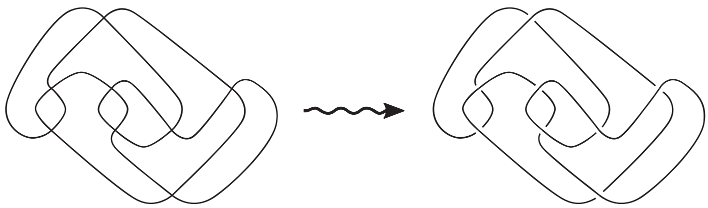


Figure 6: The link diagram $\mathcal{L}(T_+, T_-)$ obtained from $M(\Gamma(T_+, T_-))$.

2.3 Jones' subgroup $\overrightarrow{F}$

Jones defined the subgroup $\overrightarrow{F}$ of Thompson's group F to be

$$\overrightarrow{F} := \{(T_+, T_-) \in F \mid \Gamma(T_+, T_-) \text{ is 2-colorable}\}.$$

This group $\overrightarrow{F}$ is called *Jones' subgroup* or *oriented Thompson's group*. Now, 2-colorableness can be defined for any graph. A graph G is *2-colorable* if there exists a map $f\colon V(G)\,(:=\{\text{all vertices in } G\}) \longrightarrow \{+, -\}$ such that whenever two vertices $v_1, v_2 \in V(G)$ are joined by an edge, $f(v_1) \neq f(v_2)$ holds. A map f is called a *coloring* or *labeling*. By convention, we assume that the vertex $(0,0)$ of $\Gamma(T_+, T_-)$ has the color $+$. Figure 7 is an example of a 2-colorabel Γ-graph.

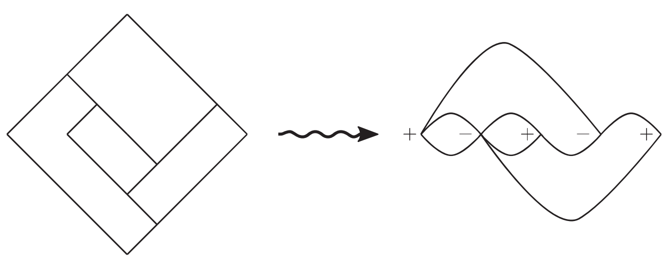


Figure 7: A 2-colorable graph.

For any $(T_+, T_-) \in \overrightarrow{F}$, the link $\mathcal{L}(T_+, T_-)$ is naturally oriented as follows: we apply the checkerboard coloring to the diagram $\mathcal{L}(T_+, T_-)$, that is, we paint regions of $\mathcal{L}(T_+, T_-)$ with black or white so that adjacent regions are different colors. By convention, the color of the unbounded region is white. Then we obtain the checkerboard surface $\mathcal{S}(T_+, T_-)$

in $\mathbb{R}^3$ from black regions and its boundary is $\mathcal{L}(T_+, T_-)$. Since the graph $\Gamma(T_+, T_-)$ is 2-colorable, the surface $\mathcal{S}(T_+, T_-)$ is orientable, and so is its boundary $\mathcal{L}(T_+, T_-)$. We assume that the regions with color $+$ are positively oriented (see Figure 8).

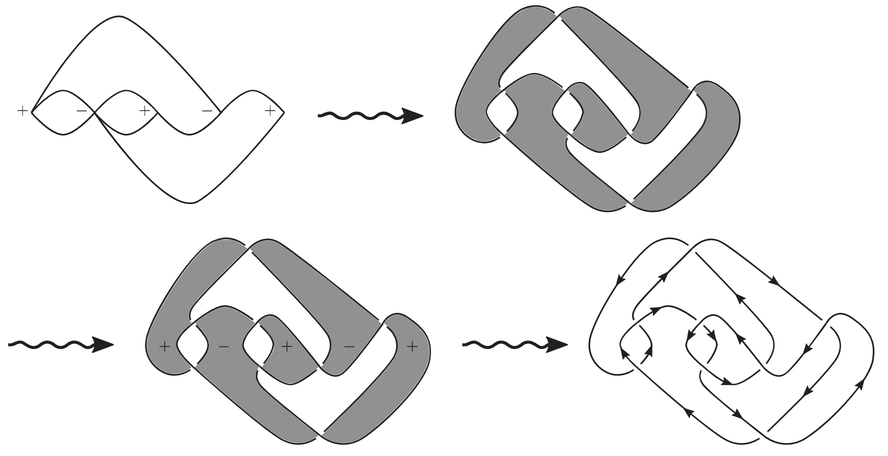


Figure 8: The checkerboard surface $\mathcal{S}(T_+, T_-)$ obtained from a element (T_+, T_-) of $\overrightarrow{F}$ is orientable. Then its boundary link $\mathcal{L}(T_+, T_-)$ is also oriented.

Theorem 2.2 ([3, Lemma 4.5, 4.6 and 4.7])**.** *Jones' subgroup $\overrightarrow{F}$ satisfies the following:*

- $\overrightarrow{F}$ *is generated by* $X := \{x_i x_{i+1} \mid i \geq 0\}$ *and* $X' := \left\{x_i^{n+1} x_{i+1} x_{i+2}^{-n} \;\middle|\; i \geq 0, n \geq 1\right\}$,
- $\overrightarrow{F}$ *is generated by* $x_0 x_1, x_1 x_2$ *and* $x_2 x_3$, *and*
- $\overrightarrow{F} \cong \left\langle y_0, y_1, y_2, \dots \;\middle|\; y_i^{-1} y_j y_i = y_{j+2} \ (i < j)\right\rangle$,

where $y_i := x_i x_{i+1}$ *for any* $i \geq 0$ *(see Figure 9).*

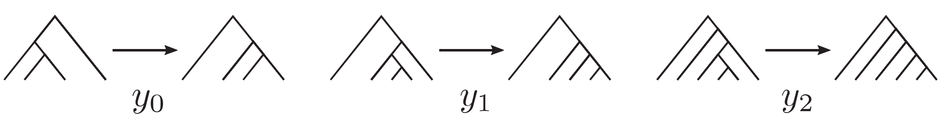


Figure 9: The generators of $\overrightarrow{F}$.

3 Examples and Main result

3.1 Examples

By direct calculation, we see that all links obtained from elements of $\overrightarrow{F}$ with ≤ 5 leaves are trivial. Thus, we have to consider elements with ≥ 6 leaves to obtain non-trivial links.

(1) Hopf link

The element $y_0^2 y_1^{-1} \in \overrightarrow{F}$ has 6 leaves. By Jones' construction, we obtain the Hopf link (see Figure 10).

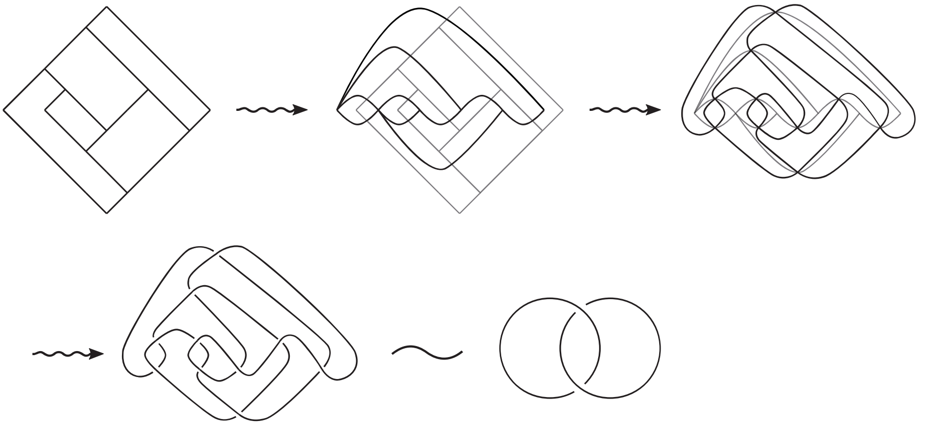


Figure 10: $y_0^2 y_1^{-1}$ creates the Hopf link.

(2) Figure-eight knot

We obtain a Γ-graph realizing a non-trivial link above. Thus, we consider adding vertices and edges to outside of this graph while preserving 2-colorable to construct other non-trivial links. In Figure 11, we add one vertex and two edges to the Γ-graph $\Gamma(y_0^2 y_1^{-1})$. Then this graph represents $y_0^2 y_1^{-2} \in \overrightarrow{F}$ and yields the figure-eight knot.

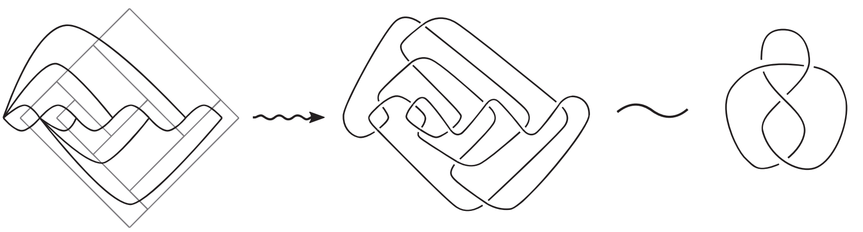


Figure 11: $y_0^2 y_1^{-2}$ creates the figure-eight knot.

(3) Whitehead link

We also consider adding one vertex and two edges to $\Gamma(y_0^2 y_1^{-2})$ as in Figure 12. Then we obtain $y_3^2 y_1^{-2} \in \overrightarrow{F}$ and this yields the Whitehead link.

Continuing adding vertices and edges as above, we obtain a sequence of 2-colorable Γ-graphs (i.e. elements of $\overrightarrow{F}$). In general, the following holds:

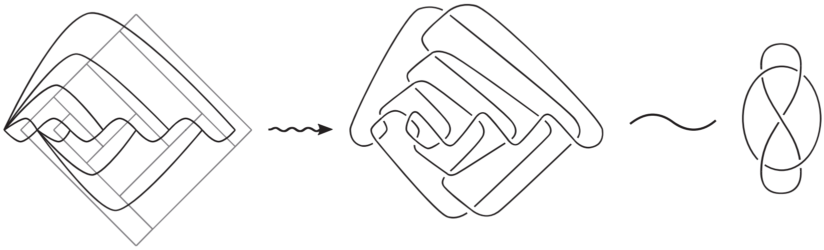
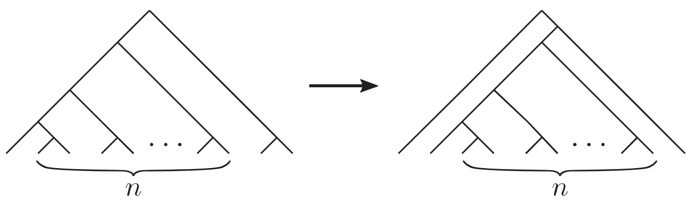
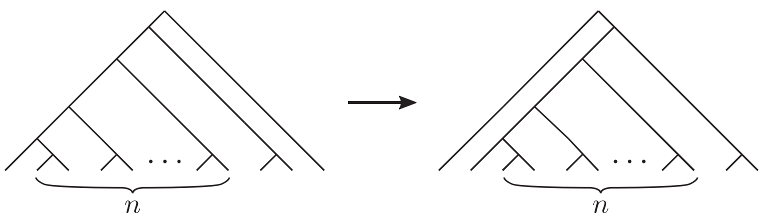


Figure 12: $y_0^3 y_1^{-2}$ creates the Whitehead link.

Lemma 3.1. *For any $n \geq 1$,*

$y_0^n y_1^{-n} =$ 

and

$y_0^{n+1} y_1^{-n} =$ 

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Their Γ-graphs are of the forms

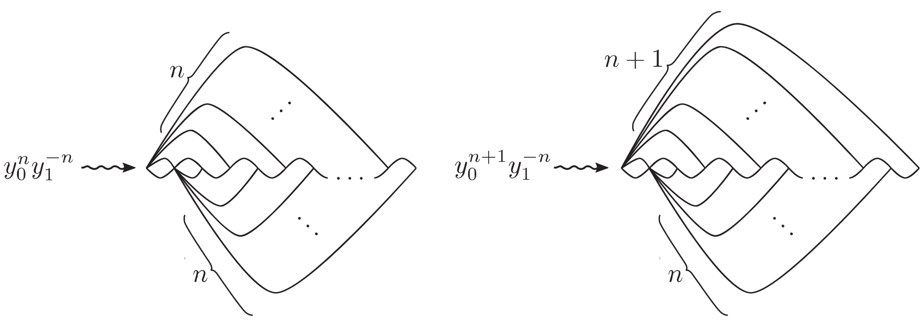

Table 1 lists names and properties of links obtained from the sequence (for $k = 0, 1, \ldots, 5$), where for a link L, let $\mu(L)$ be its number of components, $c(L)$ its crossing number, $g(L)$ its genus, and $b(L)$ its braid index.

k	g_k	L_k	$\mu(L_k)$	$c(L_k)$	alternating	fibered	$g(L_k)$	$b(L_k)$
0	$y_0^2 y_1^{-1}$	L_2a_1	2	2	YES	YES	0	2
1	$y_0^2 y_1^{-2}$	4_1	1	4	YES	YES	1	3
2	$y_0^3 y_1^{-2}$	L_5a_1	2	5	YES	YES	1	3
3	$y_0^3 y_1^{-3}$	L_6a_4	3	6	YES	YES	1	3
4	$y_0^4 y_1^{-3}$	L_7a_1	2	7	YES	YES	2	3
5	$y_0^4 y_1^{-4}$	8_{18}	1	8	YES	YES	3	3

Table 1: Examples for $k = 0, 1, \ldots, 5$

3.2 Main result

Theorem 3.2. *Let $\{g_k\}_{k\geq 0}$ be the subset of $\overrightarrow{F}$ given by*

$$g_k := \begin{cases} y_0^{m+2} y_1^{-m-1} & (k = 2m) \\ y_0^{m+2} y_1^{-m-2} & (k = 2m+1) \end{cases},$$

and L_k the link obtained from g_k by Jones' construction. Then

- $\mu(L_k) = \begin{cases} 1 & (k = 6m+1, 6m+5) \\ 2 & (k = 2m) \\ 3 & (k = 6m+3) \end{cases},$
- *$c(L_0) = 2$, and $c(L_k) = k + 3$ for any $k \geq 1$,*
- *for any $k \geq 0$, L_k is alternating and fibered,*
- *$b(L_0) = 2$, and $b(L_k) = 3$ for any $k \geq 1$,*
- $g(L_k) = \begin{cases} m & (k = 2m) \\ 3m+1 & (k = 6m+1, 6m+3) \\ 3m+3 & (k = 6m+5) \end{cases}$, *and*
- *we can easily obtain a minimal genus flat Seifert surface of L_k from $\mathcal{S}(g_k)$.*

Finally, we explain a way to obtain a minimal genus flat Seifert surface. We consider the case $k = 2$. Reducing some crossings in the link diagram L_k, its checkerboard surface, denoted by $\mathcal{S}_0(g_k)$, is also orientable (see Figure 13). Namely, this surface is also a Seifert surface of L_k. In fact the surface $\mathcal{S}_0(g_k)$ is flat and has minimal genus.

References

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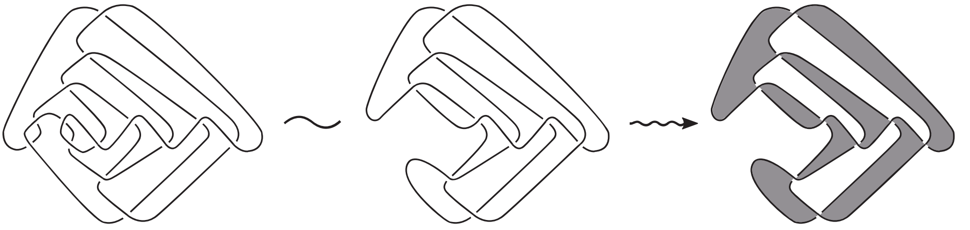

Figure 13: Reducing some crossings and the orientable checkerboard surface $\mathcal{S}_0(g_k)$ ($k = 2$).

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Graduate School of Mathematical Sciences
The University of Tokyo
Tokyo 153-8914
JAPAN
E-mail address: takano@ms.u-tokyo.ac.jp

東京大学大学院数理科学研究科　高野 暁弘