Concordance for higher dimensional welded objects and their Milnor invariants

Benjamin AUDOUX
Aix Marseille Univ, CNRS, Centrale Marseille
Jean-Baptiste MEILHAN
Univ. Grenoble Alpes, CNRS, Institut Fourier
and
Akira YASUHARA
Faculty of Commerce, Waseda University

This note is an extract from the article [3]; in particular, the reader is referred to the original paper for all proofs. We work on the smooth category.

1 Cut-diagrams

In this section we introduce the notion of cut-diagrams, which are in some sense 'expansion plans' of diagrams of codimension 2 embeddings.

1.1 Cut-diagrams for diagram of links

Before providing a general definition, we introduce cut-diagrams for classical link diagrams and surface link diagrams.

1.1.1 Classical link diagrams

A classical knot diagram is a generic immersion $S^1 \looparrowright \mathbb{R}^2$, with a finite number of transverse double points. Each double point is endowed with the over/under information on the two preimages.

These under points, which we shall call *cut points*, split S^1 into a number of arcs, called *regions*. To each cut point is associated the sign of the corresponding crossing, and the region containing the preimage of the over point. The (1–dimensional) *cut-diagram* for a classical link diagram is thus a union of copies of S^1 with a collection of cut points endowed with a sign and labeled by their associated region. See Figure 1 for a couple of examples.

Remark 1.1. Cut points can be thought of as heads of arrows in a Gauss diagram, the label being the arc where the tail is attached. Note however that the relative position of tails attached to a same arc is not specified in this language. Hence 1–dimensional cut-diagrams should rather be thought of as Gauss diagrams modulo the moves which exchanges two adjacent tails on an arc, i.e., welded objects.

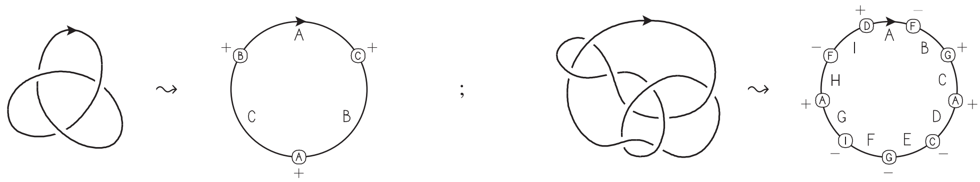


Figure 1: From a knot diagram to a 1–dimensional cut-diagram.
Here, and in forthcoming figures, regions are named with capital letters, and labels on cut points are given by circled nametags

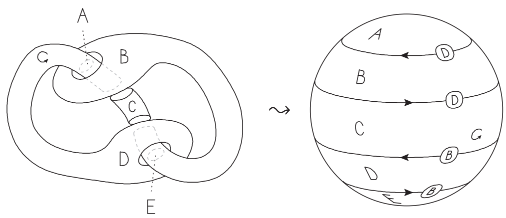


Figure 2: From a knotted surface diagram to a 2–dimensional cut-diagram.

1.1.2 Surface link diagrams

Now, consider a diagram for a surface link in 4–space. This is a generic map $\Sigma \looparrowright \mathbb{R}^3$ of a surface Σ, with lines of transverse double points which may meet at triple points and/or end at branch points. As in link diagrams, each double point is endowed with the over/under information on the two preimages (see the left-hand side of Figure 2, 3 and 4 for some examples). Each line of double points has an orientation such that the tuple of a positive normal vector to the overpassing region, a positive normal vector to the underpassing region, and a positive tangent vector to the line of double points agrees with the ambient orientation of $\mathbb{R}^3$.

The under preimages of double points form a union P of immersed circles and/or intervals in Σ, which splits Σ into *regions*. Note that the set P is known as the *lower deck* in the literature (see for example [5, § 4.1]). Each triple point of the surface diagram provides an over/under information at the corresponding crossing of P, which is encoded as in usual knot diagrams by splitting P into arcs, that we call *cut arcs*. Each cut arc inherits an orientation, which is the orientation of the corresponding line of double points, and we label it by the region containing the preimage with highest coordinate at the corresponding line of double points. The data of the oriented and labeled cut arcs in Σ forms a (2–dimensional) *cut-diagram over* Σ for the given surface link. An example is given in Figure 2 in the case $\Sigma = S^2$.

We stress that the labeling respects some natural conditions, summarized in Definition 1.4, which are inherited from the topological nature of the surface diagram. We investigate these conditions below, by analysing the local cut-diagrams arising from triple and branch points.

Definition 1.2. An ordered pair of regions (A, B) is called *C–adjacent*, for some region C, if there exists a path $\gamma : [0, 1] \to \Sigma$ such that $\gamma\big([0, 1/2)\big) \subset A$, $\gamma\big((1/2, 1]\big) \subset B$, and $\gamma(1/2)$ is a

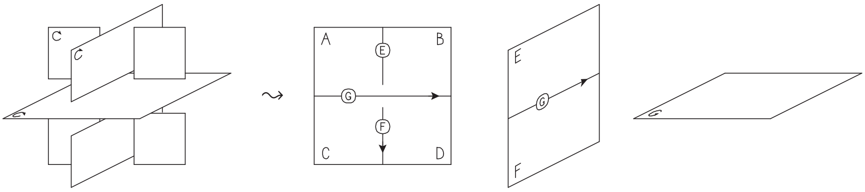


Figure 3: From broken diagrams to cut-diagrams: triple point

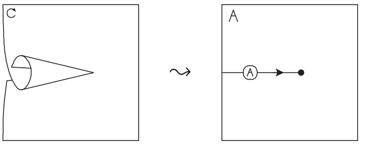


Figure 4: From broken diagrams to cut-diagrams: branch point

regular point of a cut arc labeled by C, such that the tangent vector $\gamma'(1/2)$ is a positive normal vector for this cut arc. We define similarly the notion of *C–adjacency* for two cut arcs which are separated by a double point of a C–labeled cut arc.

In the local example of Figure 3, the pair of regions (E, F) is G–adjacent (middle sheet), and we observe that likewise, the E–labeled and F–labeled cut arcs are G–adjacent (leftmost sheet). This yields a local cut-diagram with a boundary points in the interior of the surface Σ.

Definition 1.3. An *internal endpoint* of a cut-diagram over Σ is a point of a boundary component of some cut arc, which is in the interior of Σ. A cut arc containing an internal endpoint is called *terminal*.

In the local example of Figure 4, the terminal cut arc is labeled by A, the region it is adjacent to.

Definition 1.4. A labeling of the cut arcs by regions is *admissible* if it satisfies the following *labeling conditions*:

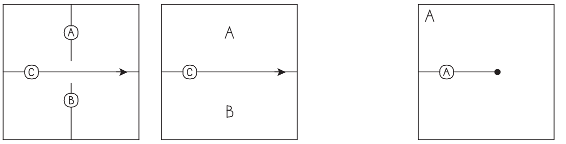
(1) any pair of A–adjacent cut arcs, for some region A, is labeled by two A–adjacent regions;

(2) a terminal cut arc containing an internal endpoint adjacent to some region A, is labeled by A.

These two conditions are locally represented in Figure 5.

1.2 Abstract cut-diagrams

The notion of cut-diagrams associated with link or surface diagrams can be generalized to an abstract notion.



First condition

Second condition

Figure 5: Illustrations of the labeling conditions

1.2.1 The 2–dimensional case

Let us begin with the 2–dimensional case. Let Σ be some oriented surface, possibly with boundary. Consider a, non necessarily properly, immersed oriented 1–manifold in Σ, together with an over/under decoration at each transverse double point, as in usual link diagrams. This immersed 1–manifold splits Σ into pieces called *regions*, and the under crossings split the immersed manifold into *cut arcs*, with the induced orientation. Label each cut arc by some region, so that the labeling conditions of Definition 1.4 are satisfied. The result, regarded up to ambient isotopy of Σ, is called a 2–dimensional *cut-diagram* over Σ.

As discussed in the preceding section, there is a canonical cut-diagram for a given diagram of a surface link.

1.2.2 The general case

We now give a general definition for cut-diagrams in any dimension.

Let Σ be an n–dimensional oriented compact manifold, possibly with boundary. The following generalized notion of diagrams for embedded n–dimensional manifolds originates from the work of D. Roseman in [13, 14]; see also [12] for a good review of this notion. A diagram on Σ is the image of a map $\varphi : P \looparrowright \Sigma$, where P is an oriented $(n-1)$–dimensional manifold, which is in generic position in the sense of [12, § 6.1], equipped with a globally coherent over/under data on double points. Roughly speaking, φ is an immersion, except at an $(n-2)$–dimensional set of branch points. If $\partial\Sigma \neq \emptyset$, we moreover require that $\varphi(P)$ meets transversally $\partial\Sigma$, so that $\varphi_{|\varphi^{-1}(\partial\Sigma)}$ induces a $(n-2)$–dimensional diagram on $\partial\Sigma$.

For any such diagram, the connected components of $\Sigma \setminus P$ are called *regions*, and the connected components of {highest preimages of φ} $\subset P$ are called *cut domains*, although we shall keep calling them *cut points* and *cut arcs* when dealing specifically with the dimensions one and two, respectively. Note that the notions of *A–adjacency* and *internal endpoint*, of Definitions 1.2 and 1.3, apply in any dimension, so that the labeling conditions in Definition 1.4 make sense. There, as well as in Definitions 1.2 and 1.3, the terms *cut arc* should be replaced by *cut domain*. Moreover, in Definition 1.3, an internal endpoint is assumed to be a *regular* point in the boundary of some cut domain. Observe also that for $n = 1$, these conditions are vacuous.

Definition 1.5. An n–dimensional *cut-diagram* over Σ is the data C of a diagram on Σ, with an admissible labeling up to ambient isotopy of Σ.

As seen in the previous subsections, a 2–dimensional cut-diagram can be naturally associated to any diagram of a knotted surface in 4–space, although not any cut-diagram arises in this way.

Given a diagram of an embedded n–dimensional manifold, the procedure generalizes naturally to produce an n–dimensional cut-diagram. This leads to the following notion.

Definition 1.6. We say that a cut-diagram is *topological* if it comes from the diagram of a codimension 2 embedding.

1.3 Cut-concordance

We consider the situation where two cut-diagrams over n–dimensional manifold Σ cobound a cut-diagram over $\Sigma \times [0,1]$. This yields general notion of concordance for cut-diagrams.

Definition 1.7. Two cut-diagrams C_0 and C_1 over Σ, are *cut-concordant* if there is a cut-diagram C over $\Sigma \times [0,1]$ which intersects $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ as C_0 and C_1, respectively. This means that, for $\varepsilon \in \{0,1\}$:

- there is an orientation preserving diffeomorphism $\psi_\varepsilon : \Sigma \to \Sigma \times \{\varepsilon\}$ sending C_ε to $C \cap \big(\Sigma \times \{\varepsilon\}\big)$, where $C \times \big(\Sigma \cap \{\varepsilon\}\big)$ is given the inward induced orientation if $\varepsilon = 0$ and the outward one if $\varepsilon = 1$;
- the following diagram commutes:

$$\begin{tikzcd} \big\{\text{cut-domains of } C_\varepsilon\big\} \arrow[r, "\text{labelling}"] \arrow[d, "\psi^{\mathrm{c}}_\varepsilon"] & \big\{\text{regions of } C_\varepsilon\big\} \arrow[d, "\psi^{\mathrm{r}}_\varepsilon"] \\ \big\{\text{cut-domains of } C\big\} \arrow[r, "\text{labelling}"] & \big\{\text{regions of } C\big\} \end{tikzcd},$$

  where $\psi^{\mathrm{c}}_\varepsilon$ (resp. $\psi^{\mathrm{r}}_\varepsilon$) is an 'inculsion map' that sends a cut-domain c_ε (resp. a region r_ε) of C_ε to the unique cut-domain d (resp. region r) of C such that $\psi_\varepsilon(d_\varepsilon) \subset d$ (resp. $\psi_\varepsilon(r_\varepsilon) \subset r$).

Moreover, if $\partial\Sigma \neq \emptyset$, we also require that $C \cap \big(\partial\Sigma \times [0,1]\big) \cong \big(C_0 \cap \partial\Sigma\big) \times [0,1]$. We also say that C is a *cut-concordance* between C_0 and C_1.

Cut-concordance defines a natural equivalence relation on cut-diagrams, thus providing a coherent theory for studying these objects. Moreover, this notion generalizes the topological notion of concordance in the following sense.

Proposition 1.8. ([3, Proposition 5.2]) *Two topological cut-diagrams, associated to two concordant codimension* 2 *embeddings, are cut-concordant.*

2 Peripheral systems of cut-diagrams

Let Σ be an n–dimensional compact manifold with connected components $\Sigma_1, \cdots, \Sigma_\ell$, and let C be a cut-diagram over Σ.

2.1 The group of a cut-diagram

From this point on, we shall make use of the following convention: given two elements a, b of some group, their commutator is defined as $[a,b] := a^{-1}b^{-1}ab$, and the conjugate of a by b is given by $a^b := a^{-1}ba$.

Definition 2.1. The *group of* C is the group $G(C)$ generated by its regions, with the *Wirtinger relation* $B^{-1}A^C$ for any regions A, B and C such that (A, B) is C–adjacent. We call each reagion a *meridian* of C, and in particular an *i-th meridian* if it belongs to Σ_i.

Notation 2.2. We denote by $\bar{F}$ the free group generated by all meridians of C, and by W the normal closure of the Wirtinger relations in $\bar{F}$. Hence we have $G(C) = \bar{F}/W$.

Remark 2.3. Diagrams of surface links give Wirtinger presentations of the fundamental group of their exterior, see e.g. [4]. It is straightforwardly verified that the group of a topological 2–dimensional cut-diagram, agrees with the fundamental group of the exterior of the surface link, and this is actually true for topological cut-diagrams of any dimension. Moreover, in that case, the above notion of meridian agrees with the classical topological notion of meridian.

Definition 2.4. The *lower central series* $\left(G_q\right)_{q\in\mathbb{N}}$ of a group G, is the descending series of subgroup defined inductively by $G_1 := G$ and $G_{q+1} := [G, G_q]$. For $q \in \mathbb{N}$, the *q-th nilpotent quotient* of G is the quotient $N_qG := G/G_q$ by the q-th term of its lower central series.

For any $q \in \mathbb{N}$, the *q-th nilpotent group* of C is the nilpotent quotient $N_qG(C)$ of $G(C)$, and we have

$$N_qG(C) \cong \bar{F}/\bar{F}_q \cdot W.$$

The following theorem provides a combinatorial Stallings Theorem [17, Thm. 5.1] in the case of a topological concordance between two codimension 2 embeddings, and a generalization to cut-concordances.

Theorem 2.5. ([3, Corollary 5.9]) *Given a cut-concordance C between two cut-diagrams C_0 and C_1, the inclusion maps ψ_0^r and ψ_1^r of Definition 1.7 induce isomorphisms for all $q \geq 1$:*

$$N_qG(C_0) \xrightarrow{\simeq} N_qG(C) \xleftarrow{\simeq} N_qG(C_1).$$

2.2 Generic path and associated words

We say that an oriented path $\gamma : [0, 1] \to \Sigma$ is *generic* if $\gamma(0), \gamma(1) \notin C$ and $\gamma\big((0, 1)\big)$ meets C only transversally, in a finite number of regular points of C.

Let γ be a generic path on Σ. Then the word $\widetilde{w}_\gamma$ associated to γ is defined by

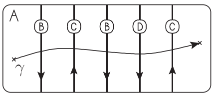
$$\widetilde{w}_\gamma := A_1^{\varepsilon_1} \cdots A_{|\gamma\cap C|}^{\varepsilon_{|\gamma\cap C|}},$$

where A_k is the label of the cut domain containing the k-th intersection point on γ with C, and by $\varepsilon_k = \pm 1$ the local sign of this intersection point. This local sign is 1 if the local orientation given by the orientation of γ and C agrees with the ambient orientation of Σ, and is -1 otherwise. The *normalized* word w_γ is defined by

$$w_\gamma := A^{-|\gamma|}\widetilde{w}_\gamma,$$

where A is the region where γ starts, and $|\gamma|$ is the sum of the exponents ε_i in $\widetilde{w}_\gamma$ such that A_i is in the same connected component as γ; see Figure 6 for an example.

Remark 2.6. If γ is a generic path joining two regions A and B, then the Wirtinger relations imply that $B = A^{\widetilde{w}_\gamma} = A^{w_\gamma}$ in $G(C)$. This elementary observation has two important consequences:



$\rightsquigarrow$ $\widetilde{w}_\gamma = BC^{-1}BDC^{-1}$
$w_\gamma = ABC^{-1}BDC^{-1}$ $\left(\begin{array}{c}\text{assuming } C \text{ and } D \text{ are on the same connected}\\ \text{component of } \Sigma \text{ as } A, \text{ but } B \text{ is not}\end{array}\right)$

Figure 6: An example of word associated to a path

(1) Any two meridians of a same connected component of Σ are conjugate, so that $G(C)$ is normally generated by any choice of one meridian on each connected component of Σ.

(2) If γ is a generic loop based in a region R, then $[R, \widetilde{w}_\gamma] = [R, w_\gamma] = 1$ in $G(C)$.

In view of Remark 2.6 (1), any conjugate of a meridian of $G(C)$ will also be called *meridian* by abuse of notation.

2.3 Loop-longitudes

We define a notion of longitudes, associated to homotopy classes of paths on Σ. For that, we need the following.

Lemma 2.7. ([3, Lemma 2.10]) *If two generic paths γ and γ' are homotopic in Σ relative to the boundaries, then $w_\gamma = w_{\gamma'}$ in $G(C)$.*

For each $i \in \{1, \dots, \ell\}$, pick an i-th meridian R_i and a basepoint $p_i \in \Sigma_i$ in the interior of the corresponding region. This provides in particular a preferred set of normal generators for $G(C)$. Then, thanks to Lemma 2.7, we can define the *i-th loop-longitude map*

$$\lambda_i^l \colon \begin{array}{ccc} \pi_1(\Sigma_i, p_i) & \longrightarrow & G(C) \\ [\gamma] & \longmapsto & w_\gamma \end{array},$$

where $[\gamma]$ is an element of $\pi_1(\Sigma_i, p_i)$ represented by a generic loop γ based at p_i. Observe that λ_i^l is actually a group homomorphism. Indeed, for any two loops γ_1, γ_2 based at p_i, it follows from Remark 2.6 (2) that $w_{\gamma_1 \cdot \gamma_2} = R_i^{-|\gamma_1|-|\gamma_2|}\widetilde{w}_{\gamma_1}\widetilde{w}_{\gamma_2} = R_i^{-|\gamma_1|}\widetilde{w}_{\gamma_1}R_i^{-|\gamma_2|}\widetilde{w}_{\gamma_2} = w_{\gamma_1}w_{\gamma_2}$.

Definition 2.8. The elements in $\mathrm{Im}(\lambda_i^l)$ are called *i-th (preferred) loop-longitudes* for C. They are well-defined up to the choice of p_i, that is up to a simultaneous conjugation of R_i and all i-th longitudes by some element $w \in G(C)$.

In general, C has an infinite number of loop-longitudes, and it will be useful to consider only a finite generating set of these loops.

Definition 2.9. For each component Σ_i of Σ, let $\{\gamma_{ij}\}_j$ be a collection of loops based at p_i, which represent a generating set for $\pi_1(\Sigma_i; p_i)$. We call $\{\gamma_{ij}\}_{i,j}$ *system of loop-longitudes* for Σ, and denote $w_{\gamma_{ij}}$ by w_{ij}.

Observe that any system $\{w_{ij}\}$ of loop-longitudes determines the whole loop-longitude maps λ_i^l.

2.4 Arc-longitudes

Any generic path on Σ leads to a normalized word, which represents a well-defined element of $G(C)$ by Lemma 2.7. It is hence possible to consider a wider range of longitudes. As a matter of fact, if $\partial\Sigma \neq \emptyset$, we shall also consider 'boundary-to-boundary' longitudes. So, for each $i \in \{1, \dots, \ell\}$ such that $\partial\Sigma_i \neq \emptyset$, we fix *boundary basepoints* p_{ij}, $j \in \{0, \dots, |\partial\Sigma_i| - 1\}$ on each boundary component of Σ_i. All these boundary basepoints are chosen disjoint from C. This provides an ordering of the components of $\partial\Sigma_i$, and the one containing p_{i0} should be regarded as a 'marked' boundary component of Σ_i.

We set $\mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big)$ to be the set of homotopy classes of paths from p_{i0} to some p_{ij} for $j > 0$, up to homotopy, and we define the *i-th arc-longitude map*

$$\lambda_i^\partial : \begin{array}{ccc} \mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big) & \longrightarrow & G(C) \\ [\gamma] & \longmapsto & w_\gamma \end{array}.$$

Here, $[\gamma]$ is the element of $\mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big)$ represented by a generic path γ from p_{i0} to some p_{ij}. Elements of $\mathrm{Im}(\lambda_i^\partial)$ are called *i-th (preferred) arc-longitudes*.

We stress that the well-definedness of arc-longitudes requires the choice of the basepoints $\{p_{ij}\}$, and that these are kept fixed at all time.

Remark 2.10. If Σ_i is not simply connected, for each $j > 0$, $\mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big)$ contains several element ending at p_{ij}, possibly infinitely many. However, fixing a generic path γ_i from p_i to p_{i0}, there is a simply transitive left action of $\pi_1(\Sigma_i, p_i)$ on $\mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big)$, defined as $\tau \cdot \gamma := \tau^{\gamma_i}.\gamma$. A *system of arc-longitudes*, made of one arc-longitude per $j > 0$, is hence sufficient to recover all elements of $\mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big)$.

2.5 Peripheral systems

We use the terminology *longitude* when referring to either a preferred loop- or arc-longitude. We define the *i-th longitude maps*

$$\lambda_i : \pi_1(\Sigma_i, p_i) \sqcup \mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big) \to G(C)$$

by combining λ_i^l and λ_i^∂.

All the notions defined so far can be gathered into the following.

Definition 2.11. A *peripheral system* of C is the data $\big(G(C); \{R_i\}, \{\lambda_i\}\big)$ of the group of C together with the choice of a meridian R_i and its longitude map λ_i for each i.
Two peripheral systems $\big(G; \{R_i\}, \{\lambda_i\}\big)$ and $\big(G'; \{R_i'\}, \{\lambda_i'\}\big)$ of cut-diagrams over Σ are *equivalent* if there exist an isomorphism $\varphi : G \to G'$ and elements $w_1, \dots, w_\ell \in G'$ such that, for every $i \in \{1, \dots, \ell\}$, we have $R_i' = \varphi(R_i)^{w_i}$ and

$$\lambda_i'(\gamma) = \begin{cases} \varphi(\lambda_i(\gamma))^{w_i} & \text{for every } \textit{loop } \gamma \in \pi_1(\Sigma_i; p_i), \\ \varphi(\lambda_i(\gamma)) & \text{for every } \textit{arc } \gamma \in \mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big). \end{cases}$$

We stress that the above notion of equivalence for peripheral systems involves only the loop-longitudes, and that arc-longitudes are not required to be conjugated. The endpoints p_{ij} of arc-longitudes are fixed throughout.

For every $q \geq 1$, the *q-th nilpotent peripheral system* of C is the data, associated to a peripheral system $\big(G(C); \{R_i\}, \{\lambda_i\}\big)$, of the q-th nilpotent quotient $N_qG(C)$, together with the image of each R_i in $N_qG(C)$ and the composite of each longitude map λ_i with the projection from $G(C)$ to $N_qG(C)$. Definition 2.11 naturally induces a notion of equivalence for q-th nilpotent peripheral systems.

It is well-known that equivalent classes of nilpotent peripheral systems of link in S^3 are concordance invariants. This remains true for cut-diagrams. Indeed, we have the following.

Theorem 2.12. ([3, Theorem 5.5]) *Equivalence classes of nilpotent peripheral systems for cut-diagrams are invariant under cut-concordance.*

Remark 2.13. It is natural to define the notion of (preferred) longitudes for knotted surfaces as the image, in the fundamental group of the surface complement, of either a cycle or a (canonically closed) boundary-to-boundary path on the surface, pushed out of the surface in such a way that it has homological intersection zero with the surface. It is clear that the preferred longitudes of a topological 2–dimensional cut-diagram agrees with the preferred longitudes of the underlying knotted surface. This remains true in higher dimensions.

3 Chen homomorphisms and a Chen–Milnor Theorem for cut-diagrams

For the nilpotent quotients of fundamental group of link complements, Milnor gave in [11], by using earlier works of Chen [6], a presentation with one generator per component and meridian/loop-longitude commutation relations. We call the presentation the *Chen-Milnor presentation*. We give an analogue for the groups of cut-diagrams. In the following, C denotes an n–dimensional cut-diagram over $\Sigma = \sqcup_{i=1}^{\ell}\Sigma_i$., and $C_i = C \cap \Sigma_i$ the i-th component of C.

3.1 Road networks and the Chen homomorphisms

Fix a basepoint p_i on each connected component Σ_i, away from C. Regions of C living in Σ_i will be denoted by R_{ij}, with $R_i := R_{i0}$ the region containing p_i. We also denote by $F = \langle R_i\rangle$ and $\bar{F} = \langle R_{ij}\rangle$ the free group generated by the R_i and the R_{ij}, respectively.

Definition 3.1. A *road network* α for C based at $\{p_i\}$ is the choice of a collection of oriented generic paths α_{ij}, called *roads*, running from p_i to a point in each region R_{ij} ($i = 1, ..., l$).

Following Section 2.2, a word $v_{ij} := \widetilde{w}_{\alpha_{ij}} \in \bar{F}$ can be associated to each road α_{ij}. Notice that, from the Wirtinger presentation of $G(C)$, the relation $R_{ij} = R_i^{v_{ij}}$ holds in $G(C)$ (see Remark 2.6 (1)).

Definition 3.2. We define *Chen homomorphisms* $\eta_q^\alpha : \bar{F} \longrightarrow F$ by setting, for every i, j and $q \geq 1$:

$$\eta_1^\alpha(R_{ij}) := R_i,$$
$$\eta_{q+1}^\alpha(R_i) := R_i \ \text{ and } \ \eta_{q+1}^\alpha(R_{ij}) := R_i^{\eta_q^\alpha(v_{ij})}.$$

This sequence of homomorphisms depends on the choice of road network α. However, for simplicity we will often denote η_q^α by η_q.

We derive some fundamental properties of the Chen homomorphisms η_q.

Lemma 3.3. ([3, Lemma 3.7]) *For all $q \geq 1$, and all $w \in \bar{F}$, $\eta_q(w) \equiv \eta_{q+1}(w)$ mod F_q.*

Lemma 3.4. ([3, Lemma 3.8]) *For any generator R_{ij} of $\bar{F}$, $\eta_q(R_{ij}) \equiv R_{ij}$ mod $\bar{F}_q \cdot W$.*

An immediate consequence of Lemma 3.4 is the following, showing that at the q-th nilpotent level, the choice of road network has no effect on the η_q map.

Proposition 3.5. ([3, Corollary 3.9]) *Let α, α' be two road networks for C, possibly with different basepoints. Then, for any $w \in \bar{F}$, $\eta_q^{\alpha}(w) \equiv \eta_q^{\alpha'}(w)$ mod $\bar{F}_q \cdot W$, and hence $\eta_q^{\alpha}(w) = \eta_q^{\alpha'}(w)$ in $N_qG(C)$.*

We stress that the choice of road network has more subtle consequences at the level of word representatives for elements of $N_qG(C)$.

3.2 Chen–Milnor type presentations

For each component Σ_i of Σ, pick a system of loop-longitudes $\mathcal{L}_i(\Sigma) := \{w_{ij}\}_j$ associated with a collection of loops $\{\gamma_{ij}\}_j$ based at p_i (see Definition 2.9). Then we have the following.

Theorem 3.6. ([3, Theorem 3.15]) *If C is a cut-diagram over Σ, then for each $q \in \mathbb{N}$ we have the following presentation for the q-th nilpotent quotient $N_qG(C)$ of $G(C)$:*

$$\left\langle R_1, \ldots, R_\ell \;\middle|\; F_q \,;\, \big[R_i, \eta_q(w_{ij})\big] \text{ for all } i \text{ and all } w_{ij} \in \mathcal{L}_i(\Sigma) \right\rangle .$$

Remark 3.7. Theorem 3.6 does not only provide a presentation for $N_qG(C)$ derived from a system of loop-longitudes, but it also gives, via the Chen homomorphisms, an algorithm to compute representative words for any element in $N_qG(C)$, and in particular for the longitudes in these quotient.

By combining this and Lemma 2.7, we have the following proposition.

Proposition 3.8. ([3, Proposition 3.16]) *Let C be a cut-diagram over Σ. If two generic paths γ and γ' are homotopic in Σ relative to the boundaries, then*

$$\eta_q(w_\gamma) \equiv \eta_q(w_{\gamma'}) \text{ mod } F_q \cdot V_{(q)},$$

where $V_{(q)}$ is the normal closure in F of $\{[R_i, \eta_q(w_{ij})] \mid$ for all i and all $w_{ij} \in \mathcal{L}_i(\Sigma)\}$.

4 Milnor invariants for cut-diagrams

The purpose of this section is to define Milnor invariants for any cut-diagram C on Σ. For a classical link, Milnor invariants are extracted from the coefficients in the Magnus expansion of the preferred longitudes in the nilpotent quotients of the fundamental group of the link exterior. Now, as discussed in Sections 2.3 and 2.4, the notion of (preferred) longitudes for cut-diagrams comes in two flavors: loop-longitudes and arc-longitudes. As a matter of fact, although coming from a same global construction that associates a collection of 'Milnor numbers' to any generic path on Σ (Section 4.1), Milnor invariants defined from loop and arc-longitudes have rather different behaviors, and we shall discuss them separately (Sections 4.2 and 4.3).

Let us recall some notation used in the previous sections: $\Sigma = \sqcup_{i=1}^{\ell}\Sigma_i$ is an oriented n–dimensional manifold, and C is a cut-diagram over Σ. For all i, regions of Σ_i are denoted by $\{R_{ij}\}$. We also fix, for each $i \in \{1, \dots, \ell\}$ such that $\partial\Sigma_i \neq \emptyset$, a point p_{ik} on each boundary component of Σ_i ($k \in \{0, \dots, |\partial\Sigma_i| - 1\}$), and consider the component containing p_{i0} as a marked boundary component.

4.1 Milnor numbers associated with a longitude

We choose an i-th meridian R_i for each i, and denote by $F := \langle R_i \rangle$ the free group generated by these meridians. We also pick a basepoint p_i on the region R_i. By Theorem 3.6, the above choice of an i-th meridian R_i for each component of C provides a generating set for $N_qG(C)$. We also fix a system of loop-longitudes $\mathcal{L}_i(\Sigma)$ for Σ.

Now, let γ be a generic path in $\pi_1(\Sigma_i, p_i) \sqcup \mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big)$, for some $i \in \{1, \dots, \ell\}$. Let ω_γ be a representative word in $\{R_i^{\pm 1}\}$ for the associated i-th longitude in $N_qG(C)$. In what follows, and abusing terminologies, we shall often blur the distinction between the path γ and the associated i-th longitude, seen either in $G(C)$ or in $N_qG(C)$.

Definition 4.1. The *Magnus expansion* of an element $w \in F$ is the formal power series $E(w) \in \mathbb{Z}\langle\langle \ell \rangle\rangle$ in ℓ noncommuting variables $X_1, \dots, X_\ell$, obtained by substituting each R_i by $1 + X_i$, and each R_i^{-1} by $1 - X + X^2 - \cdots$.
For any sequence $I = j_1 \cdots j_k$ of integers in $\{1, \dots, \ell\}$, we denote by $\mu_C(I; w)$ the coefficient of the monomial $X_{j_1} \cdots X_{j_k}$ in $E(w)$, meaning that

$$E(w) = 1 + \sum_{j_1 \cdots j_k} \mu_C(j_1 \cdots j_k; w) X_{j_1} \cdots X_{j_k},$$

where the sum runs over all sequences of (possibly repeated) integers in $\{1, \dots, \ell\}$.

Of course, the coefficients $\mu_C(I; \omega_\gamma)$ are not invariants for the i-th longitude γ, as they depend on the choice of the representative word ω_γ. But Theorem 3.6 tells us precisely how different representatives may differ.

In order to extract invariants from the numbers $\mu_C(I; \omega_\gamma)$, we introduce the following indeterminacies.

Definition 4.2. For any sequence Ii of integers in $\{1, \dots, \ell\}$, we define

- $m_C(Ii) := \gcd\big\{\mu_C(I; \omega_{ij}) \text{ for all } j\big\}$;
- $\Delta_C(I)$ is the greatest common divisor of all $m_C(J)$, where J is any sequence obtained from I by deleting at least one index, and possibly permuting the resulting sequence cyclically.

We emphasize the fact that the definition of $\Delta_C(I)$ only involves *loop*-longitudes on Σ_i, for all $i = 1, \dots, \ell$.

Proposition 4.3. ([3, Proposition 4.3]) *For any sequence I of length at most q and any $\gamma \in \pi_1(\Sigma_i, p_i) \sqcup \mathcal{A}\big(\Sigma_i, \{p_{ij}\}_j\big)$ for some $i \in \{1, \dots, \ell\}$, the residue class*

$$\overline{\mu}_C(I; \gamma) := \mu_C\big(I; \omega_\gamma\big) \mod \Delta_C(Ii)$$

does not depend on the representative word ω_γ.

This allows for the following definition.

Definition 4.4. The classes $\overline{\mu}_C(I;\gamma)$ are called *Milnor numbers* associated with γ.

We stress that, for Proposition 4.3, a choice of one meridian per component has been made, and $\overline{\mu}_C(I;\gamma)$ and $\Delta_C(Ii)$ may depend on this choice; this will be further discussed in the next subsection. The indeterminacy $\Delta_C(Ii)$ also depends on the choice of system of loop-longitude.

Observe also that Milnor numbers are well-defined regardless of the initially fixed value for q. Indeed, the lower central series being decreasing, any representative word for an element of $G(C)$ in $N_qG(C)$ is also a representative word for its image in $N_{q'}G(C)$, for any $q' < q$. In particular, by taking q to be sufficiently large, Milnor numbers are defined for sequences of arbitrary length.

Lemma 4.5. ([3, Lemma 4.6]) *Let I be a sequence of integers in $\{1,\dots,\ell\}$, and let $\omega_i \in F$ be a representative word of an i-th loop-longitude. Let $w_1, w_2 \in F$. Then*

$$\mu_C(I; w_1\omega_i w_2) \equiv \mu_C(I;\omega_i) + \mu_C(I; w_1w_2) \mod \Delta_C(Ii).$$

Combining this and Proposition 4.3, we have the following lemma.

Lemma 4.6. ([3, Lemma 4.7]) *For any sequence Ii of integers in $\{1,\cdots,\ell\}$, we have that* $\gcd\left\{\Delta_C(Ii), m_C(Ii)\right\}$ *and the indeterminacy $\Delta_C(Ii)$ are independent of the systems of loop-longitudes $\mathcal{L}_i(\Sigma)$, and of the choice of representative words ω_{ij}.*

4.2 Milnor invariants for loop-longitudes

We first focus on the case where γ is (an element of $\pi_1(\Sigma_i;p_i)$ representing) an i-th loop-longitude.

As mentioned in the previous section, we need to analyse the effect of replacing a basepoint by a conjugate. Such a basepoint change transforms a nilpotent peripheral system into an equivalent one; the meridian and loop-longitudes are conjugated as in Definition 2.11.

Definition 4.7. For $\lambda \in H_1(\Sigma_i)$ and any sequence I, we set

$$\overline{\mu}_C(I;\lambda) := \overline{\mu}_C(I;\gamma),$$

where γ is a generic loop representing λ.

Then we have the following proposition.

Proposition 4.8. ([3, Proposition 4.9]) *For any sequence I, and $\lambda \in H_1(\Sigma_i)$, $\Delta_C(I)$ and $\overline{\mu}_C(I;\lambda)$ do not depend on the choice of meridians $\{R_i\}$.*

This leads to the following.

Definition 4.9. For any sequence Ii of integers in $\{1,\cdots,\ell\}$, we define the *Milnor map*

$$M_C^{Ii} : H_1(\Sigma_i) \to \mathbb{Z}/\Delta(Ii)\mathbb{Z},$$

by sending any $\lambda \in H_1(\Sigma_i)$ to the residue class $\overline{\mu}_C(I;\lambda)$.

Remark 4.10. In the case of classical links, $H_1(\Sigma_i) \cong \mathbb{Z}$ is generated by a preferred i-th longitude, for each i, and we recover Milnor's link invariants as the image of the corresponding Milnor maps.

By Proposition 4.8, we obtain the following.

Theorem 4.11. ([3, Theorem 4.12]) *The Milnor maps are well-defined invariants of C.*

Furthermore, we can define numerical Milnor invariants of the cut-diagram C from loop-longitudes; they essentially record the images of Milnor maps, but will also appear in the next section as indeterminacies for Milnor invariants defined from arc-longitudes.

Definition 4.12. For any sequence I, we set

$$\nu_C(Ii) := \gcd\left\{\Delta_C(Ii), m_C(Ii)\right\}$$

and call it *Milnor loop-invariant* of C associated with the sequence I and the i-th component.

Here we use the usual convention that $\gcd(0,0) = 0$. By Lemma 4.6, and Proposition 4.8, we have the following.

Theorem 4.13. ([3, Theorem 4.14]) *The Milnor loop-invariants are well-defined invariants of C.*

Remark 4.14. As in the classical settings, Milnor invariants are ordered by their *length*, which is the number of indices in the indexing sequence. For a cut-diagram C, suppose that there is a largest $k > 0$ such that $m_C(J) = 0$ for any sequence J of less than k indices (we set $m_C(i) = 0$ for any i, as a convention). Then for any sequence I of length $k+1$ we have $\Delta_C(I) = 0$, and the *first non-vanishing* Milnor (loop-)invariants of C are the nontrivial invariants $\nu_C(I)$, which are simply given by $\nu_C(I) = m_C(I) \in \mathbb{Z}$.

Remark 4.15. Proposition 4.8 tells us that Milnor maps and Milnor loop-invariants are well-defined invariants for nilpotent peripheral systems *up to equivalence*.

4.3 Milnor invariants from arc-longitudes

We now focus on the case where γ is an i-th arc-longitude. Dealing with an i-th arc-longitude obviously assumes that Σ_i is not closed. It is hence rather natural to assume that *each* component of Σ has nonempty boundary. Note that this is in particular the case for string links[8] and concordances in 4–space [9].

Recall that we chose basepoints $\{p_{ij}\}$ on each boundary component, among which the point p_{i0} indicates a marked boundary component of Σ_i. This piece of data comes with the cut-diagram C, and is fixed, once and for all. This provides a *canonical* choice of i-th meridian, materialized by the basepoint p_{i0}, and hence well-defined Milnor numbers associated to each arc-longitude. In other words, as a corollary of Proposition 4.3, we obtain the following.

Theorem 4.16. ([3, Theorem 4.20]) *If Σ has no closed component, Milnor numbers associated to arc-longitudes are well-defined invariants of C.*

Remark 4.17. In the general case, where some component Σ_j of Σ may have empty boundary, the above result remains true, provided that we *fix* a choice of j-th meridian for Σ_j. Hence, in general, Milnor numbers associated to arc-longitudes are well-defined invariants of C, enhanced with a basepoint on each closed connected component of Σ (in addition to the already fixed boundary basepoints).

In general, there are however several, possibly infinitely many, different arc-longitudes running from p_{i0} to some other boundary basepoint p_{ij}. But using the action of loop-longitudes noted in Remark 2.10, all arc-longitudes can be recovered from a single one. More precisely, for any two arcs γ, γ' running from p_{i0} to p_{ij}, there exists by Remark 2.10 some $\tau \in \pi_1(\Sigma_i, p_{i0})$ such that $\gamma' = \tau.\gamma$.[1] By Lemma 4.5, we have for any sequence I of integers in $\{1, \dots, \ell\}$,

$$\mu_C(I;\gamma') \equiv \mu_C(I;\gamma) + \mu_C(I;\tau) \mod \Delta_C(Ii).$$

Since τ decomposes as a product of elements in a system of longitudes (and their inverses), we therefore observe that

$$\mu_C(I;\gamma') \equiv \mu_C(I;\gamma) \mod \nu_C(Ii),$$

where $\nu_C(Ii) = \gcd\left\{\Delta_C(Ii), m_C(Ii)\right\}$ was introduced in Definition 4.12.

We can thus define intrinsic Milnor invariants from arc-longitudes, as follows.

Definition 4.18. For any sequence I of integers in $\{1, \dots, \ell\}$, we set

$$\nu^\partial_C(I;ij) := \mu_C(I;\gamma_{ij}) \mod \nu_C(Ii),$$

where γ_{ij} is any path from p_{i0} to p_{ij}. We call it *Milnor arc-invariant* of C associated with the sequence I and the j-th boundary component of Σ_i.

Remark 4.19. As in Remark 4.15, we have that Milnor arc-invariants are well-defined invariants for nilpotent peripheral systems up to equivalence, since our notion of equivalence fixes the boundary, see Definition 2.11.

4.4 Concordance invariance results

By Remarks 4.15 and 4.19, Milnor invariants indexed by sequences I of length at most q indices only depend on the q-th nilpotent peripheral system. As a consequence, we obtain the following.

Corollary 4.20. ([3, Corollary 5.6]) *Milnor invariants for cut-diagrams are invariant under cut-concordance.*

Propositions 1.8 and 4.20 provide our main topological invariance result.

Corollary 4.21. ([3, Corollary 5.7]) *Milnor invariants are well-defined concordance invariants for codimension* 2 *embeddings.*

This allows for the following definition.

Definition 4.22. For any knotted manifold S we define Milnor invariants—which are, for any sequence I of indices in $\{1, \dots, \ell\}$, Milnor maps M^{Ii}_S, Milnor loop-invariants $\nu_S(Ii)$, and Milnor arc-invariants $\nu^\partial_S(I;ij)$—as the corresponding invariants for any cut-diagram associated to S.

[1] Observe that, since Σ_i has nonempty boundary, we picked here $p_i = p_{i0}$ as canonical basepoint for Σ_i, for defining this action.

4.5 Self-singular cut-concordance

We also introduce in [3, § 5.2] the notion of *self-singular concordance* for cut-diagrams. This is an equivalence relation on cut-diagrams implied by cut-concordance, and which is closely related to the topological notion of link-homotopy. (Recall that a link-homotopy is a continuous deformation during which distinct components remain disjoint, but each component may intersect itself.) We show in particular the following.

Proposition 4.23. ([3, Proposition 5.23]) *Non-repeated Milnor invariants for surface links are invariant under link-homotopy.*

Here, we say that a Milnor number $\overline{\mu}_C(I;\gamma)$ for a cut-diagram C and a path γ on the i-th component of Σ, is *non-repeated* if all the indices in Ii are pairwise distinct. This is a natural analogue of a result of Milnor in the link case [10].

5 Examples and applications

In this section, we gather several topological applications of Milnor invariants for cut-diagrams.

5.1 Milnor invariants of Spun links

Spun links refer to a classical construction due to Artin [1], which produces knotted surfaces from classical tangles as follows. Consider in $\mathbb{R}^4$ the upper 3–dimensional space $\mathbb{R}^3_+ = \{(x,y,z,0)\,|\,x,y\in\mathbb{R},\ z\geq 0\}$; the (x,y)–plane $P_{xy}=\{(x,y,0,0)\,|\,x,y\in\mathbb{R}\}$ sits as the boundary of $\mathbb{R}^3_+$. Given a 1–dimensional compact manifold X properly embedded in $\mathbb{R}^3_+$, the *Spun of X* is obtained by spinning X around P_{xy} inside $\mathbb{R}^4\supset\mathbb{R}^3_+$:

$$\mathrm{Spun}(X) := \left\{\left(x,y,z\cos\theta,z\sin\theta\right)\,|\,(x,y,z,0)\in X,\ \theta\in[0,2\pi]\right\}. \tag{5.1}$$

Our Milnor invariants are well-behaved under the Spun construction:

Lemma 5.1. ([3, Lemma 6.1]) *Let X be a tangle in $\mathbb{R}^3_+$ as above.*
If the i-th component of X is a knot, then for any sequence I we have

$$m_{\mathrm{Spun}(X)}(Ii) \equiv |\mu_X(Ii)| \mod \Delta_X(Ii).$$

If the i-th component of X is an arc, then for any sequence J we have $m_{\mathrm{Spun}(X)}(Ji) = \nu_{\mathrm{Spun}(X)}(Ji) = 0$.

Remark 5.2. When the i-th component of X is a knot, we have in particular that the first non-vanishing Milnor loop-invariants of Spun(X) are given by the first non-vanishing Milnor invariants of X by $\nu_{\mathrm{Spun}(X)}(Ii) = |\mu_X(Ii)|$.

Remark 5.3. The Spun construction can be generalized to any dimension in a straightforward way. Hence one can iterate this construction to build a codimension 2 knotted submanifold, and we have a similar result as Lemma 5.1 in any dimension.

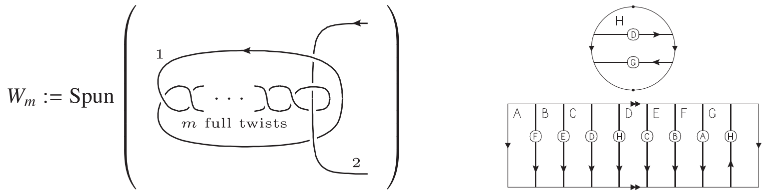
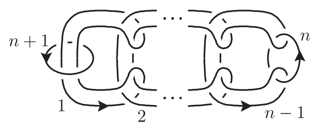


Figure 7: Definition of the knotted surfaces $W_m = \mathrm{Spun}(X_m)$, and a cut-diagram for W_3

As an application, we can give a general realization result for our generalized Milnor invariants using the Spun construction, as follows. Consider the $(n+1)$–component link M_{n+1} shown below, known as *Milnor's link*.



Milnor observed that all Milnor invariants of length $\leq n$ vanish for M_{n+1}, and that for any permutation σ in S_{n-1}, we have $\overline{\mu}_{M_{n+1}}(\sigma(1)\cdots\sigma(n-1)n\,n+1) = \begin{cases} 1 & \text{if } \sigma = \mathrm{Id}, \\ 0 & \text{otherwise.} \end{cases}$

Using Lemma 5.1 and Remark 5.2, we directly obtain a similar realization result for the first non-vanishing Milnor loop invariants $\nu(1\cdots n{+}1)$ of knotted surfaces, by considering $\mathrm{Spun}(M_{n+1})$. More generally, by iterating the Spun construction and using Remark 5.3, we obtain in this way a similar result in any dimension.

5.2 A classification result up to concordance

We now compare the relative strength of our Milnor invariants of knotted surfaces with previous invariants defined in the litterature.

Definition 5.4. For every $m \in \mathbb{N}$, we define W_m as the spun surface obtained by spinning the tangle X_m described in Figure 7.

Remark 5.5. The family W_m can be extended to negative values of m, but this amounts to spinning the mirror image of the tangle X_{-m} of Figure 7. Hence, W_{-m} is isotopic to W_m.

Many of the previously known concordance invariants of knotted surfaces cannot detect this familly of Spun links. Indeed, for all $m \in \mathbb{N}$, we have that

- the Sato–Levine invariant [16] vanishes on W_m,
- Cochran's derivation invariants [7] all vanishes on W_m,
- Saito's invariants [15] of W_m are equal for all values of $m \in \mathbb{N}$,
- since W_m is link-homotopic to a union of trivially embedded torus and sphere for all $m \in \mathbb{N}$, all link-homotopy invariants vanish on W_m.

In contrast, using Milnor loop-invariants we have the following.

Proposition 5.6. ([3, Proposition 6.6]) *For any $m_1, m_2 \in \mathbb{N}$, W_{m_1} and W_{m_2} are concordant if and only if $m_1 = m_2$.*

5.3 Link-homotopy classification results

Milnor showed in [10] that links of 3 components are classified up to link-homotopy by their non-repeated Milnor invariants of length ≤ 3. Using the good behavior of our Milnor invariants under the Spun construction, we show the following.

Proposition 5.7. ([3, Proposition 6.7]) *Let L and L' be 3–component links. Then* $\mathrm{Spun}(L)$ *and* $\mathrm{Spun}(L')$ *are link-homotopic if and ond only if L is link-homotopic to either L' or its mirror image.*

For Spun links with an arbitrary number of components, we also have the following consequence of Lemma 5.1.

Proposition 5.8. ([3, Proposition 6.9]) *Let L be an m–component link. The following are equivalent:*

(i) $\mathrm{Spun}(L)$ *is link-homotopically trivial;*

(ii) $\nu_{\mathrm{Spun}(L)}(I) = m_{\mathrm{Spun}(L)}(I) = 0$ *for any non-repeated sequence I;*

(iii) L is link-homotopically trivial.

Let $n \geq 2$ be some integer, and $p_1, \dots, p_n$ some positive integers. We call *knotted punctured spheres* any knotted surface which is a proper embedding of $\sqcup_{i=1}^n S^2_{p_i}$ into the 4–ball B^4, where $S^2_{p_i}$ is the 2–sphere with p_i holes, and such that the boundary is mapped to the trivial link with $\sum_i p_i$ components in $S^3 = \partial B^4$.

We have the following classification result, which generalizes [2, Thm. 4.8] (which corresponds to the case $p_i = 2$ for all i).

Theorem 5.9. ([3, Theorem 6.10]) *Knotted punctured spheres are classified up to link-homotopy by Milnor invariants.*

Note that knotted punctured spheres do not contain any nontrivial loop longitude, since the boundary is assumed to be a trivial link. Therefore, we actually make use here of Milnor arc-invariants.

References

[1] E. Artin. Zur isotopie zweidimensionalen flächen im R_4. *Abh. Math. Sem. Univ. Hamburg*, 4:174–177, 1926.

[2] B. Audoux, J.-B. Meilhan, and E. Wagner. On codimension two embeddings up to link-homotopy. *J. Topol.*, 10(4):1107–1123, 2017.

[3] B. Audoux, J.-B. Meilhan, and A. Yasuhara. Milnor concordance invariant for knotted surfaces and beyond. arXiv:2109.14578, 2021.

[4] J. S. Carter, S. Kamada, and M. Saito. *Surfaces in* 4*–space*, volume 142 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Low-Dimensional Topology, III.

[5] J. S. Carter and M. Saito. *Knotted surfaces and their diagrams*, volume 55. Providence, RI: American Mathematical Society, 1998.

[6] K.-T. Chen. Commutator calculus and link invariants. *Proc. Amer. Math. Soc.*, 3(4):44–55, 1952.

[7] T. D. Cochran. Geometric invariants of link cobordism. *Comment. Math. Helv.*, 60:291–311, 1985.

[8] N. Habegger and X.-S. Lin. The classification of links up to link-homotopy. *J. Amer. Math. Soc.*, 3:389–419, 1990.

[9] J.-B. Meilhan and A. Yasuhara. Link concordances as surfaces in 4–space. *to appear in Indiana Univ. Math. J.*

[10] J. Milnor. Link groups. *Ann. of Math. (2)*, 59:177–195, 1954.

[11] J. Milnor. Isotopy of links. Algebraic geometry and topology. In *A symposium in honor of S. Lefschetz*, pages 280–306. Princeton University Press, Princeton, N. J., 1957.

[12] J. H. Przytycki and W. Rosicki. Cocycle invariants of codimension 2 embeddings of manifolds. In *Knots in Poland III. Proceedings of the 3rd conference, Stefan Banach International Mathematical Center, Warsaw, Poland, July 18–25, 2010 and Będlewo, Poland, July 25 – August 4, 2010*, pages 251–289. Polish Academy of Sciences, Institute of Mathematics, 2014.

[13] D. Roseman. Projections of codimension two embeddings. In *Knots in Hellas '98. Proceedings of the international conference on knot theory an its ramifications, European Cultural Centre of Delphi, Greece, August 7–15, 1998*, pages 380–410. Singapore: World Scientific, 2000.

[14] D. Roseman. Elementary moves for higher dimensional knots. *Fundam. Math.*, 184:291–310, 2004.

[15] M. Saito. A note on cobordism of surface links in S^4. *Proc. Am. Math. Soc.*, 111(3):883–887, 1991.

[16] N. Sato. Cobordisms of semi-boundary links. *Topology Appl.*, 18:225–234, 1984.

[17] J. Stallings. Homology and central series of groups. *J. Algebra*, 2:170–181, 1965.

Benjamin AUDOUX
Aix Marseille Univ, CNRS, Centrale Marseille,
I2M, Marseille, France

Jean-Baptiste MEILHAN
Univ. Grenoble Alpes, CNRS, Institut Fourier,
F-38000 Grenoble, France

Akira YASUHARA
Faculty of Commerce, Waseda University,
1-6-1 Nishi-Waseda, Tokyo 169-8050, Japan

エクス・マルセイユ大学　オードー，ベンジャミン
グルノーブル・アルプス大学　メイヨン，ジャン・バプテスト
早稲田大学　安原 晃