An extension of Ford domain

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1 Introduction

A knot in the 3-sphere is either a torus knot, a satellite knot, or a hyperbolic knot, by the uniformization theorem for Haken manifolds due to Thurston. The complement of a hyperbolic knot admits a complete hyperbolic structure of finite volume, which is unique by the Mostow-Prasad rigidity. Thus it is important to understand the hyperbolic structures from the view point of 3-manifold theory. For that purpose, theories of canonical fundamental domains for hyperbolic manifolds are developed. A hyperbolic knot complement, with the canonical hyperbolic structure, contains a unique *cusp*, which is an end of the manifold with a neighborhood isometric to the quotient of a horoball in the hyperbolic 3-space, $\mathbb{H}^3$, by a discrete group of isometries isomorphic to $\mathbb{Z}^2$ stabilizing it. By using the information of the cusp, there are two canonical fundamental domains of the manifold. One is the *Ford domain*, and the other is the *canonical decomposition* defined by Epstein and Penner [4], where it is pointed out that the canonical decomposition is dual to the Ford domain. The assumption that the manifold is of finite volume is not necessary for the definition of Ford domain. Epstein-Penner's construction of canonical decomposition can be applied to the manifolds of infinite volume, and one can obtain a decomposition of the convex core, which is called the *EPH-decomposition* [1]. In this case, the Ford domain is dual to the Euclidean subcomplex of the EPH-decomposition.

The aim of this talk is to extend the Ford domain for a cusped hyperbolic manifold of infinite volume to the outside of hyperbolic space. The hyperbolic space is canonically regarded as a subspace of the real projective space, and the extended Ford domain is a compact convex set in the projective space. We also define a compact convex subset of the projective space from the convex set which appears in Epstein-Penner's construction, and observe that the two convex sets are polar duals to each other. We also study quasifuchsian punctured torus groups contained in a rational pleating variety, and obtain the facial structures of the convex sets (Theorem 6.2).

2 Preliminaries

The *Minkowski space* $\mathbb{E}^{1,d}$ is the real vector space of dimension $d+1$ together with the Minkowski bilinear form $\langle \cdot, \cdot \rangle$, where

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{d} x_i y_i$$

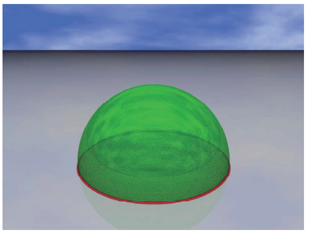

Figure 1: Isometric hemisphere: the underlying space of the upper half space model for $\mathbb{H}^3$ is the upper half space of $\mathbb{R}^3$, whose ideal boundary is identified with the one-point compactification $\widehat{\mathbb{C}}$ of the horizontal plane of height 0 identified with $\mathbb{C}$.

for $x = (x_0, \ldots, x_d)$, $y = (y_0, \ldots, y_d) \in \mathbb{E}^{1,d}$. In this paper, we often regard the underlying set of $\mathbb{E}^{1,d}$ as $\mathbb{R} \times \mathbb{R}^d$; for $x = (x_0, x_1), y = (y_0, y_1) \in \mathbb{E}^{1,d}$, $\langle x, y\rangle = -x_0 y_0 + x_1 \cdot y_1$, where $x_1 \cdot y_1$ denotes the Euclidean inner product of $x_1, y_1 \in \mathbb{R}^d$.

The *hyperboloid model* of the d-dimensional hyperbolic space $\mathbb{H}^d$ is defined by

$$\mathbb{H}^d = \{x = (x_0, x_1) \in \mathbb{E}^{1,d} \mid \langle x, x\rangle = -1,\ x_0 > 0\},$$

where the inner product on the tangent space $T_x\mathbb{H}^d$ at each $x \in \mathbb{H}^d$ is defined by the restriction of $\langle\cdot,\cdot\rangle$ on $T_x\mathbb{H}^d$. This defines the *hyperbolic metric* d on $\mathbb{H}^d$.

The subspace $L^+ = \{x = (x_0, x_1) \in \mathbb{E}^{1,d} \mid \langle x, x\rangle = 0,\ x_0 > 0\}$ is called the *positive light cone*. For $v, v' \in L^+$, let $v \sim v'$ if $v' = kv$ for some $k > 0$. Then $\sim$ is a equivalence relation on L^+. The quotient space $L^+/\sim$ is regarded as the ideal boundary $\partial\mathbb{H}^d$ of the hyperbolic space. We denote the projection by $\pi : L^+ \to \partial\mathbb{H}^d$.

For any $v \in L^+$, the set $H_v = \{x \in \mathbb{H}^d \mid \langle v, x\rangle \geq -1\}$ is a *horoball* with *center* $\pi(v) \in \partial\mathbb{H}^d$. There is a one-to-one correspondence between L^+ and the set of horoballs in $\mathbb{H}^d$ by this correspondence. For $x \in \mathbb{H}^d$ and a horoball H in $\mathbb{H}^d$, let δ be the distance between x and ∂H. The *signed* distance $d(x, \partial H)$ is defined to be δ if $x \notin H$, otherwise $-\delta$. By a calculation, we obtain $d(x, \partial H_v) = \log(-\langle v, x\rangle)$ for any $x \in \mathbb{H}^d$ and $v \in L^+$.

For $a \in \partial\mathbb{H}^d$ and $\gamma \in \mathrm{Isom}^+(\mathbb{H}^d)$ which does not stabilize a, pick a horoball H with center a and define

$$E_a(\gamma) = \{x \in \mathbb{H}^d \mid d(x, \partial H) \leq d(x, \gamma^{-1}(\partial H))\}.$$

This does not depend on the choice of H and called the *isometric hemisphere* of γ with respect to a. The terminology will be natural in the case of $d = 3$ and by employing the upper half space model for $\mathbb{H}^3$. The ideal boundary $\partial\mathbb{H}^3$ is canonically identified with the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then, for the choice $a = \infty$, $E_\infty(\gamma)$ is the Euclidean hemisphere in the upper half space with equator $\{z \in \mathbb{C} \mid |rz + s| = 1\}$ for $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. (See Figure 1.)

We introduce another model for the hyperbolic space. Define $\mathbb{K} \subset \mathbb{E}^{1,d}$ by

$$\mathbb{K}^d = \{x = (x_0, x_1) \in \mathbb{E}^{1,d} \mid x_0 = 1,\ x_1 \cdot x_1 < 1\}.$$

Then the ray emanating from the origin, O, of $\mathbb{E}^{1,d}$ which passes through a point in $\mathbb{H}^d$ intersects $\mathbb{K}^d$ at a single point. This induces a one-to-one correspondence between $\mathbb{H}^d$ and

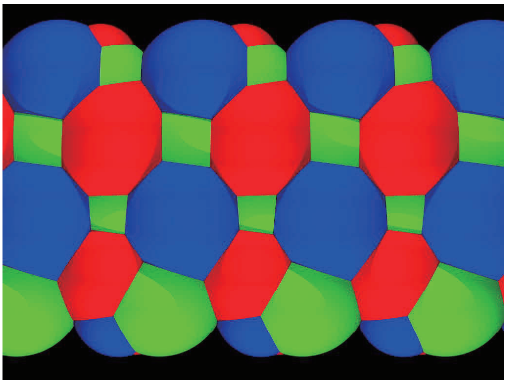

Figure 2: Ford domain for a quasifuchsian punctured torus group, looked down from ∞ in the upper half space model for $\mathbb{H}^3$

$\mathbb{K}^d$. We identify $\mathbb{H}^d$ and $\mathbb{K}^d$, and call $\mathbb{K}^d$ the *projective space model* or the *Klein model* for the hyperbolic space. The equivalence relation $\sim$ on L^+ is extended to the one on $\mathbb{E}^{1,d} \setminus \{O\}$ so that $x \sim x'$ if $x' = kx$ for some $k \neq 0$. The quotient space $(\mathbb{E}^{1,d} \setminus \{O\})/\sim$ is the d-dimensional projective space $\mathbb{RP}^d$. Let $\pi : \mathbb{E}^{1,d} \setminus \{O\} \to \mathbb{RP}^d$ be the projection. Then the restrictions of π on $\mathbb{H}^d$ and $\mathbb{K}^d$ are injective, and so the hyperbolic space is naturally embedded into the projective space. From the definition, the embedding naturally extends to the ideal boundary. In fact, the closure $\overline{\mathbb{K}^d}$ embeds into $\mathbb{RP}^d$ and is naturally identified with $\mathbb{H}^d \cup \partial\mathbb{H}^d$.

3 Ford domain and Epstein-Penner's convex hull construction

Let Γ be a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^d)$. Fix a point a in $\partial\mathbb{H}^d$, and let Γ_a be the stabilizer subgroups of Γ with respect to a. The *Ford domain* P_a of Γ with respect to a is the intersection of all $E_a(\gamma)$ for $\gamma \in \Gamma \setminus \Gamma_a$. (See Figure 2, which illustrates the Ford domain of a quasifuchsian punctured torus group, where a quasifuchsian punctured torus group is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ obtained as a quasifuchsian deformation of a fuchsian group of a once-punctured torus.) For a fundamental domain Q_a for the action of Γ_a on $\mathbb{H}^d$, the intersection $P_a \cap Q_a$ is a fundamental domain for the action of Γ on $\mathbb{H}^d$.

Epstein and Penner [4] introduced a certain ideal polyhedral decomposition of a cusped hyperbolic manifold of finite volume, which is closely related to Ford domain. In what follows, we give a quick review of their construction, which uses the Minkowski space model.

Let M be a d-dimensional complete hyperbolic manifold with a unique cusp. Then there exists a discrete subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^d)$ such that $\mathbb{H}^d/\Gamma$ is isometric to M. Let $p : \mathbb{H}^d \to \mathbb{H}^d/\Gamma = M$ be the projection. We obtain a family of horoballs in $\mathbb{H}^d$ with disjoint interior as the inverse image of a neighborhood of the cusp by p. The neighborhood of cusp is isometric to H_v/Γ_a, where H_v is the horoball in the family corresponding to $v \in L^+$ with $\pi(v) = a$, and Γ_a is the stabilizer subgroup of Γ with respect to a. We assume that the cardinality of Γ_a is infinite. Let $\widehat{B}$ be the subset of L^+ consisting of the points corresponding to the family of horoballs. Then $v \in \widehat{B}$ as H_v is such a horoball. One of the key observations in [4] is that $\widehat{B}$ is a discrete subset of $\mathbb{E}^{1,d}$. Define $\widehat{C}$ to be the closed convex hull of $\widehat{B}$ in $\mathbb{E}^{1,d}$.

It is assumed in [4] that the volume of M is finite. Then any ray in $\mathbb{E}^{1,d}$ emanating from O which passes through a point of $\mathbb{H}^d$ intersects the boundary of $\widehat{C}$ at a single point. By this correspondence, the facial structure of $\partial\widehat{C}$ induces a locally finite tessellation on $\mathbb{H}^d$ with finite sided ideal polyhedra. Since the tessellation is Γ-invariant, it descends to an ideal polyhedral decomposition of M. The decomposition is called the Epstein-Penner decomposition or the *canonical decomposition* of M.

One can proceed a similar construction even if the volume of M is infinite (see [1]). Then one obtains a certain decomposition of a subspace of M whose closure is equal to the convex core. In this case the decomposition is called the *EPH-decomposition*.

In [4], it is pointed out that their construction is dual to the Ford domain. In what follows, we explain how the Ford domain is obtained from the set $\widehat{B} \subset L^+$, which may suggest their main idea. In the Minkowski space model, the set $E_a(\gamma)$ is determined by using the Minkowski bilinear form. Since $\gamma^{-1}H_v = H_{\gamma^{-1}v}$, the inequality $d(x, \partial H_v) \leq d(x, \gamma^{-1}(\partial H_v))$ holds if and only if $\langle \gamma^{-1}v - v, x\rangle \leq 0$. Thus $E_a(\gamma) = \{x \in \mathbb{H}^d \mid \langle \gamma^{-1}v - v, x\rangle \leq 0\}$. Since M has only one cusp, $\widehat{B} = \{\gamma v \mid \gamma \in \Gamma\}$, and

$$P_a = \bigcap_{w \in \widehat{B}} \{x \in \mathbb{H}^d \mid \langle w - v, x\rangle \leq 0\}. \tag{1}$$

For cusped manifolds of infinite volume, the Ford domain is dual to the Euclidean subcomplex of EPH-decomposition [1].

4 Extended Ford domain

In this section, we introduce two convex sets $\widehat{T}_a$ and $\widehat{D}_a$ in $\mathbb{E}^{1,d}$ which naturally arise from the equality (1). Each of them has a structure of closed convex cone whose base is a compact convex set, under a certain additional condition. The bases of $\widehat{T}_a$ and $\widehat{D}_a$, regarded as subsets of $\mathbb{R}^d$, are defined to be T_a and D_a, and we call D_a the “extended” Ford domain. We can see that the pair $\widehat{T}_a$, $\widehat{D}_a$ (resp. T_a, D_a) is polar duals to each other as convex sets.

Definition 4.1. Define the subsets $\widehat{D}_a$ and $\widehat{T}_a$ of $\mathbb{E}^{1,d}$ by

$$\widehat{D}_a = \{x \in \mathbb{E}^{1,d} \mid \langle w - v, x\rangle \leq 0 \text{ for any } w \in \widehat{B}\},$$
$$\widehat{T}_a = \{u \in \mathbb{E}^{1,d} \mid v + ku \in \widehat{C} \text{ for some } k > 0\}.$$

From the definition, $\widehat{D}_a$ is a closed convex cone in $\mathbb{E}^{1,d}$ with apex O. We can see that $\widehat{T}_a$ is also a closed convex cone in $\mathbb{E}^{1,d}$ with apex O by the following lemma, whose proof uses the fact that $\widehat{B}$ is a discrete subset of $\mathbb{E}^{1,d}$.

Lemma 4.2. *$\widehat{T}_a$ is a closed subset of $\mathbb{E}^{1,d}$.*

The set $\widehat{D}_a$ is an extension of the Ford domain in the following sense.

Proposition 4.3. *$\widehat{D}_a \cap \mathbb{H}^d$ is equal to P_a.*

The *tangent cone* of $\widehat{C}$ at $v \in \partial\widehat{C}$ is the closure of the union of rays emanating from v and intersecting $\widehat{C}$ in at least one point distinct from v. By Lemma 4.2, $\widehat{T}_a$ is the image of the tangent cone by the parallel translation which maps the apex to O.

For a closed convex cone A in $\mathbb{E}^{1,d}$, the *polar dual*, A°, of A is defined by

$$A^\circ = \{x \in \mathbb{E}^{1,d} \mid \langle u, x\rangle \leq 0 \text{ for any } u \in A\}.$$

It is a standard knowledge in the theory of convex sets that $A^{\circ\circ} = (A^\circ)^\circ = A$.

Proposition 4.4. *$\widehat{T}_a$ and $\widehat{D}_a$ are polar duals to each other, namely, the equalities $\widehat{T}_a^\circ = \widehat{D}_a$ and $\widehat{D}_a^\circ = \widehat{T}_a$ hold.*

From now on, we further assume that the convex hull, $\mathrm{Hull}(\Lambda_\Gamma)$, of the limit set Λ_Γ of Γ in $\mathbb{H}^d$ contains an interior point.

Definition 4.5. Let $D_a = \widehat{D}_a \cap \mathbb{R}^d_1$ and $T_a = \widehat{T}_a \cap \mathbb{R}^d_1$, where $\mathbb{R}^d_1 = \{x \in \mathbb{R}^d \mid (1, x) \in \mathbb{E}^{1,d}\}$. We call D_a the *extended Ford domain* for Γ with respect to a.

Proposition 4.6. *By choosing the universal cover $p : \mathbb{H}^d \to M$ appropriately, both D_a and T_a are compact, and $\widehat{D}_a$ and $\widehat{T}_a$ are the convex cones over D_a and T_a with apex O, respectively.*

For a compact convex set A in $\mathbb{E}^d$, the *polar dual*, A°, of A is defined by

$$A^\circ = \{x \in \mathbb{E}^d \mid u \cdot x \leq 1 \text{ for any } u \in A\}.$$

It is another standard knowledge in the theory of convex sets that $A^{\circ\circ} = (A^\circ)^\circ = A$.

Proposition 4.7. *T_a and D_a are polar duals to each other, namely, the equalities $T_a^\circ = D_a$ and $D_a^\circ = T_a$ hold.*

5 Facial structure of T_a

A convex set has a facial structure. Let K be a convex set in $\mathbb{R}^d$. A *face* of K is a convex subset $F \subset K$ such that each segment in K whose interior intersects F is contained entirely in F. The example of a face which comes up immediately will be an *exposed face* defined as follows. A hyperplane W is a *support plane* to K if $W \cap \partial K \neq \emptyset$ and W bounds a half space containing K. An exposed face of K is the intersection of K and a support plane. It is known that the family of relative interiors of the faces of K gives a partition of K, which is called the facial structure of K. (For details, see [8] for example.)

As stated in Proposition 4.7, the sets T_a and D_a are dual to each other as convex sets. Our interest is in their facial structures.

Definition 5.1. Let $\mathcal{F}(D_a)$ and $\mathcal{F}(T_a)$ be the sets of faces of D_a and T_a, respectively.

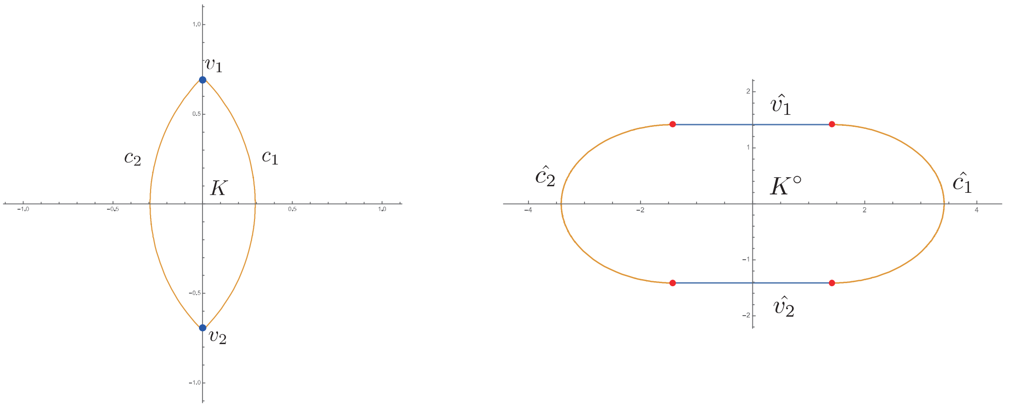

Figure 3: Polar dual convex sets: K and K° are the convex sets bounded by Jordan curves depicted in the above pictures. Some of the 0-dimensional faces of K° are not "visible" from the faces of K.

One may expect that Proposition 4.7 implies a one-to-one correspondence between $\mathcal{F}(D_a)$ and $\mathcal{F}(T_a)$. Even though we know the following general property, the duality of convex sets are more complicated. (See Example 5.3 below for a simple but "nontrivial" example.)

Proposition 5.2. *Let K be a compact convex set in $\mathbb{R}^d$ which contains the origin of $\mathbb{R}^d$ as an interior point. Then there is a one-to-one inclusion-reversing correspondence between the sets of exposed faces of K and K°.*

For a reference to the above proposition, see [7, Lemma 23.10] for example, where the term "face" is used for "exposed face".

Example 5.3. Let K be the convex set illustrated in the left picture of Figure 5.3, which is the region bounded by the union of the two curves c_1 and c_2 sharing the endpoints v_1 and v_2. We can see that the singleton consisting of each point in the boundary is a 0-dimensional exposed face of K. In particular, K has uncountably many faces. The polar dual K° of K is illustrated in the right picture of Figure 5.3, which is the region bounded by the union of four curves $\hat{c}_1$, $\hat{c}_2$, $\hat{v}_1$ and $\hat{v}_2$. We can see that the singleton consisting of each $\hat{v}_1$, $\hat{v}_2$, and interior points of $\hat{c}_1$ and $\hat{c}_2$ is a 0-dimensional exposed face, and the segments $\hat{v}_1$ and $\hat{v}_2$ are 1-dimensional exposed face, whereas the four singletons $\hat{v}_i \cap \hat{c}_j$ $(i, j \in \{1, 2\})$ are *unexposed* faces. By the correspondence of Proposition 5.2, each interior point of c_1 (resp. c_2) corresponds to an interior point of $\hat{c}_1$ (resp. $\hat{c}_2$), whereas the 0-dimensional exposed face $\{v_1\}$ (resp. $\{v_2\}$) corresponds to the 1-dimensional exposed face $\{\hat{v}_1\}$ (resp. $\{\hat{v}_2\}$). The four unexposed faces of K° are not visible from the faces of K.

The faces of T_a are described as Proposition 5.5 below.

Definition 5.4. Let $\mathcal{F}(\widehat{T}_a)$ be the set of faces of $\widehat{T}_a$. Let $\mathcal{F}_v(\widehat{C})$ be the set of faces of $\widehat{C}$ containing v, and $\mathcal{F}_v^{(\geq 1)}(\widehat{C})$ be the subset of $\mathcal{F}_v(\widehat{C})$ consisting of the faces of dimension at least 1.

Proposition 5.5. *There is a one-to-one correspondence between* $\mathcal{F}_v^{(\geq 1)}(\widehat{C})$ *and* $\mathcal{F}(T_a)$*.*

The proof of the above proposition requires the following observation.

Definition 5.6. For any $f \in \mathcal{F}_v(\widehat{C})$ and $F \in \mathcal{F}(\widehat{T}_a)$, define

$$\mathrm{ext}(f) = \{ku \in \mathbb{E}^{1,d} \mid k > 0,\ v + u \in f\}, \quad \mathrm{res}(F) = \{v + u \mid u \in F,\ v + u \in \widehat{C}\}.$$

By using the discreteness of $\widehat{B}$ in $\mathbb{E}^{1,d}$, we obtain the following lemma.

Lemma 5.7. *For any* $f \in \mathcal{F}_v(\widehat{C})$ *and* $F \in \mathcal{F}(\widehat{T}_a)$*, we have* $\mathrm{ext}(f) \in \mathcal{F}(\widehat{T}_a)$ *and* $\mathrm{res}(F) \in \mathcal{F}_v(\widehat{C})$*. Moreover, this correspondence induces a one-to-one correspondence between* $\mathcal{F}_v(\widehat{C})$ *and* $\mathcal{F}(\widehat{T}_a)$*.*

6 Rational pleating variety of punctured torus groups

The space of (once-)punctured torus groups is decomposed into pleating varieties. (See [6] for example.) Here we write a brief idea of the decomposition. Let Γ be a quasi-fuchsian punctured torus group. The boundary of the convex core $\mathrm{Hull}(\Lambda_\Gamma)/\Gamma$ of $\mathbb{H}^3/\Gamma$ has two boundary components, each homeomorphic to the punctured torus and has a complete hyperbolic structure bent along a measured geodesic lamination, which is called the *bending measured lamination*. Let μ be a pair of projective measured laminations on the punctured torus, and $\mathcal{P}(\mu)$ the space of punctured torus groups such that the pair of bending measured laminations induces μ. This space is called a *pleating variety* of punctured torus groups. When both $\mu^\pm$ are rationals, namely the supports are simple loops, $\mathcal{P}(\mu)$ is called a *rational* pleating variety.

In [1] and [2], the combinatorial structures of EPH-decompositions for punctured torus groups are studied. They proposed a conjecture which says that the combinatorial structure of the EPH-decomposition of a group in a pleating variety of punctured torus groups is completely determined by the bending measured lamination, in particular it is invariant on a pleating variety. The conjecture has been proved affirmatively by Gueritaud [5]. In what follows, we give a quick review of the combinatorial structure for a group in a rational pleating variety by using an example. (See [1, 2] and [5] for details.) Actually, we describe the combinatorial structure of the intersection of a small horosphere H with center $a \in \partial\mathbb{H}^3$ and the lift $\widetilde{\Delta}$ of the EPH-decomposition to the universal cover $\mathbb{H}^3$. Let Γ be a punctured torus group in a rational pleating variety $\mathcal{P}(\mu)$. In what follows, we assume μ is a pair of projective measured laminations whose underlying sets are simple loops with slopes ∞ and $2/5$.

Let ℓ be the geodesic in $\mathbb{H}^2$ with endpoints ∞ and $2/5$, where the set $\mathbb{Q} \cup \{\infty\}$ of slopes of simple loops on the punctured torus is canonically identified with a subset of $\partial\mathbb{H}^2$. Let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be the sequence of triangles in the Farey tessellation of $\mathbb{H}^2$ which intersects ℓ in the interior (see the left picture of Figure 4). Then we can define a simplicial complex $\mathcal{L}(\Sigma)$ embedded in the complex plane. The complex $\mathcal{L}(\Sigma)$ is the subcomplex of the cell complex illustrated in the right picture of Figure 4 consisting of the triangles. The vertices of $\mathcal{L}(\Sigma)$ are classified into 6 equivalence classes so that the vertices in each class correspond to a vertex of Σ, and are arranged in the complex plane so

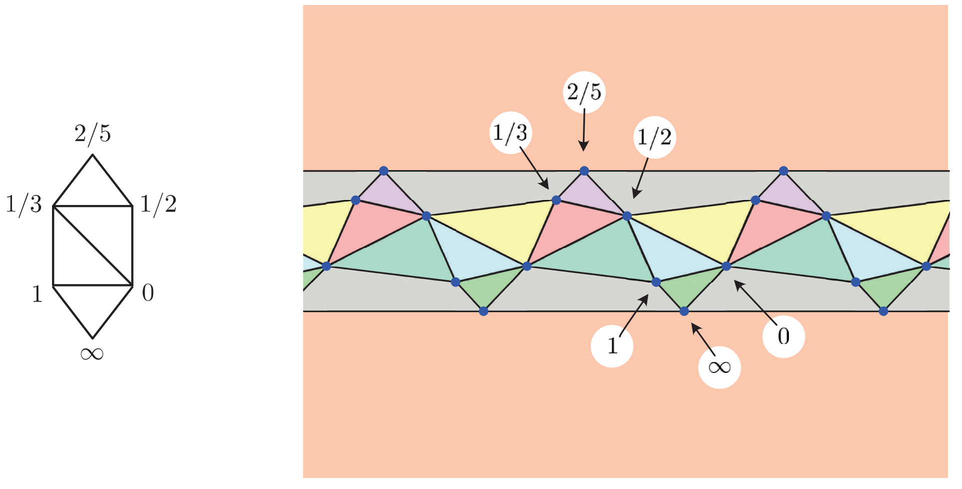

Figure 4: The facial structure of T_a

that they are invariant under the horizontal parallel translation $z \mapsto z+1$. Each triangle in Σ corresponds to a broken line consisting of the segments joining pairs of vertices corresponding to pairs of vertices in the triangle and their edges and vertices. Then the triangles of $\mathcal{L}(\Sigma)$ correspond to pairs of adjacent triangles in Σ. The triangles in $\mathcal{L}(\Sigma)$ is the intersection of H and the ideal tetrahedra in $\widetilde{\Delta}$. There are 2 more polyhedra in $\widetilde{\Delta}$ each of which projects onto a subset of $\mathbb{H}^3/\Gamma$ with nontrivial fundamental group. The intersection of the polyhedra and H forms the 2 equivalence classes of quadrangles in the right picture of Figure 4, one is arranged above $\mathcal{L}(\Sigma)$ and the other is below.

Recall that T_a is a compact convex set in $\mathbb{R}^3$, which has an interior point. Thus ∂T_a is homeomorphic to the 2-dimensional sphere. We regard the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as ∂T_a and describe its facial structure. The cell complex $\widetilde{\Delta} \cap H$ is a subcomplex of $\mathcal{F}(T_a)$. There are 3 more faces of $\mathcal{F}(T_a)$. Two of them come from the invisible faces of $\widehat{C}$, which do not appear as cells of $\widetilde{\Delta}$. Those are polygons with infinite sides, which are illustrated in the right picture of Figure 4 as unbounded regions lying above and below the cell complex $\widetilde{\Delta} \cap H$. The remaining face is the vertex $\{\infty\} \subset \widehat{\mathbb{C}}$, which is not illustrated in Figure 4. The vertex comes from the ray in $\widehat{C}$ emanating from v with direction v.

Proposition 6.1. *The facial structure of ∂T_a is as described above. In particular, the facial structure of ∂T_a is invariant on the rational pleating variety $\mathcal{P}(\mu)$. Moreover, every face of ∂T_a is exposed.*

Finally, we obtain the following theorem.

Theorem 6.2. *Every face of ∂D_a is exposed, and hence there is a one-to-one inclusion-reversing correspondence between $\mathcal{F}(T_a)$ and $\mathcal{F}(D_a)$. In particular, the facial structure of ∂D_a is invariant on the rational pleating variety $\mathcal{P}(\mu)$.*

For the proof of the above theorem, we define an exhaustion $T_a^{(1)} \subset T_a^{(2)} \subset \cdots \to T_a$

of T_a with polytopes. Then we obtain a descending sequence $D_a^{(1)} = (T_a^{(1)})^\circ \supset D_a^{(2)} = (T_a^{(2)})^\circ \supset \cdots \to T_a^\circ = D_a$. Since the polar duality of polytopes is well-understood, we can show that the exposed faces of D_a obtained as the dual to exposed faces of T_a does not contain unexposed faces.

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