Rasmussen type invariant from equivariant instanton Floer homology

Hayato Imori, Kouki Sato, and Masaki Taniguchi
Kyoto University, Meijo University
and Institute of Physical and Chemical Research

1 Introduction

This is a survey of an upcoming paper [2] whose authors are Aliakbar Daemi, Hayato Imori, Kouki Sato, Christopher Scaduto, and Masaki Taniguchi. In [2], we will give an alternative description of Kronheimer-Mrokwa's knot concordance invariant $s^\sharp$ via equivariant singular instanton homology and prove a connected sum inequality of $s^\sharp$. As its applications, we will obtain several new results on the knot concordance group and the (3-dimensional) homology cobordism group.

1.1 Four-genus and the Milnor conjecture

The 4-*genus* $g_4(K)$ of a knot $K \subset S^3$ is the minimal genus of smoothly and properly embedded orientable surfaces $\Sigma \subset D^4$ with boundary $\partial\Sigma = K$. The 4-genus has played a central role in the studies of 4-dimensional aspects of knots, while determining the 4-genus of a knot is still an open problem in general. As a special case of this problem, it was conjectured that a positive (p, q)-torus knot $T_{p,q}$ satisfies the equality $g_4(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$ (called the *Milnor conjecture*). This conjecture was first proved by Kronheimer-Mrowka [17] in 1993:

Theorem 1.1. ([17], Milnor conjecture) For a positive (p, q)-torus knot $T_{p,q}$,

$$g_4(T_{p,q}) = \frac{1}{2}(p-1)(q-1).$$

Kronheimer-Mrowka's proof of Theorem 1.1 is due to Yang-Mills gauge theory. In particular, they consider solutions to a non-linear partial differential equation (anti-self-dual equation) which have singularities along embedded surfaces in 4-manifolds. Such solutions are called *singular instantons*. The technique of singular instantons requires subtle analysis over the complement of embedded surfaces.

On the other hand, Rasmussen [23] gave an alternative proof of Theorem 1.1 only due to combinatorial arguments, by introducing a new *knot concordance invariant*. Here we review the definition of knot concordance.

Definition 1.2. Let K_0 and K_1 be oriented knots in S^3. Then K_0 and K_1 are *concordant* if there exists an smoothly and properly embedded oriented annulus $S \subset [0,1] \times S^3$ such that $S \cap (\{0\} \times S^3) = K_0$ and $S \cap (\{1\} \times S^3) = K_1$.

Note that two knots K_0 and K_1 are concordant if and only if $g_4(K_0 \# (-K_1^*)) = 0$, where $-K_1^*$ is a mirror image of K_1 with reversed orientation and $\#$ denotes the connected sum operation. Definition 1.2 defines an equivalence relation on the set of oriented knots in S^3, called *knot concordance*. The quotient set

$$\mathcal{C} := \{ \text{ oriented knots in } S^3 \} / \sim$$

forms an abelian group under the connected sum operation. The abelian group $\mathcal{C}$ is called the *knot concordance group*.

In [23], Rasmussen introduced a homomorphism $s : \mathcal{C} \to 2\mathbb{Z}$ based on a combinatorial construction in Khovanov homology theory. Rasmussen's s-invariant satisfies the following 4-genus bound;

$$|s(K)| \leqslant 2g_4(K). \tag{1}$$

Moreover, if K is a positive knot (i.e. K admits a knot diagram with no negative crossing), then

$$s(K) = 2g_4(K) = 2g_3(K), \tag{2}$$

where $g_3(K)$ is the Seifert genus of K. These equalities give an alternative proof of Theorem 1.1. A homomorphism $\mathcal{C} \to \mathbb{R}$ satisfying (1) and (2) is called a *slice torus invariant* ([19, 18]). In general, slice torus invariants are *not* unique. Ozsváth-Szabó [21] also introduced another slice torus invariant $2\tau : \mathcal{C} \to 2\mathbb{Z}$ by using Heegaard Floer theory, and it is proved in [11] that s and 2τ are not equal.

1.2 Kronheimer-Mrowka's $s^\sharp$-invariant

An analogue of Rasmussen's s-invariant can be constructed from one of the flavors of knot homology groups from singular instantons. Kronheimer-Mrowka [15] introduced a $\mathbb{Z}$-valued concordance invariant $s^\sharp$ using singular instanton theory, and they first claimed that $s^\sharp$ is equal to s. Later, Gong [10] pointed out that $s^\sharp$ and s are not equal for all positive torus knots.

Here we review the construction of the $s^\sharp$-invariant and Gong's result. Kronheimer-Mrowka [16] introduced a knot homology group $I^\sharp(K)$, called *framed instanton Floer homology*. Roughly speaking, $I^\sharp$ is a functor from the category of knots and surface cobordisms (smoothly and properly immersed and oriented surfaces Σ in $[0,1] \times S^3$ with $\partial\Sigma = -K \sqcup K'$) to the category of $\mathbb{Z}/4$-graded modules over the ring $\mathbb{Q}[u^{\pm 1}]$. The functor $I^\sharp$ has the following properties:

(i) $I^\sharp(K)$ has rank 2 as $\mathbb{Q}[u^{\pm 1}]$-module.

(ii) $I^\sharp(K)$ has homogeneous generators $\mathbf{z}_\pm$ whose $\mathbb{Z}/4$-gradings differ by 2.

(iii) $I^\sharp(\Sigma)$ is a degree $2g(\Sigma)$ map mod 4.

(iv) Let $\Sigma : K \to K'$ be a surface cobordism. If Σ' is obtained from Σ by a positive twist move or a finger move, then

$$I^\sharp(\Sigma') = (1 - u^2) I^\sharp(\Sigma).$$

If Σ' is obtained from Σ by a negative twist move, then

$$I^\sharp(\Sigma') = I^\sharp(\Sigma).$$

(v) Let $\Sigma_2 \colon U \to U$ be a trivial genus 2 cobordism between unknots. Then

$$I^\sharp(\Sigma_2)\mathbf{u}_\pm = 4u^{-1}(u-1)^2\mathbf{u}_\pm,$$

where $\mathbf{u}_\pm$ are generators of $I^\sharp(U)$ given in (ii).

Let λ be a solution to $\lambda^2 = u^{-1}(u-1)^2$, which is virtually corresponding to trivial genus 1 cobordism. Then λ can be seen as an element

$$\lambda = (u-1) - \frac{1}{2}(u-1)^2 + \cdots$$

of the ring $\mathbb{Q}[[u-1]]$, and we see $\mathbb{Q}[[u-1]] = \mathbb{Q}[[\lambda]]$. We change the coefficient ring of $I^\sharp(K)$ from $\mathbb{Q}[u^{\pm 1}]$ to $\mathbb{Q}[[\lambda]]$. For a given knot K, consider an embedded surface cobordism $\Sigma : U \to K$. Then the image of two generators $\mathbf{u}_\pm$ of $I^\sharp(U)$ by $I^\sharp(\Sigma)$ (modulo torsion) can be written in the form

$$I^\sharp(\Sigma)\mathbf{u}_\pm = \begin{cases} \lambda^{m_\pm(\Sigma)}\mathbf{z}_\pm & \text{if } g(\Sigma) \text{ is even} \\ \lambda^{m_\mp(\Sigma)}\mathbf{z}_\mp & \text{if } g(\Sigma) \text{ is odd} \end{cases}$$

up to multiplication by a unit of $\mathbb{Q}[[\lambda]]$. Now, $s^\sharp$-invariant is defined by

$$s^\sharp(K) := 2g(\Sigma) - (m_+(\Sigma) + m_-(\Sigma)).$$

Obviously, from its definition, the $s^\sharp$-invariant gives the following lower bound for the 4-genus.

Proposition 1.3. ([15]) For any knot $K \subset S^3$,

$$s^\sharp(K) \leqslant 2g_4(K).$$

Since Rasmussen's s-invariant is also expressed as above, the $s^\sharp$-invariant was first expected to be equal to the s-invariant. However, Gong [10] gave the following calculations of the $s^\sharp$-invariant for algebraic knots (including all positive torus knots) and showed that $s^\sharp$ and s are not equal.

Theorem 1.4. ([10]) For any quasi-positive knot K,

$$2g_4(K) - 1 \leqslant s^\sharp(K) \leqslant 2g_4(K).$$

In particular, for any algebraic knot K,

$$s^\sharp(K) = 2g_4(K) - 1.$$

In fact, Theorem 1.4 implies that $s^\sharp$ is not even a homomorphism on the knot concordance $\mathcal{C}$, unlike the s-invariant. In [10], Gong posed the following question.

Question 1.5. ([10]) Is there an universal constant $C > 0$ such that $|s^\sharp(K\#K') - s^\sharp(K) - s^\sharp(K')| \leqslant C$?

The main difficulty solving Question 1.5 was the lack of a connected sum formula in $I^\sharp$-theory; namely, we could not compare $I^\sharp(K\#K')$, $I^\sharp(K)$ and $I^\sharp(K')$ directly. As stated in Section 2.1, we solved Question 1.5 by interpreting $s^\sharp$ in terms of *equivariant instanton Floer theory*, in which we have a nice connected sum formula.

1.3 Framed vs. equivariant instanton Floer homology

In this section, we give a quick review of the difference between framed instanton theory and equivariant instanton theory. Roughly speaking, instanton knot Floer homology theories are infinite dimensional analogues of Morse homology, where we consider the Chern-Simons functional over an infinite dimensional space $\mathcal{B}$;

$$CS : \mathcal{B} \to S^1.$$

The Chern-Simons functional is supposed to be an analogue of Morse functions over the finite dimensional smooth manifolds. We would like to define a chain complex $(C_*(K), d)$, where $C_*(K)$ is a graded module generated by the critical point set of CS, and d is a differential given by the counting of 'gradient flow lines' of CS. However, in general, there are 'bad critical points' on the infinite dimensional space $\mathcal{B}$, which are caused by quotient singularities. To avoid this technical difficulty, there are two options to define instanton Floer homology groups:

- Framed instanton homology $I^\sharp$ using twisted $SO(3)$-bundles with no reducible critical points;
- Equivariant description of chain complexes $\widetilde{C}$ corresponding to the Morse-Bott situations of a framed configuration.

From the latter view point, Daemi-Scaduto [3] defined $\mathcal{S}$-complexes $\widetilde{C}(K)$ of knots via singular $SU(2)$ connections. The $\mathcal{S}$-complexes of knots admit the following connected sum formula.

Theorem 1.6. ([3]) For two knots K and K' in S^3,

$$\widetilde{C}_*(K\#K') \simeq \widetilde{C}_*(K) \otimes \widetilde{C}_*(K').$$

In the proof of a connected sum inequality for $s^\sharp$, we use Theorem 1.6.

2 Main results

2.1 Main theorems

We succeeded to give an alternative definition of $s^\sharp$-invariant in terms of $\mathcal{S}$-complexes, which is described as the divisibility of a certain homology class. Moreover, this view point enables us to give a $\mathbb{Z}$-valued slice torus invariant $\widetilde{s}$ of knots, which approximates the Kronheimer-Mrowka's $s^\sharp$-invariant via the framework in [3].

Theorem 2.1. ([2]) There exists a homomorphism $\widetilde{s}\colon \mathcal{C} \to \mathbb{Z}$ such that $2\widetilde{s}$ is a slice torus invariant and

$$|s^\sharp(K) - 2\widetilde{s}(K)| \leqslant 1. \tag{3}$$

General properties of slice torus invariants [19, Proposition 3.3] and [18, Corollary 5.9.] allow us to compute $\widetilde{s}$ for several families of knots.

Corollary 2.2. ([2]) If a knot K is quasi-positive, then

$$\widetilde{s}(K) = g_4(K).$$

If K is an alternating knot, then

$$\widetilde{s}(K) = -\frac{\sigma(K)}{2},$$

where the convention of the knot signature σ is chosen so that $\sigma(K) < 0$ for positive knots K.

Here we note that the approximation (3) gives an answer to Question 1.5. Indeed, for any pair of knots K_1 and K_2, we have

$$\begin{aligned} &|s^\sharp(K_1\#K_2) - s^\sharp(K_1) - s^\sharp(K_2)| \\ &\leqslant |s^\sharp(K_1\#K_2) - \widetilde{s}(K_1\#K_2)| + |s^\sharp(K_1) - \widetilde{s}(K_1)| + |s^\sharp(K_1) - \widetilde{s}(K_1)| \leqslant 3. \end{aligned}$$

In [2], we will give the best possible answer to Question 1.5.

Theorem 2.3. ([2]) For any pair of knots K_1 and K_2, we have

$$|s^\sharp(K_1\#K_2) - s^\sharp(K_1) - s^\sharp(K_2)| \leqslant 1.$$

As another application, we prove a relation between $\widetilde{s}$ and the Frøyshov invariant of the 1-surgery:

Theorem 2.4. ([2]) For any knot $K \subset S^3$, if $\widetilde{s}(K) > 0$ then $h(S^3_1(K)) < 0$.

Note that a similar relation holds for the τ-invariant and the d-invariant in Heegaard Floer theory. (This can be understood via the ν^+-invariant [14]. See [24, Section 2.2] for more details.)

2.2 Definition of Rasmussen type invariants

In this section, we describe the construction of $s^\sharp$ and $\widetilde{s}$ in terms of equivariant instanton knot Floer homology. The following are ingredients which are needed in our construction:

- An $\mathcal{S}$-complex $(\widetilde{C}, \widetilde{d}, \chi)$, which is a $\mathbb{Z}/4$-graded module over $R := \mathbb{Q}[[\lambda]]$ with two differentials $\widetilde{d}$ and χ, associated to a pair of integral homology 3-sphere Y and a knot K.

- *The large and small equivariant complexes* $(\widehat{\mathbf{C}}, \check{\mathbf{C}}, \bar{\mathbf{C}})$, $(\widehat{\mathfrak{C}}, \check{\mathfrak{C}}, \bar{\mathfrak{C}})$ induced from an $\mathcal{S}$-complex $(\widetilde{C}, \widetilde{d}, \chi)$, where $\widehat{\mathbf{C}} = (\widetilde{C} \otimes R[x], \widetilde{d} + x\chi)$ and $\operatorname{rank}_{R[x]} H_*(\widehat{\mathbf{C}}) = 1$.
- *Special cycles* in $\widehat{\mathbf{C}}$, which are cycles compatible with cobordism maps and connected sums, and whose homology classes are canonical up to torsion.
- *The framed complex* $(C^\sharp, d^\sharp) := \widehat{\mathbf{C}}/(x^2 - 4\lambda^2)$, which recovers Kronheimer-Mrowka's knot homology group $I^\sharp(K)$.
- *The twisted* $\mathcal{S}$*-complex* $(\widetilde{C}, \widetilde{d} + 2\lambda\chi) = \widehat{\mathbf{C}}/(x - 2\lambda)$, whose homology group satisfies a Künneth formula for connected sums.

Relations of these objects are depicted in Figure 1. Now, our constructions of $s^\sharp$ and $\widetilde{s}$ are described as follows. Let (Y, K) be a pair of a homology 3-sphere Y and a knot K in Y. Consider the framed complex $(C^\sharp(Y, K), d^\sharp) = \widehat{\mathbf{C}}/(x^2 - 4\lambda^2)$. For a sufficiently large integer $n \in \mathbb{Z}_{>0}$, two elements $\xi^\sharp_\pm(n)$ in $H_*(C^\sharp(Y, K), d^\sharp)/\mathrm{Tor}$ are uniquely determined from a special cycle in $\widehat{\mathbf{C}}$. Now we define

$$s^\sharp_\pm(Y, K) := \min\{n - m \mid \xi^\sharp_\pm(n) = \lambda^m \exists y \in H_*(C^\sharp(Y, K), d^\sharp)/\mathrm{Tor}\}$$

and

$$s^\sharp(Y, K) := s^\sharp_+(Y, K) + s^\sharp_-(Y, K).$$

Next, we consider a twisted $\mathcal{S}$-complex $(\widetilde{C}(Y, K), \widetilde{d} + 2\lambda\chi) = \widehat{\mathbf{C}}/(x - 2\lambda)$. For a sufficiently large integer $n \in \mathbb{Z}_{>0}$, an element $\widetilde{\xi}(n)$ in $H_*(\widetilde{C}, \widetilde{d} + 2\lambda\chi)$ is uniquely determined from a special cycle in $\widehat{\mathbf{C}}$. Define

$$\widetilde{s}(Y, K) := \min\{n - m \mid \widetilde{\xi}(n) = \lambda^m \exists y \in H_*(\widetilde{C}, \widetilde{d} + 2\lambda\chi)/\mathrm{Tor}\}.$$

Here we remark that original $s^\sharp$ was defined only for knots in S^3, and so our construction enables us to generalize $s^\sharp$ to an arbitrary pair (Y, K). For the case $Y = S^3$, we drop Y from the above notations.

2.3 Proof of Theorem 2.1

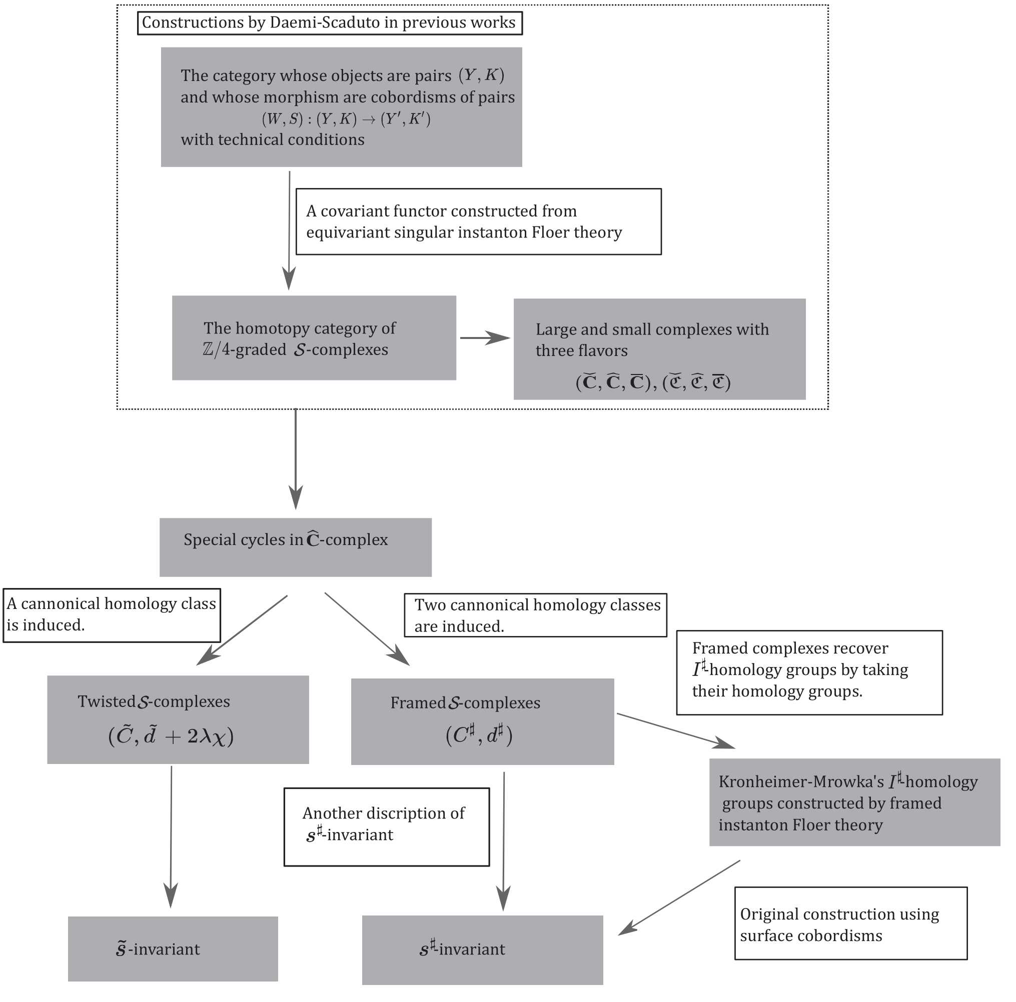
Here we show a sketch of the proof of Theorem 2.1. By construction, we have an exact sequence

$$(\widetilde{C}, \widetilde{d} - 2\lambda\chi) \xrightarrow{\iota} (C^\sharp, d^\sharp) \xrightarrow{\pi} (\widetilde{C}, \widetilde{d} + 2\lambda\chi), \tag{4}$$

and $\operatorname{rank}_{R[x]} H_*(\widehat{\mathbf{C}}) = 1$ implies $\operatorname{rank}_R H_*(\widetilde{C}, \widetilde{d} \pm 2\lambda\chi) \geqslant 1$. Moreover, if $Y = S^3$, then we have $\operatorname{rank}_R H_*(C^\sharp(K), d^\sharp) = \operatorname{rank}_R I^\sharp(K) = 2$. These imply $\operatorname{rank}_R H_*(\widetilde{C}, \widetilde{d} + 2\lambda\chi) = 1$, and now the Künneth formula of $(\widetilde{C}, \widetilde{d} + 2\lambda\chi)$ for connected sums gives the additivity of $\widetilde{s}$ (among knots in S^3). Moreover, the projection π in (4) satisfies $\pi(\xi^\sharp_+(n)) = \widetilde{\xi}(n)$ and $\pi(\xi^\sharp_-(n)) = 2\lambda\widetilde{\xi}(n)$. Such observations give

$$\max\{s^\sharp_+(K) - 1, s^\sharp_-(K)\} \leqslant \widetilde{s}(K) \leqslant \min\{s^\sharp_+(K), s^\sharp_-(K) + 1\},$$

which leads to the approximation (3). Now, by the additivity of $\widetilde{s}$ and the approximation (3), we have the following description of $\widetilde{s}$-invariant.



Constructions by Daemi-Scaduto in previous works
The category whose objects are pairs (Y,K) and whose morphism are cobordisms of pairs $(W,S):(Y,K)\to(Y',K')$ with technical conditions
A covariant functor constructed from equivariant singular instanton Floer theory
The homotopy category of $\mathbb{Z}/4$-graded $\mathcal{S}$-complexes
Large and small complexes with three flavors $(\widetilde{\mathbf{C}},\widehat{\mathbf{C}},\overline{\mathbf{C}}),(\widetilde{\mathfrak{C}},\widehat{\mathfrak{C}},\overline{\mathfrak{C}})$
Special cycles in $\widehat{\mathbf{C}}$-complex
A cannonical homology class is induced.
Two cannonical homology classes are induced.
Framed complexes recover $I^\sharp$-homology groups by taking their homology groups.
Twisted $\mathcal{S}$-complexes $(\tilde{C},\tilde{d}+2\lambda\chi)$
Framed $\mathcal{S}$-complexes $(C^\sharp,d^\sharp)$
Kronheimer-Mrowka's $I^\sharp$-homology groups constructed by framed instanton Floer theory
Another discription of $s^\sharp$-invariant
Original construction using surface cobordisms
$\tilde{s}$-invariant
$s^\sharp$-invariant

Figure 1:

Proposition 2.5. ([2]) For any knot $K \subset S^3$, we have

$$2\widetilde{s}(K) = \lim_{n\to\infty} \frac{s^\sharp(nK)}{n}.$$

Combining this description with Proposition 1.3 and Theorem 1.4, we have

$$2g_4(K) \geqslant \frac{2g_4(nK)}{n} \geqslant \left|\frac{s^\sharp(nK)}{n}\right| \to |2\widetilde{s}(K)| \quad (n \to \infty)$$

for any knot K, and

$$\frac{1}{n} \geqslant \left|\frac{s^\sharp(nK)}{n} - \frac{2g_4(nK)}{n}\right| \to |2\widetilde{s}(K) - 2g_4(K)| \quad (n \to \infty)$$

for any positive knot K, where the equalities

$$2g_4(nK) = s(nK) = ns(K) = 2ng_4(K)$$

follows from the equality (2). These imply the slice torus property of $2\widetilde{s}$ and complete the proof of Theorem 2.1.

3 Topological Applications

In this section, we show several applications of the main theorems to the homology cobordism group of homology 3-spheres and the knot concordance group.

3.1 Application to the homology cobordism group

First, let us recall the definition of the *homology cobordism group of homology 3-spheres.*

Definition 3.1. Let Y_0 and Y_1 be oriented integral homology 3-spheres. We say Y_0 and Y_1 are *homology cobordant* if there exists smooth compact oriented 4-manifold W such that $\partial W = -Y_0 \sqcup Y_1$ and $H_*(W, Y_i; \mathbb{Z}) = 0$.

The homology cobordism defines an equivalence relation on the set of oriented integral homology 3-spheres. The quotient set

$$\Theta^3_{\mathbb{Z}} := \{\text{oriented integral homology 3-spheres}\}/\sim$$

forms an abelian group under the connected sum operation. The abelian group $\Theta^3_{\mathbb{Z}}$ is called *the homology cobordism group of homology 3-spheres.*

A counterpart of the homology cobordism group in the topological category can be also defined. However, it follows from Freedman's work [7] that the topological homology cobordism group is trivial. On the other hand, it is known that the homology cobordism group in the smooth category has rich structures. For example, Donaldson's diagonalization theorem [5] implies that the Poincaré homology 3-sphere $\Sigma(2,3,5)$ has infinite order in $\Theta^3_{\mathbb{Z}}$. Moreover, by the use of orbifold Yang-Mills gauge theory, Fintushel-Stern [6] and Furuta [9] proved that the family $\{S^3_{1/n}(T_{p,q})\}_{n>0}$ generates a $\mathbb{Z}^\infty$-subgroup in $\Theta^3_{\mathbb{Z}}$, where $S^3_{p/q}(K)$ denotes the p/q-surgery of S^3 along a knot $K \subset S^3$. Recently, Dai-Hom-Stofreggen-Truong [4] use involutive Heegaard Floer homology theory to prove that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$-summand. However, the whole structure of $\Theta^3_{\mathbb{Z}}$ is still mysterious.

Nozaki-Sato-Taniguchi [20] proposed a criterion when positive $1/n$-surgeries of S^3 along a knot are linearly independent in $\Theta^3_{\mathbb{Z}}$, in terms of Frøyshov's h-invariant [8].

Theorem 3.2. ([20]) If $h(S^3_1(K)) < 0$, then $\{S^3_{1/n}(K)\}_{n\in\mathbb{Z}_{>0}}$ are linearly independent in the homology cobordism group $\Theta^3_{\mathbb{Z}}$.

Although the assumption $h(S^3_1(K)) < 0$ looks simple enough, it is a difficult problem to determine even the sign of the h-invariant in general, since the h-invariant is defined by counting solutions of a non-linear PDE. From this view point, Theorem 2.4 provides us a new powerful detection of the negativity of $h(S^3_1(K))$. Indeed, we obtain several large classes of knots whose positive $1/n$-surgeries are linearly independent in $\Theta^3_{\mathbb{Z}}$.

Theorem 3.3. ([2]) Suppose that K belongs to one of the following two classes of knots;

(i) alternating knots with negative signatures, and

(ii) non-slice quasi-positive knots.

Then $\{S^3_{1/n}(K)\}_{n\in\mathbb{Z}_{>0}}$ are linear independent in $\Theta^3_{\mathbb{Z}}$.

Proof. For an alternating knot K with negative signature, we have

$$\tilde{s}(K) = -\frac{1}{2}\sigma(K) > 0$$

by Corollary 2.2. Similarly, for a non-slice quasi-positive knots K, we have

$$\tilde{s}(K) = g_4(K) > 0.$$

Now, Theorem 2.4 implies that $h(S^3_1(K)) < 0$, and Theorem 3.2 completes the proof. □

3.2 Application to satellite operations

Next, we show an application to the knot concordance group. For knots $K \subset S^3$ and $P \subset S^1 \times D^2$, let $P(K)$ denote the satellite knot with companion K and pattern P. Satellite operations descend to maps on the concordance group;

$$P : \mathcal{C} \to \mathcal{C}, \quad [K] \mapsto [P(K)]$$

which is also denoted by the same notation P. In general, $P : \mathcal{C} \to \mathcal{C}$ is not a homomorphism. The following conjecture is proposed by Hedden and Pinzón-Caicedo [12].

Conjecture 3.4. ([12]) The image of non-constant satellite operations has infinite rank.

Here "has infinite rank" means that the image of the satellite operation generates an infinite rank subgroup in $\mathcal{C}$. The problem can be separated into two cases;

(i) Winding number of the pattern P is zero.

(ii) Winding number of the pattern P is non-zero.

Conjecture 3.4 for the case (ii) is completely solved in [12].

Proposition 3.5. ([12]) Let P be a pattern of non-zero winding number. Then the satellite operation $P : \mathcal{C} \to \mathcal{C}$ has infinite rank.

The case (ii) can be treated using the Tristram-Levine signature function. On the other hand, Conjecture 3.4 for the case (i) is more difficult since it contains cases where all elements in the image of P are topologically slice (e.g. the Whitehead doubling, denoted by Wh). Hedden and Pinzón-Caicedo [12] approached the case (i) by applying Yang-Mills gauge theory over the double branched covering along a knot, and they gave the following partial answer to their conjecture.

Theorem 3.6. ([12]) Let P be a pattern with winding number 0 and $\Sigma_2(P(U))$ be a double branched cover along a knot $P(U)$. If the meridian ∂D^2 has framed lift to $\Sigma_2(P(U))$ with non-zero $\mathbb{Q}$-linking nmber, then $P : \mathcal{C} \to \mathcal{C}$ has infinite rank.

As an application of $\widetilde{s}$-invariant, we give another partial answer to Conjecture 3.4.

Theorem 3.7. ([2]) Suppose that a pattern $P \subset S^1 \times D^2$ satisfies the following properties;

(i) there is no 3-ball in $S^1 \times D^2$ containing P,

(ii) if K is strongly quasi-positive then $P(K)$ is also strongly quasi-positive, and

(iii) $P(U)$ is a trivial knot.

Then the satellite operation $P : \mathcal{C} \to \mathcal{C}$ has infinite rank.

More concretely, we can show that $\{P(\mathrm{Wh}(T_{p,p+nq}))\}_{n\geqslant 0}$ are linearly independent in $\mathcal{C}$ for any pattern P in Theorem 3.7 and any coprime integers $p, q > 1$.

Here we show a construction of an infinite family of patterns in Theorem 3.7. Let $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^n$ be sequences of positive integers with $n \leqslant m+1$. Then a pattern $P_{(\{a_i\},\{b_i\})}$ is obtained from the tangle diagram in Figure 2 by identifying the right and left vertical edges of the border. Here, the boxes containing $-a_i$ (resp. $-b_i$) are corresponding to a_i times (resp. b_i times) negative full twists. For example, $P_{(\emptyset,\{1\})}$ is the Whitehead doubling and $P_{(\{1,\cdots,1\},\emptyset)}$ is the $(m+1,1)$-cabling. For given sequences $\{a_i\}_{i=0}^m$ and $\{b_i\}_{i=0}^n$ with $n \leqslant m+1$, the pattern $P_{(\{a_i\},\{b_i\})}$ satisfies all assumptions in Theorem 3.7.

We also note that if $P_1, P_2, \ldots, P_n$ are patterns which satisfy the assumptions in Theorem 3.7, then their composition $P_n \circ \cdots \circ P_1$ also satisfies the same assumptions. In particular, if at least one of $P_1, \ldots, P_n$ has trivial winding number then the composition $P_n \circ \cdots \circ P_1$ also has trivial winding number. Moreover, one can see that if at least two of $P_1, \ldots, P_n$ have trivial winding number, then $P := P_n \circ \cdots \circ P_1$ has trivial $\mathbb{Q}$-linking number. Even for such a pattern P, Theorem 3.7 can detect a linearly independent family in the image of P, which cannot be detected by Theorem 3.6.

Sketch of proof of Theorem 3.7. We only show the main ideas to prove Theorem 3.7. The first important step is to define a real-valued knot concordance invariant $r_0(K) \in [0, \infty]$ which is a variant of the r_s-invariant for homology 3-spheres defined in [20], where $s \in [-\infty, 0]$. The construction of $r_0(K)$ is essentially the same as $r_0(Y)$, and hence $r_0(K)$ satisfies properties similar to [20, Theorem 1.1]. The second important step is to detect

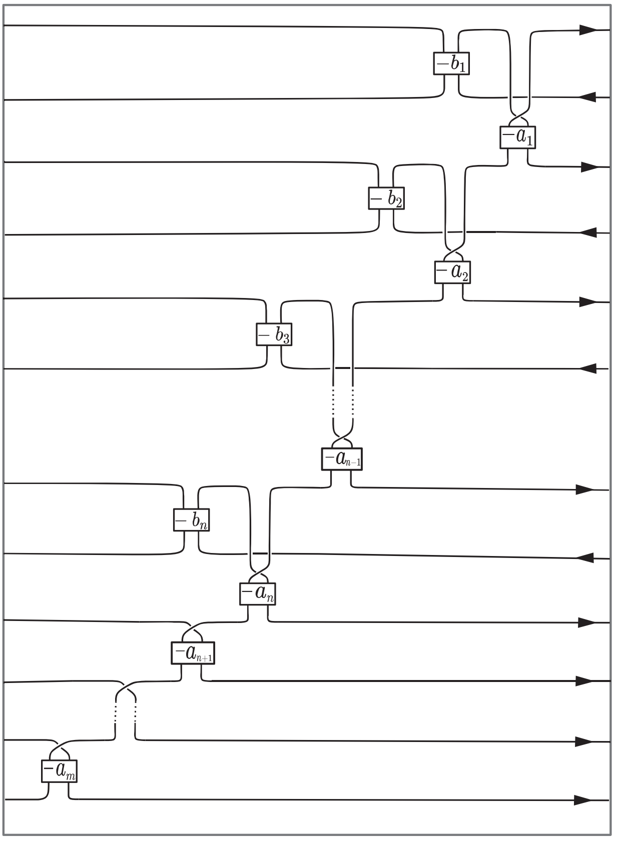


Figure 2: A diagram of tangle corresponding to the pattern $P_{(\{a_i\},\{b_i\})}$

the non-triviality of $r_0(K)$. For that purpose, we establish a local equivalence theory of $\mathcal{S}$-complexes so that $r_0(K)$ is connected with $\widetilde{s}(K)$, where $\widetilde{s}$ is non-trivial for any strongly quasi-positive knot. $\square$

4 Open problems

As seen in previous sections, there are several slice-torus invariants [21, 23, 18, 1] derived from different theories, and we expect that comparing $\widetilde{s}$ with other slice torus invariants improves the understanding of the relations among those theories. In particular, the difference $\widetilde{\varepsilon}(K) := 2\widetilde{s}(K) - s^\sharp(K) \in \{-1, 0, 1\}$ can be regarded as an analogous invariant to the *epsilon-invariant* $\varepsilon(K)$ in Heegaard Floer theory [13].

Indeed, it will be shown in [2] that $\widetilde{\varepsilon}$ shares the properties (1), (3) and (6) in [13, Proposition 3.6] with ε. From this point of view, we conjecture the following relations between knot concordance invariants from equivariant singular instanton Floer theory and those from Heegaard Floer theory:

Conjecture 4.1. ([Daemi-Imori-Sato-Scaduto-Taniguchi]) For any knot K, we have:

$$\widetilde{s}(K) = \tau(K), \qquad s^\sharp(K) = \nu(K) - \nu(-K^*),$$

where τ is the *tau-invariant* [21] and ν is the *nu-invariant* [22]. In particular, we have $\widetilde{\varepsilon}(K) = \varepsilon(K)$.

For all quasi-positive knots and alternating knots, the first equality $\widetilde{s}(K) = \tau(K)$ follows from the fact that $\widetilde{s}$ and τ are slice-torus invariants [18]. Moreover, Gong's calculations $s^\sharp(K) = 2g_4(K) - 1$ for algebraic knots give an affirmative answer to the second equality $s^\sharp(K) = \nu(K) - \nu(-K^*)$ for algebraic knots. (This will be extended in [2] to all quasi-positive knots with knot signature negative.) On the other hand, the following is an open problem, which is the remaining part of analogues of [13, Proposition 3.6]:

Problem 4.2. ([Daemi-Imori-Sato-Scaduto-Taniguchi]) Does $\widetilde{\varepsilon}$ satisfy the following properties?

(i) If $\widetilde{\varepsilon}(K) = 0$, then $\widetilde{s}(K) = 0$.

(ii) If $|\widetilde{s}(K)| = g_4(K)$, then $\widetilde{\varepsilon}(K) = \mathrm{sgn}(\widetilde{s}(K))$.

(iii) If K is homologically thin in Heegaard Floer theory, then $\widetilde{\varepsilon}(K) = \mathrm{sgn}(\widetilde{s}(K))$.

Here we note that the definition of $s^\sharp$ is analogous to the Rasmussen invariant rather than τ, and this causes difficulty proving results of $\widetilde{s}$ analogous to τ. Recently, Baldwin and Sivek [1] defined the *instanton tau-invariant* $\tau^\sharp(K)$ and the *instanton nu-invariant* $\nu^\sharp(K)$ in terms of cobordism maps of framed instanton Floer homology obtained via surgeries along K. In particular, the definitions of $\tau^\sharp$ and $\nu^\sharp$ are analogous to τ and ν respectively (while the value of $\nu^\#(K)$ is similar to $\nu(K) - \nu(-K^*)$). Moreover, one can expect to relate $\tau^\sharp$ and $\nu^\sharp$ to $s^\sharp$ in terms of framed instanton Floer homology theory. Based on these observations, we propose the following conjecture:

Conjecture 4.3. ([Daemi-Imori-Sato-Scaduto-Taniguchi]) For any knot K, we have:

$$\widetilde{s}(K) = \tau^\sharp(K), \qquad s^\sharp(K) = \nu^\sharp(K).$$

References

[1] John A. Baldwin, Steven Sivek. *Framed instanton homology and concordance*, J. Topol., 14(4):1113–1175, 2021.

[2] Aliakbar Daemi, Hayato Imori, Kouki Sato, Christopher Scaduto, and Masaki Taniguchi. in preparation.

[3] Aliakbar Daemi and Christopher Scaduto. *Equivariant aspects of singular instanton Floer homology*, preprint, arXiv:1912.08982, 2019.

[4] Irving Dai, Jennifer Hom, Matthew Stoffregen, and Linh Truong. *An infinite-rank summand of the homology cobordism group*, arXiv, 2018.

[5] Simon K Donaldson. *An application of gauge theory to four-dimensional topology*, J. Differential Geom., 18(2):279-315, 1983.

[6] Ronald Fintushel and Ronald J. Stern. *Instanton homology of Seifert fibred homology three spheres*, Proc. London Math.Soc. 61(1):109-137, 1990.

[7] Michael Hartley Freedman. *The topology of four-dimensional manifolds*, J. Differential Geometry 17(3):357-453, 1982.

[8] Kim A. Frøyshov. *Equivariant aspects of Yang-Mills Floer theory*, Topology 41(3):525-552, 2002.

[9] Mikio Furuta. *Homology cobordism group of homology 3-spheres*, Invent. Math. 100(2):339-355,1990.

[10] Sherry Gong. *On the Kronheimer-Mrowka concordance invariant*, J. Topol., 14(1):1–28, 2021.

[11] Matthew Hedden and Philip Ording. *The Ozsváth-Szabó and Rasmussen concordance invariants are not equal*, Amer. J. Math. 130(2):441–453, 2008.

[12] Matthew Hedden and Juanita Pinzón-Caicedo. *Satellites of infinite rank in the smooth concordance group*, Invent. Math., 225(1):131–157,2021.

[13] Jennifer Hom. *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, J. Topol., 7(2):287–326, 2014.

[14] Jennifer Hom, Zhongtao Wu. *Four-ball genus bounds and a refinement of the Ozváth-Szabó tau invariant*, J. Symplectic Geom., 14(1):305–323, 2016.

[15] Peter Kronheimer, Tomasz Mrowka. *Gauge theory and Rasmussen's invariant*, J. Topol., 6(3):659–674, 2013.

[16] Peter Kronheimer, Tomasz Mrowka. *Knot homology groups from instantons*, J. Topol. 4(4):835-918, 2011.

[17] Peter Kronheimer, Tomasz Mrowka. *Gauge theory for embedded surfaces. I* Topology, 32(4): 773–826, 1993.

[18] Lukas Lewark. *Rasmussen's spectral sequences and the sl_N-concordance invariants*, Adv. Math., 260:59–83, 2014.

[19] Charles Livingston. *Computations of the Ozsváth-Szabó knot concordance invariant*, Geom. Topol., 8:735–742, 2004.

[20] Yuta Nozaki, Kouki Sato, and Masaki Taniguchi, *Filtered instanton floer homology and the homology cobordism group* arXiv, 2019.

[21] Peter S. Ozsváth, Zoltán Szabó. *Knot Floer homology and the four-ball genus*, Geom. Topol., 7:615–639, 2003.

[22] Peter S. Ozsváth, Zoltán Szabó. *Knot Floer homology and rational surgeries* Algebr. Geom. Topol., 11(1):1–68, 2011.

[23] Jacob Rasmussen. *Khovanov homology and the slice genus*, Invent. Math., 182(2):419–447, 2010.

[24] Kouki Sato. *Topologically slice knots that are not smoothly slice in any definite 4-manifold*, Algebr. Geom. Topol., 18(2):827–837, 2018.

京都大学理学研究科　井森　隼人
Hayato Imori
Department of Mathematics, Kyoto University
Kitashirakawa, Sakyo-ku, Kyoto, 606-8502, Japan
E-mail address: `imori.hayato.67m@st.kyoto-u.ac.jp`

名城大学理工学部数学科　佐藤　光樹
Kouki Sato
Department of Mathematics, Meijo University
1-501 Shiogamaguchi, Tempaku-ku, Nagoya, 468-8502, Japan
E-mail address: `satokou@meijo-u.ac.jp`

理化学研究所数理創造プログラム　谷口　正樹
Masaki Taniguchi
Institute of Physical and Chemical Research, iTHEMS
2-1 Hirosawa, Wako, Saitama 351-0198, Japan
E-mail address: `masaki.taniguchi@riken.jp`