A short review on constraints of cosmetic surgery of knots in S^3

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1 Introduction

For a knot K in the 3-sphere S^3, let $E(K)$ be the complement of an open tubular neighborhood of K. A *slope* is a non-trivial, unoriented simple closed curve on $\partial E(K)$.

As is well-known, there is a natural one-to-one correspondence between the set of slopes and the set $\mathbb{Q} \cup \{\infty = \frac{0}{1}\}$, where the meridian μ of K corresponds to $\infty = \frac{0}{1}$. In the following, by slope we always mean a non-meridional slope and we regard a slope as a rational number $r = \frac{p}{q} \in \mathbb{Q}$, where p, q are coprime integers taken so that $p \geq 0$.

Let $S^3_K(r)$ be the r-surgery on K. For an oriented, closed 3-manifolds M and N, we denote by $M \cong N$ if M and N are *orientation-preservingly* homeomorphic. We also denote by $-M$ the 3-manifold M with opposite orientation.

Definition 1. Two Dehn surgeries $S^3_K(r)$ and $S^3_K(r')$ on the same knot K along two inequivalent slopes r and r' are *purely cosmetic* (resp. *chirally cosmetic*) if $S^3_K(r) \cong S^3_K(r')$ (resp. $S^3_K(r) \cong -S^3_K(r')$).

Here two slopes r, r' are *equivalent* if there is an orientation-preserving homeomorphism $f : E(K) \to E(K)$ that sends the slope r to r'. For a non-trivial knot in K in S^3, by Gordon-Luecke theorem [4] two slopes r and r' are equivalent if and only if $r = r'$.

It is conjectured that there are no purely cosmetic surgeries.

Conjecture 1 (Cosmetic surgery conjecture [12, Problem 1.81 (A)])**.** *Every knot K does not have purely cosmetic surgeries.*

On the other hand, there are two families of chirally cosmetic surgeries on non-trivial knots in S^3.

(a) For an amphicheiral knot K, $S^3_K(m/n) \cong -S^3_K(-m/n)$.

(b) For the $(2, r)$-torus knot K, $S^3_K(\frac{2r^2(2m+1)}{r(2m+1)+1}) \cong -S^3_K(\frac{2r^2(2m+1)}{r(2m+1)-1})$ for any $m \in \mathbb{Z}$.

It is natural to ask whether there exist other chirally cosmetic surgeries or not. To study the question, it is useful to classify chirally cosmetic surgeries into the following three types.

0-type $S^3_K(r) \cong -S^3_K(-r)$.

+-type $S^3_K(r) \cong -S^3_K(r')$ such that $rr' > 0$.

$-$-type $S^3_K(r) \cong -S^3_K(r')$ such that $rr' < 0$ and $r + r' \neq 0$.

Our naive expectation that (a) and (b) are only the chirally cosmetic surgeries of knots in S^3 is stated in the following forms.

Conjecture 2. *Let K be a non-trivial knot in S^3.*

(i) K admits a chirally cosmetic surgery of 0-type if and only if K is amphicheiral.

(ii) K admits a chirally cosmetic surgery of $+$-type if and only if K is a $(2,p)$-torus knot.

(iii) K never admits a chirally cosmetic surgery of $-$-type.

In this note we review various constraints on chirally cosmetic surgeries. Although our main object is chirally cosmetic surgeries, to make the situation and difference clearer, we will also review constraints on purely cosmetic surgeries.

We omit the computational part, the examination of to what extent known constraints are able to rule out (chirally) cosmetic surgeries. For the results and discussion of these issues, we refer to [7].

2 Summary of constraints of cosmetic surgeries

Let v be an invariant of 3-manifold (in our purpose, it is sufficient to use an invariant of rational homology spheres). For a 3-manifold M is obtained by r-surgery on a knot K in S^3, we have a rational surgery formula

$$v(M) = f(v'(K), r)$$

that describes $v(M) = v(S^3_K(r))$ in terms of the surgery slope r and invariants $v'(K)$ of the knot K. Moreover, usually v behaves nicely with respect to orientation reversal, in the sense that it satisfies the equality

$$v(-M) = \pm v(M)$$

Thus if $S^3_K(r) \cong \pm S^3_K(r')$ then the equality $v(S^3_K(r)) = \pm v(S^3_K(r'))$ leads to non-trivial relation among r, r' and $v'(K)$, giving a non-trivial constraint for purely or chirally cosmetic surgery $S^3_K(r) \cong \pm S^3_K(r')$.

We give a short survey on several constraints to cosmetic surgeries obtained in this strategy.

2.1 The 1st homology group

Let $v_0(K) = |H_1(M;\mathbb{Z})|$ be the order of the 1st homology group. The rational surgery formula is given as $v_0(S^3_K(p/q)) = p$. Therefore we get

Theorem 2. *If $S^3_K(p/q) \cong \pm S^3_K(p'/q')$ then $p = p'$.*

Thus in the following it is sufficient to consider the case $p = p'$.

2.2 The Casson-Walker and the Casson-Gordon invariant

Let $v_1 = \lambda$ be the Casson-Walker invariant and τ be the total Casson-Gordon invariant (see [2] for the definition).

The rational surgery formula of the Casson-Walker invariant and the total Casson-Gordon invariant is

$$v_1(S^3_K(p/q)) = \frac{q}{p}a_2(K) + s(q,p), \quad \tau(S^3_K(p/q)) = -4ps(q,p) - \sigma(K,p).$$

Here

- $a_2(K)$ is the coefficient of z^2 of the Conway polynomial $\nabla_K(z)$ of K.
- $s(q,p)$ is the Dedekind sum.
- $\sigma(K,p) = \sum_{\omega^p=1} \sigma_\omega(K)$ is the total p-signature, where $\sigma_\omega(K)$ is the Levine-Tristram signature at ω.

Therefore from these invariants we get the following constraints which is discussed and used in [6].

Theorem 3.
- *If $S^3_K(p/q) \cong S^3_K(p/q')$, then $a_2(K) = 0$.*
- *If $S^3_K(p/q) \cong -S^3_K(p/q')$ then $12(q+q')a_2(K) = 6p(s(q,p) + s(q',p)) - 3\sigma(K,p)$.*

One should note that already at this point, the constraint for purely cosmetic surgery and chirally cosmetic surgery diverge; the constraint for purely comsetic surgery says that certain invariant (in the current setting a_2) is trivial, whereas for chirally cosmetic surgery, the constraints is more complicated.

2.3 Rank of the Heegaard Floer homology

Since the Heegaard Floer homology is an extremely strong and useful invariant, it is natural to expect that the Heegaard Floer homology brings various information on cosmetic surgeries.

In [14] a rational surgery formula of Heegaard Floer homology was given. However, since the Heegaard floer homology is a complicated object having various structures, using the rational surgery formula in the full strength is not so easy. Usually, to make the argument manageable, one restricts attentions to simpler invariants that are derived from the Heegaard floer homology.

For example, by looking at the rank of the Heegaard Floer homology, the rational formula becomes much simpler;

Proposition 4. *[14, Proposition 9.6]*

$$\mathit{rank}\,\widehat{HF}(S^3_K(\frac{m}{n})) = \begin{cases} p + qC_K & (\frac{p}{q} \geq 2\nu(K)-1)) \\ -p + (4\nu(K)-2)q + qC_K & (0 \leq \frac{p}{q} \leq 2\nu(K)-1) \\ p - (4\nu(K)-2)q - qC_K & (q < 0, \nu(K) > 0) \\ p - qC_K & (q < 0, \nu(K) = 0) \end{cases}$$

Here C_K is a certain quantity determined by the knot floer chain complex of K, and $\nu(K)$ is an invariant called the nu-invariant.

From this surgery formula, we get the following constraints.

Proposition 5. *Let K be a non-trivial knot. If $S^3_K(p/q) \cong -S^3_K(p/q')$ ($\frac{p}{q} > \frac{p}{q'}$), then the following holds.*

(i) $\frac{p}{q} > 0$.

(ii) If $\frac{p}{q'} > 0$, then $S^3_K(p/q)$ is an L-space and $C_K = 0$. In particular, $\frac{p}{q}, \frac{p}{q'} \geq 2g(K) - 1$.

(iii) If $0 > \frac{p}{q'}$ and $\nu(K) = 0$, then $q + q' = 0$.

(iv) If $0 > \frac{p}{q'}$ and $\nu(K) > 0$, then $q + q' > 0$. Moreover,

(iv-a) $\dfrac{q+q'}{p} = \dfrac{2}{4\nu(K) - 2 + C_K}$ *when* $\frac{p}{q} \leq 2\nu(K) - 1$.

(iv-b) $\dfrac{q+q'}{p} = \dfrac{(4\nu(K)-2)(-q')}{pC_K}$ *when* $\frac{p}{q} \geq 2\nu(K) - 1$ *(in this case $C_K \neq 0$).*

With some additional arguments, we get the following useful reformulation of the above constraint that avoids to use the ν-invariant.

Corollary 6. *[7] Let K be a knot in S^3. Assume that K admits a chirally cosmetic surgery $S^3_K(\frac{p}{q}) \cong -S^3_K(\frac{p}{q'})$. Then*

$$\left|\frac{q+q'}{p}\right| < \begin{cases} \frac{2}{2g(K)-1} & (C_K = 0) \\ \frac{2}{C_K} & (C_K > 0) \end{cases}$$

Since $g(K) = 1$ and $C_K = 0$ happens if and only if K is the trefoil (whose chirally cosmetic surgeries are classified), as a consequence, we rule our chirally cosmetic surgeries of $\pm$-type producing integral homology sphere or homology $\mathbb{R}P^3$.

Corollary 7. *[7] If K admits a chirally cosmetic surgery $S^3_K(\frac{p}{q}) \cong -S^3_K(\frac{p}{q'})$, then $|\frac{q+q'}{p}| < 1$. In particular, if K admits a chirally cosmetic surgery of $\pm$-type, then $|p| > 2$.*

To use Corollary 6 effectively, we need to know the value C_K. When K is quasi-alternating, we have a direct formula of C_K.

Lemma 8. *Assume that K is quasi-alternating Then*

$$C_K = \frac{1}{2}(\det(K) - 2|\sigma(K)| - 1)$$

2.4 Ni-Wu's result on purely cosmetic surgery

To proceed further, it is useful to use the following remarkable restriction for purely cosmetic surgery.

Theorem 9. *(Ni-Wu [15]) If $S^3_K(p/q) \cong S^3_K(p/q')$, then $q = -q'$.*

This constraint is obtained from the rational surgery formula of the d-invariant, together with some preceding results obtained by Heegaard Floer homology, Casson-Walker invariants. Here we give a quick outline;

From the surgery formula of d-invariant, if $q > 0$, we have an inequality

$$d(S^3_K(p/q)) \leq d(S^3_K(L(p,q))) \tag{1}$$

Similarly, when $q' < 0$ we have the following opposite inequality

$$d(S^3_K(p/q)) \geq d(S^3_K(L(p,q))). \tag{2}$$

It is known that $d(S^3_K(L(p,q))) = 4p\lambda(L(p,q))$. Now we utilize the following know facts;

(a) If $S^3_K(p/q) \cong S^3_K(p/q')$ then $qq' < 0$

(b) If $S^3_K(p/q) \cong S^3_K(p/q')$ then $\lambda(L(p,q)) = \lambda(L(p,q'))$.

This shows that the if $S^3_K(p/q) \cong S^3_K(p/q')$, then the both inequalities (1),(2) must be equality. This gives more restriction for the Heegaard Fleor homology leading to the conclusion $q = -q'$ (and several additional constraints).

This argument does not immediately extend to chirally cosmetic surgery – although we can take the property (a) to hold by restricting the chirally cosmetic surgery of $-$-type, (b) is not true in general. Moreover, since $d(-M) = -d(M)$, even if we assume that (a) and (b), we do not get the conclusion that both (1),(2) are equalities.

As we will see the lacking of this strong constraints for slopes makes it harder to treat chirally cosmetic surgeries.

2.5 More on purely cosmetic surgery

In [5], by taking account of grading of Heegaard floer homology (recall that d-invariant is understood as a grading of particular element) Hanselman proved the following quite strong constraints.

Theorem 10 (Hanselman [5]). *Let K be a non-trivial knot and $th(K)$ be the Heegaard Floer thickness of K. If $S^3_K(\frac{p}{q}) \cong S^3_K(\frac{p}{q'})$ for $q \neq q'$, then either*

- $\{\frac{p}{q}, \frac{p}{q'}\} = \{2, -2\}$ *and* $g(K) = 2$, *or,*
- $\{\frac{p}{q}, \frac{p}{q'}\} = \{\frac{1}{q}, -\frac{1}{q}\}$ *for some* $0 < q \leq \frac{th(K)+2g(K)}{2g(K)(g(K)-1)}$.

Thus contrary to the chirally cosmetic surgery case (Corollary 7), for purely cosmetic surgery, the remaining possibility is purely cosmetic surgery producing an integral homology sphere or a homology $\mathbb{R}P^3$.

Hanselman's constraint and author's work on quantitative refinement of Birman-Menasco finiteness theorem [11] shows the following finiteness theorem of purely cosmetic surgery.

Theorem 11. *[10] Let $b(K)$ be the braid index of K. If $g(K) \geq \frac{3}{2}b(K)$ then K does not admit a purely cosmetic surgery. In particular, for given $b > 0$, these are only finitely many knots with braid index b that admit purely cosmetic surgeries.*

2.6 The finite type invariant of degree two

In the theory of finite type invariant of 3-manifolds, the rank of homology groups nad the Casson-Walker invariant λ can be understood as the finite type invariant of degree zero and one, respectively.

Thus it is natural to explore the next finite type invariant, the finite type invariant of degree two (up to scalar multiple, these is only one (primitive) finite type invariant of degree two). For the degree two finite type invariant τ_2, we have the following surgery formula [8].

$$\lambda_2(S^3_K(p/q)) = \frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{8}\frac{q^2}{p^2} - v_3(K)\frac{q}{p} + \frac{a_2(K)}{48}\left(1 - \frac{1}{p^2}\right) + \lambda_2(L(p,q)).$$

Here

- $a_4(K)$ is the coefficient of z^2 of the Conway polynomial $\nabla_K(z)$ of K.
- $v_3(K)$ is the primitive finite type invariant of degree 3, normalized so that it takes value $\frac{1}{4}$ for the right-handed trefoil. By using the Jones polynomial, $v_3(K)$ is written as
$$v_3(K) = -\frac{1}{144}V'''_K(t) - \frac{1}{48}V''_K(t)$$

This leads to the following constraints for purely and chirally comsetic surgeries.

Theorem 12. *[8, Corollary 1.3 (iii)] Let K be a knot. If K admits a chirally cosmetic surgery of $\pm$-type; $S^3_K(\frac{p}{q}) \cong -S^3_K(\frac{p}{q'})$ with $q+q' \neq 0$, then either (i) or (ii) holds.*

(i) $v_3(K) = 0$ and $7a_2(K)^2 - a_2(K) - 10a_4(K) = 0$.

(ii) $v_3(K) \neq 0$ and $\frac{p}{q+q'} = \frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{8v_3(K)}$.

2.7 Combining the constraint

For chirally surgeries $S^3_K(p/q) \cong -S^3_K(p/q')$, the Heegaard Floer constraint says that $\frac{q+q'}{p}$ is bounded by certain constant (that depends on K), whereas the degree two finite type invariant constraint says that $\frac{q+q'}{p}$ is written by combinations of certain knot invariants (provided $v_3(K) = 0$).

By combining these two constraints, we get more practical constraints. Among them, when we further assume that K is (quasi)-alternating, we get the following useful formula.

Theorem 13. *[7] Let K be a (quasi-)alternating knot. If K admits a chirally cosmetic surgery of $\pm$-type and $v_3(K) \neq 0$, then*

$$|7a_2(K)^2 - a_2(K) - 10a_4(K)| > \frac{1}{2}(|\det(K)| - 2|\sigma(K)| - 1)|4v_3(K)|$$

3 Some discussions

We close this short note by some related discussions.

3.1 Higher order finite type invariant and the LMO invariants

It is natural to try to extend and explore finite type invariants constraints of degree $3, 4, \ldots$.

More generally, to treat *all* the finite type invariants at the same time, it is natural to use the LMO invariant [13] of rational homology spheres.

The rational surgery formula of LMO invariants was given by Bar-Natan and Lawrence [1]; the LMO invariant of $S^3_K(p/q)$ is computed from the slope p/q, together with the Kontsevich invariant of K. Indeed, the result in the previous section, the rational surgery formula of the degree two finite type invariant λ_2, is obtained by explicitly writing the degree two part of the LMO invariant by using the rational surgery formula of the LMO invariant. Thus theoretically one can use and treat the LMO invariant to study cosmetic surgeries.

However, due to the several reasons currently it is hard to use the LMO invariant in full generality.

- The LMO invariant and Kontsevich invariant takes value in a space of Jacobi diagrams, which is a graded vector space. The structure of such spaces are complicated – even for the dimension is unknown, except small degree part.
- The computation of Kontsevich invariant is also hard. Even though one can write the relevant information of Kontsevich invariants by using other famous invariants (like the Conway polynomial, Jones polynomial) whenever the degree is small, getting exact relation and formula requires additional works.

For degree 3 part, although the computations become more complicated, when we focus on chirally cosmetic surgery of 0-type, we get a simple and still practical constraint.

Theorem 14. *[8] If $S^3_K(p/q) \cong -S^3_K(-p/q)$, then $v_5(K) = 0$. Here $v_5(K)$ is a certain finite type invariant of degree 5, which can be written by using the Kauffman polynomial (see Appendix of [6] for an explicit formula)*

Although it is hard to write down and use the surgery formula of higher degree part of the LMO invariant, it is still desirable to explore the theoretical aspect of the LMO invariant and cosmetic surgeries. We refer to [8] for more discussions on LMO invariant and cosmetic surgeries.

3.2 Hyperbolic geometry

In a different direction, using hyperbolic structure, Futer-Purcell-Schliemer showed that if a non-amphicheiral hyperbolic knot admits a (chirally) cosmetic surgery $S^3_K(r) \cong -S^3_K(r')$, then the pair of such slopes (r, r') must be a certain finite set which is explicitly computable (see [3] for more precise statement). This result allows us to check, for a given hyperbolic knot, whether it admits (chirally) cosmetic surgery or not.

3.3 Alternating knots

It turns out our constraint Theorem 13 is useful, in the sense that when the crossing number is large, it excludes the of chirally cosmetic surgery in many cases.

This ‘in many cases’ can be explained by heuristic order estimates; The left hand side of the constraint Theorem 13 can grow polynomially with respect to the number of crossings c, whereas, the right hand side, the determinant, can grow exponentially.

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