Problems on Low-dimensional Topology, 2022

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This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 25–27, 2022.

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1 Cosmetic surgery on knots

(Tetsuya Ito)[2]

For a knot K in the 3-sphere S^3 and a slope $r \in \mathbb{Q} \cup \{\infty = \frac{0}{1}\}$, we denote by $S^3_K(r)$ the r-surgery on K. Two Dehn surgeries $S^3_K(r)$ and $S^3_K(r')$ on the same knot K are *purely cosmetic* if $r \neq r'$ and $S^3_K(r) \cong S^3_K(r')$, and *chirally cosmetic* if $r \neq r'$ and $S^3_K(r) \cong -S^3_K(r')$. Here for an oriented, closed 3-manifolds M and N we denote by $M \cong N$ if M and N are *orientation-preservingly* homeomorphic, and $-M$ means the 3-manifold M with opposite orientation.

Although it is conjectured that purely cosmetic surgery does not exist unless K is the unknot (the *cosmetic surgery conjecture*), there are two families of chirally cosmetic surgeries on non-trivial knots.

(a) For an amphicheiral knot K, $S^3_K(r) \cong -S^3_K(-r)$.

(b) for a $(2,n)$-torus knot K we have $S^3_K(\frac{2n^2(2m+1)}{n(2m+1)+1}) \cong -S^3_K(\frac{2n^2(2m+1)}{n(2m+1)-1})$ for any $m \in \mathbb{Z}$.

Thus it is natural to ask whether there exist other chirally cosmetic surgeries or not. To study the question, it is useful to separate chirally cosmetic surgeries into the following three types.

0-type $S^3_K(r) \cong -S^3_K(-r)$.

+-type $S^3_K(r) \cong -S^3_K(r')$ such that $rr' > 0$.

−-type $S^3_K(r) \cong -S^3_K(r')$ such that $rr' < 0$ and $r + r' \neq 0$.

Our naive expectation is stated in the following forms.

Conjecture 1.1 (T. Ito)**.** *Let K be a non-trivial knot in S^3.*

(i) *K admits a chirally cosmetic surgery of 0-type if and only if K is amphicheiral.*

(ii) *K admits a chirally cosmetic surgery of +-type if and only if K is a $(2,n)$-torus knot.*

(iii) *K never admits a chirally cosmetic surgery of −-type.*

Seemingly manageable problems on chirally cosmetic surgeries

Although at the moment, the conjecture seems to be difficult to solve in general, we list several related questions which seems to be with in reach, with a possible strategy and relevant background results/arguments.

Question 1.2 (T. Ito)**.** *Does K have no chirally cosmetic surgery of −-type if K is non-prime ?*

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Chirally cosmetic surgeries of +-type are L-space surgeries [44]. Since L-space knots are prime [29], non-prime knot never admits a chirally cosmetic surgery of +-type.

Certainly the JSJ decomposition structure of $S^3_K(r)$ will provide severe restrictions for a non-prime knot K to admit chirally cosmetic surgeries. See [48] where the non-existence of purely cosmetic surgeries for non-prime knot was proven. See also [24], where the author used the same strategy to study chirally cosmetic surgeries of cable knots.

Question 1.3 (T. Ito)**.** *Is Conjecture 1.1 true for 2-bridge knots, or, alternating knot of genus two ?*

In [21] we confirmed the conjecture for genus one alternating knots. Thanks to various nice features of 2-bridge knots, using constraints in [22] we can show a large portion of 2-bridge knots admit no chirally cosmetic surgeries. The question will be solved if one can specify a class of 2-bridge knots that fails to satisfy known constraints, and when we check the conjecture for such 'bad' classes (optimistically, if such a bad classes are finite then using the results in [13] one can show they indeed admit no chirally cosmetic surgeries).

Similarly, alternating knots of genus two is effectively enumerated by using generator-twisting method [47] and again it is easy to see a large portion of 2-bridge knots admit no chirally cosmetic surgeries by using results in [22] so a strategy similar to 2-bridge knots will work.

Question 1.4 (T. Ito)**.** *Is Conjecture 1.1 true for the L-space knot ?*

Among L-space knots, $(2,n)$ torus knots are special in the sense that there are the only L-space knots which are alternating. The aforementioned fact that all the chirally cosmetic surgeries of +-type are L-space surgeries is obtained by just looking at the rank of the Heegaard Floer homology. Since L-space knots and L-spaces are the most simplest class in a theory of Knot/Heegaard Floer homology, it is feasible to use more finer structure of the Heegaard Floer theory (*i.e.*, (absolute) grading, the d-invariants) to get additional restrictions. For example, Varvarezos [49] showed that L-space knot admits no chirally cosmetic surgery of −-type by incorporating the grading informations. Similar techniques and arguments will work, and at least, will bring more restrictions for chirally cosmetic surgeries of +-types for L-space knots.

From the LMO invariant to cosmetic surgery and back

One approach to attack purely/chirally cosmetic surgery is to use the LMO invariant Z^{LMO} which is obtained by the 'Aarhus integral' of the Kontsevich invariant $Z(K)$ of K (see [23] for details).

It is conjectured that the Kontsevich invariant Z distinguishes all the knots, and that a knot K is amphicheiral if and only if the odd degree part of $Z(K)$ vanishes. Although these conjectures are the most important open problems, a partial solution will bring a progress on cosmetic surgeries.

For example, in the LMO invariant point of view Conjecture 1.1 (i) can be rephrased as follows.

Question 1.5 (T. Ito). *If* $Z^{LMO}\big(S^3_K(r)\big) = Z^{LMO}\big(-S^3_K(-r)\big)$, *then is the odd degree part of* $Z(K)$ *vanish ?*

In [23] the author confirmed the conjecture for some small degree parts by direct computations, producing several useful constraints for chirally cosmetic surgeries.

Conversely, partial results on (purely or chirally) cosmetic surgery bring natural questions on the LMO invariants. It is known that the LMO invariant fails to distinguish *rational* homology spheres (however, we should mention that known examples are Seifert fibered space), whereas it is conjectured that the LMO invariant distinguishes *integral* homology spheres.

Using the Heegaard Floer homology, we have seen that $S^3_K(1/n) \not\cong -S^3_K(1/m)$ for any m, n, unless K is amphicheiral and $m = -n$ [22]. Thus in a light of the aforementioned conjecture, it is natural to ask

Question 1.6 (T. Ito). *Is* $Z^{LMO}\big(S^3_K(1/n)\big) \neq Z^{LMO}\big(-S^3_K(1/m)\big)$, *unless* K *is amphicheiral and* $m = -n$ *?*

As for the purely cosmetic surgery, a situation is more interesting; Hanselmann showed that unless K has genus two, a pure cosmetic surgery of K must be of the form $S^3_K(1/n) \cong S^3_K(-1/n)$ for some n [18] (indeed he gave more detailed constraints). Thus the following special case of the conjecture that the LMO invariant distinguishes integral homology spheres is of great importance.

Question 1.7 (T. Ito). *Is* $Z^{LMO}\big(S^3_K(1/n)\big) \neq Z^{LMO}\big(S^3_K(-1/n)\big)$, *unless* K *is not the unknot?*

2 Mellin-Barnes integrals and the beta invariant of 3-manifolds

(Andrew Kricker)

These questions arose in joint work with Craig Hodgson and Rafael Siejakowski [19] studying asymptotic behaviour of the Garoufalidis-Kashaev meromorphic 3D index [14]. Our asymptotic analysis found connections to many interesting topological invariants, many of which are familiar like the volume and twisted Reidemeister torsion. The discussion here concerns a topological invariant which arose in our analysis which appears to be new and whose significance is not yet understood.

Let $\mathcal{T}$ be an N-tetrahedron ideal triangulation of a connected, oriented three-manifold M whose ideal boundary is a torus. In Section 9 of [19] we introduce the Mellin-Barnes integrals associated to $\mathcal{T}$ and combine them to define the β invariant. This invariant appears to capture contributions to the asymptotics arising from a collection of boundary parabolic $\mathrm{PSL}(2,\mathbb{R})$-representations of M. There are many interesting open questions related to the nature and topological significance of this invariant.

The starting point of the construction of β is a combinatorial structure on $\mathcal{T}$ called a $\mathbb{Z}_2$-taut angle structure. Let $Q(\mathcal{T})$ denote the collection of quad types of $\mathcal{T}$. Recall, for example, that a solution of Thurston's gluing equations is a function

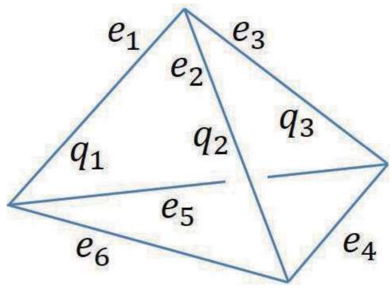
$z\colon Q(\mathcal{T}) \to \mathbb{C}$ satisfying Thurston's equations. A $\mathbb{Z}_2$-taut angle structure is a function $\omega\colon Q(\mathcal{T}) \to \{+1, -1\}$ such that the set of assignments within each tetrahedron is $\{+1, +1, -1\}$ and such that the product of assignments around any edge of the triangulation is $+1$.

We say that a $\mathbb{Z}_2$-taut angle structure $\omega\colon Q(\mathcal{T}) \to \{+1, -1\}$ is obtained from a real solution of Thurston's equations $z\colon Q(\mathcal{T}) \to \mathbb{R}$ if

$$\omega(q) = \frac{z(q)}{|z(q)|}. \tag{1}$$

Assume that $\alpha : Q(\mathcal{T}) \to \mathbb{R}$ is a strict angle structure on $\mathcal{T}$ with vanishing boundary angle holonomy. For example, if Thurston's equations on M have a solution then the imaginary parts of the shape parameters provide such a structure. (In general, this assumption can be removed by an analytic continuation argument.)

To build β, we start by introducing the **Mellin-Barnes integral** $\mathcal{I}_{\mathcal{MB}}(\mathcal{T}, \omega, \alpha)$, which is a function of the triangulation $\mathcal{T}$, a $\mathbb{Z}_2$-taut angle structure ω on $\mathcal{T}$, and a strict angle structure α on $\mathcal{T}$ (which is required for the definition but which we prove it does not depend on). $\mathcal{I}_{\mathcal{MB}}(\mathcal{T}, \omega, \alpha)$ is given by a state-integral expression. A **state** $x\colon E(\mathcal{T}) \to \mathbb{R}$ will be an assignment of a real number $x(e)$ to each edge e of the triangulation. In the state-integral each edge variable will be integrated over $\mathbb{R}$ except for an arbitrarily chosen edge which will be fixed at zero. The integrand will be a product of one factor for every ideal tetrahedron Δ in $\mathcal{T}$. Consider the following typical ideal tetrahedron where the edges in this picture are labelled with the symbols e_i, and the q_i are the quad types. Note that an edge of the triangulation may be identified with more than one edge in this tetrahedron. The function $x(e_i)$ will indicate the value of the state x on the edge of this tetrahedron labelled e_i. (Here for brevity we will be a bit imprecise with notation.)



If $\omega(q_1) = +1$, $\omega(q_2) = +1$, and $\omega(q_3) = -1$, then the factor of the integrand associated to this tetrahedron Δ is

$$\mathcal{B}^{\Delta}_{\omega,\alpha} = B(A_1, A_2)$$

where

- $A_1 = \frac{\alpha(q_1)}{\pi} + ix(e_2) + ix(e_5) - ix(e_3) - ix(e_6)$
- $A_2 = \frac{\alpha(q_2)}{\pi} + ix(e_3) + ix(e_6) - ix(e_1) - ix(e_4)$

and where $B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$ is Euler's beta function, which is based on the Gamma function $\Gamma(z)$ of a complex variable z.

Note that the two arguments of the Beta function correspond to the two quad types labelled $+1$ by the $\mathbb{Z}_2$-taut angle structure ω. Furthermore, note that the argument corresponding to some quad type is a complex number whose real part is the angle assigned that quad type by α, and whose imaginary part is exactly what you obtain for that quad type from the edge labels by doing leading-trailing deformations.

We define, for some fixed choice of edge e:

$$\mathcal{I}_{\mathcal{MB}}(\mathcal{T}, \omega, \alpha) = \frac{1}{(2\pi)^{N-1}} \int_{x:E(\mathcal{T})\to\mathbb{R},\, x(e)=0} \prod_{\Delta\in\mathcal{T}} \mathcal{B}^{\Delta}_{\omega,\alpha}\, dx.$$

These integrals do not necessarily converge. We prove in [19] that when they do converge they do not depend on the choice of the fixed edge e or the choice of the strict angle structure α.

To build a topological invariant from these building blocks we just sum up these Mellin-Barnes integrals over a suitable collection Ω_{taut} of $\mathbb{Z}_2$-taut angle structures. The resulting function we call the **Beta invariant**.

$$\beta(\mathcal{T}) = \sum_{\omega\in\Omega_{\text{taut}}} \mathcal{I}_{\mathcal{MB}}(\mathcal{T}, \omega). \tag{2}$$

This is discussed in detail in Section 9 of [19]. This collection Ω_{taut} is quite natural. To explain: a $\mathbb{Z}_2$-taut angle structure is also an S^1-valued angle structure. We prove in [19] that the manifold of S^1-valued angle structures with trivial peripheral angle holonomy in general may have several components, each of them a torus. The collection Ω_{taut} is exactly the set of $\mathbb{Z}_2$-taut angle structures appearing on a canonical component of this manifold. In the important special case the manifold has a strict angle structure with trivial peripheral angle holonomy (for example if the manifold has a complete hyperbolic structure) then it is exactly the component that structure appears on.

The beta invariant is only defined when the improper integrals in the definition converge. Otherwise we say it is undefined. Our numerical investigations lead us to believe:

Conjecture 2.1 (C. Hodgson, A. Kricker, R. Siejakowski)**.** *For every finite ideal triangulation $\mathcal{T}$ of a connected, oriented three-manifold M the Mellin-Barnes integrals involved in the definition of $\beta(\mathcal{T})$ converge and hence $\beta(\mathcal{T})$ is defined.*

In [19] we prove the following.

Theorem Assume $\mathcal{T}$ and $\mathcal{T}'$ are two ideal triangulations for M that are related by a Pachner 2-3 move. If $\beta(\mathcal{T})$ is defined, then so is $\beta(\mathcal{T}')$, and they are equal. Hence β is a topological invariant of M which we can denote $\beta(M)$.

This new topological invariant $\beta(M)$ is quite mysterious and fascinating. Given that it is appearing as the leading term in a contribution to an asymptotic expansion of a quantum invariant, and comparing this to some familiar stories in quantum topology, we are naturally led to wonder whether it can be re-expressed in terms of geometric and topological invariants with a clear interpretation.

Question 2.2 (C. Hodgson, A. Kricker, R. Siejakowski)**.** *Can $\beta(M)$ be expressed in terms of elementary geometric and topological invariants of M?*

We can be more detailed about this. Our numerical investigations lead us to expect the following. Recall that the β invariant is expressed as a sum over $\mathbb{Z}_2$-taut angle structures of the corresponding Mellin-Barnes integrals (see Equation 2). If some $\mathbb{Z}_2$-taut angle structure $\omega \in \Omega_{\mathrm{taut}}$ is obtained from a boundary parabolic $\mathrm{PSL}(2,\mathbb{R})$-representation (via a real solution of Thurston's equations, as explained above in Equation 1) then the corresponding Mellin-Barnes integral $\mathcal{I}_{\mathcal{MB}}(\mathcal{T},\omega)$ should be some topological invariant of the corresponding boundary parabolic $\mathrm{PSL}(2,\mathbb{R})$-representation. On the other hand, if there is no such representation yielding ω then that Mellin-Barnes integral should be zero.

The following is an interesting first step towards building this picture:

Conjecture 2.3 (C. Hodgson, A. Kricker, R. Siejakowski)**.** *Let $\mathcal{T}$ be an ideal triangulation of M and suppose that $\omega \in \Omega_{\mathrm{taut}}$ is a $\mathbb{Z}_2$-taut angle structure for which $\mathcal{I}_{\mathcal{MB}}(\mathcal{T},\omega) \neq 0$. Then there exists a real solution $z : Q(\mathcal{T}) \to \mathbb{R}\backslash\{0,1\}$ of Thurston's edge consistency and completeness equations yielding ω via $\omega(q) = \frac{z(q)}{|z(q)|}$.*

Luo has introduced an optimization problem which generates solutions to Thurston's equations over $\mathbb{R}$ [33]. We expect that Luo's theory will be important ingredient in understanding these questions.

3 On graded modules of Y_n-equivalence filtration on the homology cylinders

(Yuta Nozaki)

Let $\Sigma_{g,1}$ be a connected oriented compact surface of genus g with one boundary component and let $\mathcal{IC} = \mathcal{IC}_{g,1}$ be the monoid of homology cylinders over $\Sigma_{g,1}$. Goussarov [16] and Habiro [17] introduced clasper surgery and the Y_n-equivalence relation on $\mathcal{IC}$ for positive integers n. We write $Y_n\mathcal{IC}$ for the submonoid consisting of $M \in \mathcal{IC}$ which is Y_n-equivalent to the trivial homology cylinder. The quotient set $Y_n\mathcal{IC}/Y_{n+1}$ by the Y_{n+1}-equivalence is a finitely generated abelian group, which attracts considerable attention in low-dimensional topology. In particular, this group is closely related to the nilpotent quotient of the Torelli group, the Goussarov-Habiro conjecture about finite-type invariants of homology cylinders, and the homology cobordism group of homology cylinders. We study $Y_n\mathcal{IC}/Y_{n+1}$ via the surgery map $\mathfrak{s}\colon \mathcal{A}_n^c \to Y_n\mathcal{IC}/Y_{n+1}$, where $\mathcal{A}_n^c$ denotes the $\mathbb{Z}$-module of connected Jacobi diagrams with n trivalent vertices subject to the AS, IHX, and self-loop relations. Here, each univalent vertex of a Jacobi diagram is colored by an element of the set $\{1^+, 1^-, \ldots, g^+, g^-\}$. Since $\mathfrak{s}$ is surjective except $n = 1$, the group $Y_n\mathcal{IC}/Y_{n+1}$ is described by $\mathcal{A}_n^c$ and the kernel $\operatorname{Ker}\mathfrak{s}$. Moreover, it is known that $\mathfrak{s}$ is an isomorphism over $\mathbb{Q}$, and thus, $\operatorname{Ker}\mathfrak{s}$ is contained in the torsion subgroup $\operatorname{tor}\mathcal{A}_n^c$. The structure of the group $Y_n\mathcal{IC}/Y_{n+1}$ is determined for $n = 1, 2, 3, 4$ in [37, 38, 40, 41], and there are no torsion elements of order greater than 2 at this stage. Therefore, the following question naturally arises.

Question 3.1 (Y. Nozaki, M. Sato, M. Suzuki)**.** *Are there torsion elements in* $Y_n\mathcal{IC}/Y_{n+1}$ *of order greater than* 2*?*

This is closely related to the following purely combinatorial question.

Question 3.2 (Y. Nozaki, M. Sato, M. Suzuki)**.** *Are there torsion elements in* $\mathcal{A}_n^c$ *of order greater than* 2*?*

For example, if there is an element in $Y_n\mathcal{IC}/Y_{n+1}$ of order 3, then $\mathcal{A}_n^c$ must have an element of order 3. It is worth mentioning here that similar questions about another type of Jacobi diagram were posed by Stanford [42, Conjecture 2.2 and Question 2.4]. Note that there is no obvious relation between these questions and Question 3.2 since the Poincaré-Birkhoff-Witt isomorphism is defined over not $\mathbb{Z}$ but $\mathbb{Q}$. See also [42, Conjectures 10.8 and 10.15] for related questions in the case $g = 0$.

Remark 3.3. Let $\mathcal{A}_{n,l}^c$ denote the submodule of $\mathcal{A}_n^c$ generated by Jacobi diagrams whose first Betti numbers are l. Then it is known that, for $l = 0, 1$, the module $\operatorname{tor}\mathcal{A}_{n,l}^c$ is generated by torsion elements of order 2 arising from line symmetry of Jacobi diagrams.

Question 3.4 (Y. Nozaki, M. Sato, M. Suzuki)**.** *Are* $\operatorname{tor}\mathcal{A}_n^c$ *and* $\operatorname{Ker}\mathfrak{s}_n$ *included in the submodule generated by symmetric Jacobi diagrams* (*in the sense of* [41, Section 3.3])*?*

Next, we consider the restriction $\mathfrak{s}_{n,0}$ of $\mathfrak{s}$ to the 0-loop part $\mathcal{A}_{n,0}^c$. In [40], it is shown that $\operatorname{Ker}\mathfrak{s}_{2n+1,0} \subset \operatorname{Im}(\Delta_{n,0} \colon \mathcal{A}_{n,0}^c \to \mathcal{A}_{2n+1,0}^c)$ for $n \geq 1$ (see [40, Definition 3.4] or [5, Definition 40] for the definition of $\Delta_{n,0}$). We can also check that $\operatorname{Ker}\mathfrak{s}_{2n+1,0} = \operatorname{Im}\Delta_{n,0}$ when $n = 1, 2$.

Question 3.5 (Y. Nozaki, M. Sato, M. Suzuki)**.** *Does* $\operatorname{Ker}\mathfrak{s}_{2n+1,0} = \operatorname{Im}\Delta_{n,0}$ *hold for any* n*?*

This question is interesting in the light of the homology cobordism group ([5]).

4 Rasmussen type invariant from equivariant instanton Floer homology of knots

(Hayato Imori, Kouki Sato, and Masaki Taniguchi)

In [30], Kronheimer and Mrowka defined a $\mathbb{Z}$-valued knot concordance invariant $s^\sharp(K)$ which is defined in terms of the framed instanton Floer homology $I^\sharp(K)$ of knots. The definition of $s^\sharp$ can be seen as an instanton theoretic analogue of the *Rasmussen invariant* [45]. In the upcoming paper [8], the following theorem will be proven:

Theorem (Daemi-Imori-Sato-Scaduto-Taniguchi [8]) For any knot K, the limit $\widetilde{s}(K) := \lim_{n\to\infty} \frac{1}{2n} s^\sharp(\#_n K)$ exists and is an integer. Moreover, $\widetilde{s}(K)$ satisfies the following properties:

(i) $\widetilde{s}(K\#K') = \widetilde{s}(K) + \widetilde{s}(K')$ for any two knots K and K';

(ii) $\widetilde{s}(K) \leq g_4(K)$ for any knot K, where $g_4(K)$ denotes the smooth 4-genus of K;

(iii) $\widetilde{s}(T_{p,q}) = g_4(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$ for any positive torus knot $T_{p,q}$;

(iv) $|2\widetilde{s}(K) - s^\sharp(K)| \leq 1$ for any knot K.

Our approach for proving the theorem is to use Daemi–Scaduto's equivariant singular instanton Floer homology [9]. Here we note that a real-valued concordance invariant satisfying t he properties (i)–(iii) above is called a *slice-torus* invariant [32, 31]. There are several slice-torus invariants [43, 45, 31, 2] derived from different theories, and we expect that comparing $\widetilde{s}$ with them improves the understanding of the relations among these theories. In particular, the difference $\widetilde{\varepsilon}(K) := 2\widetilde{s}(K) - s^\sharp(K) \in \{-1, 0, 1\}$ can be regarded as an analogous invariant to the *epsilon-invariant* $\varepsilon(K)$ in Heegaard Floer theory [20]. Indeed, it will be shown in [8] that $\widetilde{\varepsilon}$ shares the properties (1), (3) and (6) in [20, Proposition 3.6] with ε. From this point of view, we conjecture the following relations between knot concordance invariants from equivariant singular instanton Floer theory and those from Heegaard Floer theory:

Conjecture 4.1 (Daemi-Imori-Sato-Scaduto-Taniguchi). *For any knot K, we have:*

$$\widetilde{s}(K) = \tau(K), \qquad s^\sharp(K) = \nu(K) - \nu(K^*),$$

where K^* is the mirror image of K, τ is the tau-invariant defined in [43] and ν is the nu-invariant defined in [44]. In particular, we have $\widetilde{\varepsilon}(K) = \varepsilon(K)$.

For all quasi-positive knots and alternating knots, the first equality $\widetilde{s}(K) = \tau(K)$ follows from the fact that $\widetilde{s}$ and τ are slice-torus invariants [31]. Moreover, Gong [15] proves the equality $s^\sharp(K) = 2g_4(K) - 1$ for any algebraic knot K, which gives a partial answer to the second equality $s^\sharp(K) = \nu(K) - \nu(K^*)$ for algebraic knots. (This will be extended in [8] to all quasi-positive knots with knot signature negative.) On the other hand, the following is an open problem, which is the remaining part of analogues of [20, Proposition 3.6]:

Problem 4.2 (Daemi-Imori-Sato-Scaduto-Taniguchi). *Does $\widetilde{\varepsilon}$ satisfy the following properties?*

(i) *If $\widetilde{\varepsilon}(K) = 0$, then $\widetilde{s}(K) = 0$.*

(ii) *If $|\widetilde{s}(K)| = g_4(K)$, then $\widetilde{\varepsilon}(K) = \mathrm{sgn}(\widetilde{s}(K))$.*

(iii) *If K is homologically thin in Heegaard Floer theory, then $\widetilde{\varepsilon}(K) = \mathrm{sgn}(\widetilde{s}(K))$.*

Here we note that the definition of $s^\sharp$ is analogous to the Rasmussen invariant rather than τ, and this causes difficulty proving results of $\widetilde{s}$ analogous to τ. Recently, Baldwin and Sivek [2] defined the *instanton tau-invariant* $\tau^\sharp(K)$ and the *instanton nu-invariant* $\nu^\sharp(K)$ in terms of cobordism maps of framed instanton Floer homology obtained via surgery along knots. In particular, the definitions of $\tau^\sharp$ and $\nu^\sharp$ are analogous to τ and ν respectively. Moreover, one can expect to relate $\tau^\sharp$ and $\nu^\sharp$ to $s^\sharp$ in terms of framed instanton Floer homology theory. Based on these observations, we propose the following conjecture:

Conjecture 4.3 (Daemi-Imori-Sato-Scaduto-Taniguchi). *For any knot K, we have:*

$$\widetilde{s}(K) = \tau^\sharp(K), \qquad s^\sharp(K) = \nu^\sharp(K) - \nu^\sharp(K^*).$$

5 Nielsen realization and relative genus bounds for 4-manifolds

(Hokuto Konno)

Nielsen realization problem in dimension 4

Given a smooth manifold X and a given (finite) subgroup G of the mapping class group $\pi_0(\mathrm{Diff}(X))$, the *Nielsen realization problem* for G asks whether there exists a (group-theoretic) section $s : G \to \mathrm{Diff}(X)$ of the natural map $\mathrm{Diff}(X) \to \pi_0(\mathrm{Diff}(X))$ over G. If there is a section $s : G \to \mathrm{Diff}(X)$, we say that G is *realizable* in $\mathrm{Diff}(X)$.

This problem was originally considered in dimension 2, and Kerckhoff [25] proved that any finite subgroup of the mapping class group of an oriented closed surface Σ_g is realizable. (On the other hand, Morita [39] proved that the whole mapping class group $\pi_0(\mathrm{Diff}(\Sigma_g))$ is not realizable if the genus g is large enough.)

In contrast, Raymond and Scott [46] showed that, in every dimension ≥ 3, there are nilmanifolds for which the Nielsen realization fails. Focusing on dimension 4 and simply-connected manifolds, it was recently proved by Baraglia and the author [3] and Farb and Looijenga [11] that the Nielsen realization fails for $K3$, and the author [27] generalized these results to more general spin 4-manifolds with negative signature. All of these results [46, 3, 11, 27] show that certain order 2 subgroups of the mapping class group are not realizable.

Question 5.1 (H. Konno). *Is there a 4-manifold X that admits a non-realizable odd order subgroup G of $\pi_0(\mathrm{Diff}(X))$?*

In dimension 4, positive results to Nielsen realization, namely results saying that some class of subgroups of $\pi_0(\mathrm{Diff}(X))$ are realizable, are not many for the moment. For example, Mostow's rigidity yields a positive result for hyperbolic manifolds of dimension ≥ 3, even not only for dimension 4. Farb and Looijenga [11] gave a criterion in terms of the intersection form to check a given order 2 subgroup G of $\pi_0(\mathrm{Diff}(K3))$ is realizable. Can one generalize this to more 4-manifolds and more general subgroups? Namely:

Question 5.2 (H. Konno)**.** *For a given 4-manifold X, is there a criterion in terms of the intersection form to check a given subgroup G of $\pi_0(\mathrm{Diff}(X))$ is realizable?*

Relative genus bounds

Let X be a closed oriented smooth 4-manifold and consider the punctured 4-manifold $X \backslash D^4$. Let K be a knot in $S^3 = \partial(X \backslash D^4)$. Fixing a homology class $\alpha \in H_2(X;\mathbb{Z}) \cong H_2(X \backslash D^4, S^3; \mathbb{Z})$, one may ask the minimum of the genera $g(S)$ of compact oriented surfaces S in X that are bounded by K and represent α. This is a relative version of the classical minimal genus problem for closed surfaces in a closed 4-manifold. Let us consider this problem mainly in the smooth category, and focus on giving lower bonds on $g(S)$, which we call *relative genus bounds.* To get an interesting result as a *relative* genus bound, such genus bounds are supposed to be described in terms of certain knot invariants.

There are two typical situations where one may have strong relative genus bounds:

1. The first situation is when X is a definite 4-manifold. In this case, one may have genus bounds based on diagonalization-type results, such as Ozsváth–Szabó's genus bound using their τ-invariant [43]. The Rasmussen invariant s also may be effective, not only for $X = S^4$ (see [36]).

2. The other is when X has non-trivial gauge-theoretic invariant, such as Seiberg–Witten or Bauer–Furuta invariants, or relative versions of them. In this case, one may have the adjunction inequality as a strong genus bound. See [35] for a summary of this.

The above two situations have been studied extensively, however, only the followings seem to appear as relative genus bounds that can be applied to a 4-manifold without any assumptions on the intersection form and on gauge-theoretic invariants:

(i) Genus bound obtained by applying Manolescu's relative 10/8-inequality [34] to the double branched covering of X along S. (See, for example, [28, Theorem A.1].) This genus bound is described in terms of Manolescu's κ-invariant [34] of the branched covering of S^3 along K.

(ii) Genus bound obtained by applying a relative 10/8-type inequality for a 4-manifold with involution [28, Theorem 1.1] by Miyazawa, Taniguchi, and the author to the double branched covering of X along S equipped with the covering involution. (See [28, Theorem 1.4].) This genus bound is described in terms of a variant of the κ-invariant, which is defined in [28] and takes the covering involution of the branched cover into account, of the branched covering of S^3 along K.

(iii) Genus bounds obtained from the G-signature theorem and the Levine–Tristram signature (see [35, Subsections 3.2 and 3.3] for a summary). Note that these bounds can be applied also to locally flat topological embeddings, and the smooth structure of X is not reflected.

Question 5.3 (H. Konno, J. Miyazawa, M. Taniguchi)**.** *Is there a relative genus bound that can be applied to a 4-manifold without any assumptions on the intersection form and on gauge-theoretic invariant, other than the bounds (i), (ii), (iii) listed above?*

6 Gromov hyperbolicity for fine curve graphs

(Erika Kuno)[3]

Let S_g be a closed orientable surface of genus g. In [4], Bowden, Hensel, and Webb introduced a new curve graph $C^\dagger(S_g)$ called the *fine curve graph* of S_g whose vertex is a smoothly-embedded essential simple closed curve (we simply call it a curve), and whose edge is a pair of disjoint curves. They proved that, similar to the ordinary curve graphs, the graph $C^\dagger(S_g)$ is Gromov hyperbolic.

Since the diffeomorphism groups (or the homeomorphism groups) on surfaces act naturally on fine curve graphs, we can study these groups via fine curve graphs. For a closed smooth manifold M, let $\mathrm{Diff}_0(M)$ denote the group of diffeomorphisms on M which are isotopic to the identity. In [4], they proved that several natural norms on $\mathrm{Diff}_0(S_g)$ (such as the commutator length and the fragmentation norm) are unbounded if $g \geq 1$. This contrasts with the case of higher dimensional manifolds. This result is obtained by constructing a nontrivial quasimorphism on $\mathrm{Diff}_0(S_g)$. Here, a function $\phi\colon G \to \mathbb{R}$ on a group G is called a *quasimorphism* if its *defect*

$$D(\phi) = \sup_{g,h \in G} \left|\phi(gh) - \phi(g) - \phi(h)\right|$$

is finite. A quasimorphism ϕ is *homogeneous* if $\phi(g^n) = n\phi(g)$ for every $g \in G$ and $n \in \mathbb{Z}$. Let $QH(G)$ denote the $\mathbb{R}$-vector space of homogeneous quasimorphisms on G. Naturally, the space $H^1(G) = H^1(G;\mathbb{R})$ of homomorphisms from G to $\mathbb{R}$ is a linear subspace of $QH(G)$. The quotient space $QH(G)/H^1(G)$ is called *the space of nontrivial quasimorphisms* and is denoted by $\widetilde{QH}(G)$. Bowden, Hensel, and Webb [4] also proved that the space $\widetilde{QH}(\mathrm{Diff}_0(S_g))$ of nontrivial quasimophisms on $\mathrm{Diff}_0(S_g)$ is infinite dimensional if $g \geq 1$.

Let N_g be a closed nonorientable surface of genus g. Kimura and the author [26] generalized the above results of Bowden, Hensel, and Webb [4] to N_g of genus $g \geq 3$. Namely, the fine curve graph $C^\dagger(N_g)$ is Gromov hyperbolic and the space of nontrivial quasimorphisms $\widetilde{QH}(\mathrm{Diff}_0(N_g))$ on $\mathrm{Diff}_0(N_g)$ is infinite dimensional for $g \geq 3$. Moreover, for $g = 2$ we proved that the fine curve graph $C^\dagger(N_2)$ is Gromov hyperbolic. However, as far as the author is aware, for $g = 1$ there is no proof whether the fine curve $C^\dagger(N_1)$ is Gromov hyperbolic, and for $g \leq 2$ we could not construct a nontrivial quasimorphism on $\mathrm{Diff}_0(N_g)$. We remark that for nonorientable surfaces N_g of genus $g \leq 2$, we modify the definition of the fine curve

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graph $\mathcal{C}^\dagger(N_g)$ so that two vertices form an edge if the corresponding curves intersect at most once.

Question 6.1 (E. Kuno). *For $g = 1$, is $C^\dagger(N_1)$ Gromov hyperbolic? What is the dimension of the space of nontrivial quasimorphisms $\widetilde{QH}(\mathrm{Diff}_0(N_g))$ on $\mathrm{Diff}_0(N_g)$ for $g \leq 2$?*

7 "Extended" Ford domains of Kleinian groups

(Hirotaka Akiyoshi)

The *Ford domain* of a Kleinian group Γ is the common exterior of the isometric hemispheres, $D(\Gamma) = \bigcap_{\gamma \in \Gamma - \Gamma_\infty} E(\gamma) \subset \mathbb{H}^3$, where the upper half space model is used for the hyperbolic space $\mathbb{H}^3$, Γ_∞ denotes the stabilizer subgroup of Γ with respect to $\infty \in \partial\mathbb{H}^3$, and $E(\gamma)$ denotes the exterior of the isometric hemisphere of γ. When $\mathbb{H}^3/\Gamma$ is a hyperbolic manifold with a single cusp, $D(\Gamma)$ is canonically determined by setting ∞ to be a parabolic fixed point. In this case, by choosing a fundamental domain S_∞ for the action of Γ_∞ on $\mathbb{H}^3$, $D(\Gamma) \cap S_\infty$ is a fundamental domain for the action of Γ on $\mathbb{H}^3$. If moreover M is of finite volume, then $D(\Gamma)$ is geometric dual to the canonical decomposition, the ideal polyhedral decomposition defined by Epstein-Penner [10]. Epstein-Penner's convex hull construction can be applied to a cusped manifold of infinite volume to define the *EPH-decomposition* [1].

Let us consider a natural extension of the Ford domain to the outside of $\mathbb{H}^3$ as follows. The hyperbolic space $\mathbb{H}^3$ can be regarded as a subspace of the real projective space $\mathbb{RP}^3$ via projective (or Klein) model, where every totally geodesic plane in $\mathbb{H}^3$ is the intersection of a projective plane and $\mathbb{H}^3$. The *extended Ford domain* $\widehat{D}(\Gamma)$ is the region in $\mathbb{RP}^3$ bounded by the planes supporting faces of the Ford domain such that $\widehat{D}(\Gamma) \supset D(\Gamma)$.

Our naive question is the following.

Question 7.1 (H. Akiyoshi). *What is the geometric meaning of the extended Ford domain?*

As is mentioned in my talk, the extended Ford domain is closely related to the EPH-decomposition.

Problem 7.2 (H. Akiyoshi). *Describe the relationship between the combinatorial structures of the extended Ford domain and the EPH-decomposition for a quasifuchsian manifold, or for more general cusped hyperbolic manifolds of infinite volume.*

A good starting point might be Furokawa's example of once-punctured Klein bottle groups [12, Section 6]. He found a continuous path of once-punctured Klein bottle groups which produces an extra connected component in the intersection of the Ford domain with $\partial\mathbb{H}^3$.

Problem 7.3 (H. Akiyoshi). *Study the extended Ford domains for Furokawa's example, and compare the combinatorial structures for groups before and after producing the extra connected component.*

I also expect that one can employ extended Ford domain to the theory of convex projective structures on a manifold with cusps, and vice versa. (There is a huge theory of convex projective structures. See [7] or [6] for example which seems to be closely related to this topic.)

Question 7.4 (H. Akiyoshi)**.** *Is the developed image* $\bigcup_{\gamma \in \Gamma} \widehat{D}(\Gamma)$ *properly convex in* $\mathbb{RP}^3$*? For such a group* Γ*, how are the hyperbolic structure of* $\mathbb{H}^3/\Gamma$ *and the convex projective structure of* $\Big(\bigcup_{\gamma \in \Gamma} \widehat{D}(\Gamma)\Big)/\Gamma$ *related?*

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