Recent Results on Reflection Principles in Second-Order Arithmetic

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1 Introduction

We survey recent results on reflection in second-order arithmetic. The reflection principles we consider can be roughly divided into two categories: semantic reflection and syntactic reflection.

We first sketch what we mean by semantic and syntactic reflection. A semantic reflection principle is of the form

if φ is true then there is a model which satisfies φ.

We consider both variations in the form of φ and in the properties of the models. A syntactic reflection principle is of the form

if φ is provable then φ is true.

Again, we consider variations in the form of φ; but we also consider variations on the system φ is provable.

We use standard notations for second-order arithmetic. Definitions for the systems considered in this survey can be found in Simpson [13]. We work in RCA_0 unless stated otherwise. We use $\mathbb{N}$ for the set of natural numbers inside models and ω for the "real" set of natural numbers.

There are many variations for reflection principles in reverse mathematics. Precise definitions are in the main text but we explain the notation here. Ref is used for syntactic reflection; $\omega\mathsf{Ref}$ and $\beta\mathsf{Ref}$ are used for ω- and β-model reflection, respectively; $\omega_X\mathsf{Ref}$ is used for syntactic reflection for ω-logic, with $X \in \{P, R, I\}$. To restrict the principle to a class of formulas Γ, we use Γ as a prefix: Γ-Ref. We append (T) to restrict the reflection to formulas provable in T or models satisfying T. If there is no theory after a reflection principle we may consider T empty. Append $^-$ and $^{--}$ to consider only formulas without set parameters and formulas without any parameters, respectively.

2 Lifting results from first-order logic

Frittaion [5] studied the relation between reflection and induction in second-order arithmetic. His results transfer classical results in first-order arithmetic to second-order arithmetic. Yokoyama [14] studied the relation between reflection and variations of the Paris-Harrington theorem in second-order arithmetic.

Kreisel and Lévy [6] studied the connection between reflection and induction in first-order arithmetic. Formally let $\mathsf{Ref}(T)$ formalize the schema "if φ is provable in T then φ is true" and $\mathsf{TI}(\varepsilon_0)$ is transfinite induction up to ε_0. Kreisel and Lévy showed that

$$\mathsf{EA} + \mathsf{Ref}(\mathsf{EA}) \equiv \mathsf{PA}; \text{ and}$$
$$\mathsf{PA} + \mathsf{Ref}(\mathsf{PA}) \equiv \mathsf{PA} + \mathsf{TI}(\varepsilon_0),$$

where EA is Kalmár's elementary arithmetic and PA is Peano arithmetic. A finer analysis of the equivalence between reflection and induction was given by Leivant [7]:

$$\begin{aligned}\mathsf{EA} + \mathsf{I}\Pi_n &\equiv \mathsf{EA} + \Sigma_{n+1}\text{-}\mathsf{Ref}(\mathsf{EA}) \\ &\equiv \mathsf{EA} + \Pi_{n+2}\text{-}\mathsf{Ref}(\mathsf{EA}),\end{aligned}$$

where Γ-$\mathsf{Ref}(\mathsf{EA})$ is the reflection schema $\mathsf{Ref}(\mathsf{EA})$ restricted to formulas in Γ and $\mathsf{I}\Pi_n$ is the induction schema for Π_n formulas.

Frittaion [5] showed that the relation between reflection and induction still holds in second-order arithmetic. He proved that if T_0 is a finitely axiomatizable second-order arithmetic theory and T is obtained by adding full induction to T_0, then

$$T_0 + \mathsf{Ref}(T_0) \equiv T; \text{ and}$$
$$T_0 + \mathsf{Ref}(T) \equiv T_0 + \mathsf{TI}(\varepsilon_0).$$

He also extended Leivant's result to second-order arithmetic: if T_0 is a Π^1_{k+2} finitely axiomatizable extension of RCA_0, then

$$\Pi^1_{n+1}\text{-}\mathsf{Ref}(T_0) \equiv \mathsf{I}\Pi^1_n \supseteq (\mathsf{I}\Pi^1_n)^- \equiv \Sigma^1_{n+1}\text{-}\mathsf{Ref}(T_0); \text{ and}$$
$$\Pi^1_{n+1}\text{-}\mathsf{Ref}(T) \equiv \Pi^1_n\text{-}\mathsf{TI}(\varepsilon_0) \supseteq \Pi^1_n\text{-}\mathsf{TI}(\varepsilon_0)^- \equiv \Sigma^1_{n+1}\text{-}\mathsf{Ref}(T),$$

whenever $n \geq k+1$.

The inclusions above do not reverse, as uniform reflection for Π^1_n formulas does not prove local reflection for Σ^1_n formulas and *vice-versa*. More precisely, if T_0 is a Π^1_2 finitely axiomatizable extension of ACA_0 and $n \geq 1$, then

$$T_0 + \Pi^1_{n+2}\text{-}\mathsf{Ref}(T_0) \nvdash \Sigma^1_{n+2}\text{-}\mathsf{Ref}(T_0)^-,$$

and, if $T_0 + \Sigma^1_{n+1}\text{-}\mathsf{Ref}(T_0) + \Sigma^1_{n+1}\text{-}\mathsf{AC}$ is consistent, then

$$T_0 + \Sigma^1_{n+1}\text{-}\mathsf{Ref}(T_0) \nvdash \Pi^1_{n+1}\text{-}\mathsf{Ref}(T_0)^-.$$

Frittaion asks if the consistency condition above is necessary. He also asks about the relation between local reflection, induction, transfinite induction up to ε_0 and their parameter-free variants $^-$ and $^{--}$.

We now turn back to first-order arithmetic. Paris and Harrington [9] proved the equivalence between the Paris-Harrington theorem and Π_2-$\mathsf{Ref}(\mathsf{PA})$. In [14], Yokoyama characterizes variations of the Paris-Harrington theorem as reflection theorems for subsystems of second-order arithmetic. A restricted Π^1_1 formula is a formula of the form $\forall X.\theta$ where θ is Σ^0_2. Denote the class of restricted Π^1_1 formulas by $\mathrm{r}\Pi^1_1$. Yokoyama states that:

- $\overline{\mathrm{PH}}^2$, $\mathrm{It}\overline{\mathrm{PH}}^2_2$, $\mathrm{r}\Pi^1_1$-$\mathsf{Ref}(\mathsf{I}\Sigma^0_1)$, $\mathrm{r}\Pi^1_1$-$\mathsf{Ref}(\mathsf{WKL}_0 + \mathsf{RT}^2_2)$, and the well-foundedness of ω^ω are pairwise equivalent;
- $\overline{\mathrm{PH}}^3$, $\mathrm{It}\overline{\mathrm{PH}}^2$, $\mathrm{r}\Pi^1_1$-$\mathsf{Ref}(\mathsf{I}\Sigma^0_2)$, $\mathrm{r}\Pi^1_1$-$\mathsf{Ref}(\mathsf{WKL}_0 + \mathsf{RT}^2)$, and the well-foundedness of ω^{ω^ω} are pairwise equivalent;
- $\overline{\mathrm{PH}}^{n+1}$, $\mathrm{r}\Pi^1_1$-$\mathsf{Ref}(\mathsf{I}\Sigma^0_n)$, and the well-foundedness of ω_{n+1} are pairwise equivalent, for $n \geq 1$;
- $\overline{\mathrm{PH}}$, $\mathrm{It}\overline{\mathrm{PH}}^n_k$ $(n \geq 3,\ 2 \leq k \leq \infty)$, $\mathrm{r}\Pi^1_1$-$\mathsf{Ref}(\mathsf{ACA}_0)$, and the well-foundedness of ε_0 are pairwise equivalent; and
- $\mathrm{It}\overline{\mathrm{PH}}$ is equivalent to $\mathrm{r}\Pi^1_1$-$\mathsf{Ref}(\mathsf{ACA}_0')$.

Yokoyama proves the equivalence between variants of PH and well-foundedness by measuring the largeness of finite sets using ordinals; and the equivalence between variants of PH and reflection principles using the method of indicators. For definitions of the systems above see [14].

3 β-model Reflection and Sequences of β-models

In this section we consider the characterization of β-model reflection found in Simpson [13], and the author and Yokoyama's characterization of the existence of sequences of β-models by reflection principles [8].

Any subset of $\mathbb{N}$ can be viewed as a countable coded model. Let $M \subseteq \mathbb{N}$, the sets in the coded model M are $(M)_n = \{i \mid \langle i, n\rangle \in M\}$, for $n \in \mathbb{N}$. M is a countable coded β-model iff for all $e, m \in \mathbb{N}$ and $X, Y \in M$,

$\varphi_1^1(e,m,X,Y)$ holds iff $M \models \varphi_1^1(e,m,X,Y)$ holds. Here φ_1^1 is a fixed universal lightface Π_1^1 formula (see [13, Section VII.2]). β_k-model reflection for Γ formulas is the axiom schema stating that, for all $X \subseteq \mathbb{N}$ and Γ formula $\varphi(X)$, if $\varphi(X)$ holds then there is a countable coded β_k-model M such that $X \in M$ and $M \models \varphi(X)$.

For $n \geq 1$, let Π_n^1-CA_0 denote axiom system obtained by adding Π_n^1 comprehension and Σ_1^0 induction to RCA_0; and Strong Σ_n^1-DC_0 denote the axiom system obtained by adding strong Σ_n^1 dependent choice to ACA_0. It is known that Strong Σ_1^1-DC_0 and Strong Σ_2^1-DC_0 are respectively equivalent to Π_1^1-CA_0 and Π_2^1-CA_0 [13, Theorem VII.7.6]. If $V = L$, then Strong Σ_n^1-DC_0 is equivalent to Π_n^1-CA_0.

Strong Σ_n^1-DC_0 also is equivalent to the existence of coded β_n-models. Furthermore, the Strong Σ_n^1-DC_0 are equivalent to versions of β-model reflection: Strong Σ_{k+1}^1-DC_0 is equivalent to β_{k+1}-model reflection for Σ_{k+3}^1 formulas; and Σ_{k+2}^1-DC_0 is equivalent to β_{k+1}-model reflection for Σ_{k+4}^1 formulas.

In [8], the author and Yokoyama studied sequences coded models of the form:

$$\begin{aligned} X \in Y_0 \ \in \ \cdots \ \in \ & Y_n, \\ Y_0 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} & Y_n \subseteq_{\beta_e} \mathcal{N}, \end{aligned}$$

where $e \in \omega$, $i, n \in \mathbb{N}$ and $\mathcal{N}$ is the ground model. The existence of such sequences of arbitrary length $n \in \mathbb{N}$ can be characterized by reflection principles. Let $\psi_e(i,n)$ formalize the statement "there is a sequence of coded models $Y_0, \ldots, Y_n$ such that $X \in Y_k \in Y_{k+1}$, $Y_k \subseteq_{\beta_i} Y_{k+1}$ and $Y_{k+1} \subseteq_{\beta_e} \mathcal{N}$ for all $k < n$. If $e \leq i$, then $\forall n.\psi_e(i,n)$ is equivalent to Π_{e+2}^1-$\mathsf{Ref}(\Sigma_i^1$-$\mathsf{DC}_0)$.

The author and Yokoyama also proved that particular instances of this equivalence can be characterized as determinacy axioms:

- Π_2^1-$\mathsf{Ref}(\mathsf{ACA}_0)$ is equivalent to $\forall n.(\Sigma_1^0)_n$-Det_0^*;
- Π_3^1-$\mathsf{Ref}(\Pi_1^1$-$\mathsf{CA}_0)$ is equivalent to $\forall n.(\Sigma_1^0)_n$-Det; and
- Π_3^1-$\mathsf{Ref}(\Pi_2^1$-$\mathsf{CA}_0)$ is equivalent to $\forall n.(\Sigma_2^0)_n$-Det.

Note that Π_2^1-$\mathsf{Ref}(\mathsf{ACA}_0)$, Π_3^1-$\mathsf{Ref}(\Pi_1^1$-$\mathsf{CA}_0)$ and Π_3^1-$\mathsf{Ref}(\Pi_2^1$-$\mathsf{CA}_0)$ do not satisfy the hypothesis for Frittaion's theorems described in Section 2.

During the RIMS 2021 Proof Theory Workshop, Toshiyasu Arai asked about the characterization of sequences of coded models with ordinal length.

4 Iterated Reflection and ω-reflection

Pakhomov and Walsh [11] studied the relation between iterated reflection principles and ω-model reflection principles. In [10], Pakhomov and Walsh use iterated reflection principles to study the Π_1^1 proof-theoretic ordinal of theories extending ACA_0.

We now define the iterated reflection principles Π_n^1-$\mathsf{Ref}^\alpha(T)$ and Π_n^1-$\mathsf{Ref}^{\mathbf{ON}}(T)$. Fix a theory T finitely axiomatizable by a Π_2^1 formula. Define:

$$\Pi_n^1\text{-}\mathsf{Ref}^\alpha(T) := T + \{\Pi_n^1\text{-}\mathsf{Ref}(\Pi_n^1\text{-}\mathsf{Ref}^\beta(T)) \mid \beta < \alpha\},$$

and

$$\Pi_n^1\text{-}\mathsf{Ref}^{\mathbf{ON}}(T) := \forall\alpha(\mathrm{WO}(\alpha) \to \Pi_n^1\text{-}\mathsf{Ref}(\Pi_n^1\text{-}\mathsf{Ref}^\beta(T))).$$

Any set of $\mathbb{N}$ can be viewed as a countable coded ω-model. Let $M \subseteq \mathbb{N}$, the sets in the coded ω-model M are $(M)_n = \{n \mid \langle i,n\rangle \in M\}$, for $n \in \mathbb{N}$. Let Π_n^1-$\omega\mathsf{Ref}(T)$ formalize "for a Π_n^1 formula $\varphi(X)$ and a set $X \subseteq \mathbb{N}$, there is a coded ω-model M such that $X \in M$, $M \models \varphi(X)$ and $M \models T$.

Π_1^1-$\mathsf{Ref}^{\mathbf{ON}}(T)$ is equivalent to every set being contained in an ω-model of T. This can be generalized for more complex theories: if T is a Π_{n+1}^1 axiomatizable theory, Π_n^1-$\mathsf{Ref}^{\mathbf{ON}}(T)$ is equivalent to Π_n^1-$\omega\mathsf{Ref}^{\mathbf{ON}}(T)$.

Pakhomov and Walsh use this result to uniformly prove that $|\mathsf{ACA}_0^+|_{\Pi_1^1} = \phi_2(0)$, $|\Sigma_1^1\text{-}\mathsf{AC}_0|_{\Pi_1^1} = |\Pi_2^1\text{-}\mathsf{Ref}^{\varepsilon_0}(\Sigma_1^1\text{-}\mathsf{AC}_0)| = \phi_{\varepsilon_0}(0)$, $|\mathsf{ATR}_0|_{\Pi_1^1} = \Gamma_0$, and $|\mathsf{ATR}|_{\Pi_1^1} = \Gamma_{\varepsilon_0}$. Here, $|T|_{\Pi_1^1}$ is the Π_1^1 proof theoretic ordinal of T.

The connection between iterated Π^1_1-reflection and Π^1_1 proof-theoretic ordinals was studied by Pakhomov and Walsh in [10,12]. Let T, U be theories in the language of second-order arithmetic, then $T \prec_{\Pi^1_1} U$ iff U proves $\Pi^1_1\text{-}\mathsf{Ref}(T)$, the syntactic reflection principle for Π^1_1 formulas provable in T. The restriction of $\prec_{\Pi^1_1}$ to Π^1_1 sound extensions of ACA_0 is well-founded.

Let T be a Π^1_1 sound extension of ACA_0. Denote by $|T|_{\mathsf{ACA}_0}$ the rank of T in $\prec_{\Pi^1_1}$ and by $|T|_{\Pi^1_1}$ the Π^1_1 proof theoretic ordinal of T. Let ACA_0^+ denote the axiom system consisting of ACA_0 and the assertion that, for any $X \subseteq \mathbb{N}$, the ω^{th} Turing jump of X exists. If T is a Π^1_1-sound extension of ACA_0^+, then

$$|T|_{\mathsf{ACA}_0} = |T|_{\Pi^1_1}.$$

In [10], this is proved using iterated reflection principles; in [12] a proof using cut elimination is given. They also prove that, for a given ordinal notation system α, $|\Pi^1_1\text{-}\mathsf{Ref}^\alpha(\mathsf{ACA}_0)|_{\mathsf{ACA}_0} = \alpha$ and $|\Pi^1_1\text{-}\mathsf{Ref}^\alpha(\mathsf{ACA}_0)|_{\Pi^1_1} = \varepsilon_\alpha$. Furthermore, $\Pi^1_1\text{-}\mathsf{Ref}^\alpha(\mathsf{ACA}_0)$ is Π^1_1-conservative over $\Pi^1_1(\Pi^0_3)\text{-}\mathsf{Ref}^{\varepsilon_\alpha}(\mathsf{RCA}_0)$, where $\Pi^1_1(\Pi^0_3)$ denotes the class of formulas of the form $\forall X.\psi$ with $\psi \in \Pi^0_3$.

In [11], Pakhomov and Walsh also study the Π^1_2-proof theoretic ordinal of theories. The proof theoretic dilator of a theory T is the function from $\omega_1 \cup \{\infty\}$ to $\omega_1 \cup \{\infty\}$ defined by:

$$|\alpha| \mapsto |T + \mathsf{WO}(\alpha)|_{\Pi^1_1}.$$

Write $|T|_{\Pi^1_2}$ to denote the proof theoretic dilator of T. If T is a Π^1_2 axiomatizable theory such that $|T|_{\Pi^1_2} = |\varphi^+_\alpha|$ for some ordinal α, then for any β, we have

$$|\Pi^1_2\text{-}\mathsf{Ref}^\beta(T)|_{\Pi^1_2} = |\varphi^{+\omega^\beta}_\alpha|.$$

They also proved $|\Pi^1_1\text{-}\omega\mathsf{Ref}^\alpha(\mathsf{ACA}_0)|_{\Pi^1_2} = |\varphi^+_{1+\alpha}|$. Here φ_α is the α^{th} Veblen function; $\varphi^+_\alpha(\beta)$ is the least ordinal strictly above β that is value of φ_α; and $\varphi^{+\gamma}_\alpha(\beta)$ is the γ^{th} ordinal strictly above β that is value of φ_α.

As a last note, ω-model reflection is also related to transfinite induction. Let $0 < n \leq \omega$ and fix a finite axiomatization of ACA_0. Jäger and Strahm proved that $\mathsf{ACA}_0 + \Sigma^1_{n+1}\text{-}\omega\mathsf{Ref}(\mathsf{ACA}_0) \equiv \Pi^1_n\text{-}\mathsf{TI}_0$. This is a refinement of Friedman [4], who proved $\Sigma^1_\infty\text{-}\omega\mathsf{Ref}$ is equivalent to $\Pi^1_\infty\text{-}\mathsf{TI}$. A proof of this result can be found in [13, Theorem VIII.5.4]. A particular case characterizes Σ^1_1 dependent choices: $\Sigma^1_1\text{-}\mathsf{DC}$ is equivalent to $\Sigma^1_3\text{-}\omega\mathsf{Ref}$ and to $\Pi^1_1\text{-}\mathsf{TI}$ [13, VIII.5.12].

5 ω-logic Reflection, ATR_0, $\Pi^1_1\text{-}\mathsf{CA}_0$ and Transfinite Induction

In this section we consider reflection for ω-logic. ω-logic is obtained by adding the ω-rule to second-order arithmetic:

$$\frac{\varphi(0), \Gamma \quad \varphi(1), \Gamma \quad \varphi(2), \Gamma \quad \cdots}{\forall x.\varphi(x), \Gamma}.$$

Fernández-Duque [3] mentions three ways to model the statement "φ is a theorem of ω-logic":

- there is a well-founded derivation tree formalizing an ω-proof of φ;
- there is a well-ordering Λ such that φ belongs to the set of theorems of ω-logic obtained by recursion along Λ;
- φ is in the least set closed under axioms and rules of ω-logic.

Denote "φ is a theorem of ω-logic" in these ways by $[P]\varphi$, $[R]\varphi$, and $[I]\varphi$, respectively. Therefore we have three varieties of reflection for ω-logic, one for each way. If X is one of P, R or I, let $[X|A]\varphi$ mean "there is an ω-logic proof of φ using A as an oracle". $\omega_X\mathsf{Ref}(T)$ formalizes the sentence "for all $A \subseteq \mathbb{N}$, if $[X|A]\varphi$ holds then so does φ". We write $\omega_X\mathsf{Ref}$ when T is empty.

Two of the big five subsystems of second-order arithmetic have been characterized by ω-logic reflection: Cordón-Franco *et al.* [2] proved that Π^1_2-$\omega_R\mathsf{Ref}$ is equivalent to ATR_0; and Fernández-Duque [3] proves that Π^1_3-$\omega_I\mathsf{Ref}$ is equivalent to Π^1_1-CA_0.

Arai [1] has proved the equivalence between ω-logic reflection and a transfinite induction: $\mathsf{RCA}_0 + \omega_P\mathsf{Ref}$ is equivalent to $\mathsf{RCA}_0 + \Pi^1_\omega$-$\mathsf{TI}_0$.

Cordón-Franco also proved that

$$\Sigma^1_{n+1}\text{-}\omega_R\mathsf{Ref}(\mathsf{ACA}_0) \equiv \mathsf{ATR}_0 + \Pi^1_n\text{-}\mathsf{TI},$$

where Σ^1_{n+1}-$\omega_R\mathsf{Ref}(\mathsf{ACA}_0)$ is obtained by adding the axioms of ACA_0 to the ω-logic. Fernández-Duque proved an analogous result for Π^1_1-CA_0: Σ^1_{n+1}-$\omega_I\mathsf{Ref}(\mathsf{ACA}_0)$ is equivalent to Π^1_1-$\mathsf{CA}_0 + \Pi^1_n$-TI.

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