Proof search in multi-succedent sequent calculi for intuitionistic logics [*]

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Abstract

In this note, terminating and bicomplete proof search procedures with respect to the Kripke semantics are given in multi-succedent sequent calculi for intuitionistic propositional logic and fragments of intuitionistic predicate logic. G. Mints [11, 12] in his later years investigated a proof search procedure in single-succedent sequent calculus for intuitionistic predicate logic.

1   Schütte's schema

Schütte's schema in [14] is described as follows.

Given a logic calculus, e.g., a sequent calculus and a semantics for the logic, search recursively a cut-free derivation of a given sequent in a bottom-up manner. This results in a (computable but possibly infinite) deduction tree of the given sequent. If the tree is a (finite) derivation, then it tells us that the sequent is cut-free derivable in the sequent calculus. Otherwise it yields a counter model of the sequent with respect to the semantics. Let us call the division *Schütte's dichotomy*. Thus the schema shows simultaneously the completeness of the cut-free fragment of the sequent calculus with respect to the semantics and the Hauptsatz for the calculus. The schema has been successfully applied to (first-order and higher-order) classical logic calculi by K. Schütte [15].

G. Mints [11, 12] investigates a proof search procedure in single-succedent sequent calculus for intuitionistic predicate logic with respect to Kripke semantics. On the other side a multi-succedent sequent calculus for intuitionistic logic was introduced by S. Maehara [8], and it is known that it relates to semantic tableaux and Kripke semantics, cf. [17]. However a naive proof search procedure in a multi-succedent sequent calculus for intuitionistic propositional logic may not terminate in a finite number of steps since the left rule for implication might be iterated unlimitedly. For example the following proof search for the sequent sequent $(p \supset q) \supset \bot \Rightarrow q$ does not terminate. It is desirable that a proof search procedure for a decidable logic is terminating.

$$\cfrac{\cfrac{\cfrac{(p \supset q) \supset \bot, p \Rightarrow q}{\cfrac{(p \supset q) \supset \bot, p \Rightarrow p \supset q \quad \bot, p \Rightarrow q}{(p \supset q) \supset \bot, p \Rightarrow q}}}{\cfrac{(p \supset q) \supset \bot \Rightarrow p \supset q \qquad \bot \Rightarrow q}{(p \supset q) \supset \bot \Rightarrow q}}}{}$$

In this note let us give terminating and bicomplete proof search procedures under the schema for multi-succedent sequent calculi LJpm^c and LJm^c for intuitionistic propositional logic Ip and fragments of predicate logic Iq with respect to the Kripke semantics [6]. By a bicomplete proof search procedure we mean a procedure such that we can extract a counter model from failed proof search, cf. [3]. LJpm^c denotes a modified multi-succedent sequent calculus for intuitionistic propositional logic. In the calculus conclusions of minor formulas are added in succedents when a right rule is applied. It turns out that a proof search

procedure for LJpm^c is terminating. A terminating proof search for the above sequent $(p \supset q) \supset \bot \Rightarrow q$ in LJpm^c runs as follows.

$$\cfrac{\cfrac{\cfrac{(p \supset q) \supset \bot, p \Rightarrow p \supset q, q \quad \bot, p \Rightarrow q}{(p \supset q) \supset \bot, p \Rightarrow q} \ (L \supset)}{(p \supset q) \supset \bot \Rightarrow p \supset q, q} \quad \bot \Rightarrow q}{(p \supset q) \supset \bot \Rightarrow q} \ (L \supset)$$

When the left rules $(L \supset)$ are applied with the main formula $(p \supset q) \supset \bot$, the left minor formula $p \supset q$ is accompanied with its conclusion q. The leaf sequent $(p \supset q) \supset \bot, p \Rightarrow p \supset q, q$ is not a classical tautology.

Given a sequent S_0, our search procedure yields a finite $(\land, \lor)$-tree $TR(S_0)$ of finite deductions. Each leaf in the $(\land, \lor)$-tree receives a value in $\{0, 1\}$. Any formula in the sequent at a topmost node is essentially either an atom (propositional variable) or an absurdity $\bot$, and the sequent is obviously falsified when the value is 0. Otherwise the sequent is an axiom. Furthermore if the value of the whole $(\land, \lor)$-tree is 1, then by pruning, we can extract a derivation of S_0 from the tree of deductions $TR(S_0)$. Otherwise a Kripke model is readily constructed, in which S_0 is falsified.

As a corollary and by the depth-first left-to-right implementation of proof search, we see in Section 3 that the intuitionistic propositional logic is in PSPACE. The fact was first proved by R. Ladner [7].

In Section 4 we consider proof search procedures in sequent calculi LJm^c for a fragment of the intuitionistic predicate logic Iq. Proof search procedure for the fragment is terminating and bicomplete. As a corollary we see that the fragment is decidable. One fragment denoted by $\forall^{(++,-)}\exists^{(+,-)}$ is a class of formulas in a relational language in which each positively occurring universal quantifier occurs strictly positive. Mints [9] showed that the positive fragment $\forall^{(+)}\exists^{(-)}$ is decidable.

In Section 5 we consider a proof search procedure for the full intuitionistic predicate logic Iq over a finite language possibly with function symbols, and shows that Schütte's dichotomy holds for the proof search.

2 Propositional case

In this section we consider a proof search procedure in a sequent calculus LJpm^c for the intuitionistic propositional logic Ip. $Atm = \{p_i : i \in \omega\}$ denotes a countable set of atoms (propositional variables). Atoms are denoted by $p, q, r, \ldots$. Formulas are constructed recursively from atoms and the absurdity $\bot$ by means of $\lor, \land, \supset$. *Cedents* are finite set of formulas. In the calculus LJpm^c the derived objects are *sequents*, which are ordered pairs of cedents denoted $\Gamma \Rightarrow \Delta$, where Γ is the *antecedent* and Δ the *succedent* of the sequent. Though the calculus is for intuitionistic propositional logic Ip, several formulas may occur in succedents, i.e., multi-succedent sequents. p in LJpm^c stands for *propositional*, m for *multi-succedent sequents*. Moreover the super script c in LJpm^c indicates *conclusion*. The conclusion β of an implicational minor (active) formula $\alpha \supset \beta$ in succedents is augmented in each inference rule. For example a right rule for disjunction in LJpm^c is of the form

$$\frac{\Gamma \Rightarrow \Delta, \alpha \supset \beta, \beta}{\Gamma \Rightarrow \Delta, (\alpha \supset \beta) \lor \gamma}$$

Let us explain the reason why we augment conclusions in succedents. Let LJpm be a sequent calculus for the intuitionistic propositional logic Ip in which conclusions may be absent in succedents. Namely LJpm is the propositional fragment of the calculus m-G3i in [17]. A right rule for disjunction in LJpm is of the form

$$\frac{\Gamma \Rightarrow \Delta, \alpha \supset \beta}{\Gamma \Rightarrow \Delta, (\alpha \supset \beta) \lor \gamma}$$

In searching a derivation of a given sequent, we need to analyze implicational formulas $\alpha \supset \beta$ in antecedents several times.

$$\frac{\alpha \supset \beta, \Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \supset \beta, \Gamma \Rightarrow \Delta} \ (L \supset)$$

Although we can control the number of applications of the left rule $(L \supset)$ in terms of the given sequents as in [1], let us take another route. By augmenting conclusions in succedents, we see that each formula $\alpha \supset \beta$ is used as the major formula of a right rule $(R \supset)$ at most once on each branch in the searching tree. Moreover

each formula $\alpha \supset \beta$ is used as the major formula of a left rule $(L \supset)$ at most once in a part of a branch such that the part contains no right rule $(R \supset)$. In this way we can conclude that the search procedure terminates, cf. (1) and Lemma 2.9.

A formula of the form $\alpha \supset \beta$ is an *implicational* formula.

Let β_i be formulas. $\beta_1 \supset \beta_2 \supset \cdots \supset \beta_n \supset \beta_{n+1}$ denotes the formula $\beta_1 \supset (\beta_2 \supset (\cdots \supset (\beta_n \supset \beta_{n+1}) \cdots))$ in the association to the right.

Definition 2.1 Let $\alpha \equiv (\beta_1 \supset \beta_2 \supset \cdots \supset \beta_n \supset \beta_{n+1})$ $(n \geq 0)$ with a non-implicational formula β_{n+1}, and $\alpha_k \equiv (\beta_k \supset \beta_{k+1} \supset \cdots \supset \beta_n \supset \beta_{n+1})$ for $k = 1, 2, \ldots, n$. Then let $\alpha^{\mathsf{c}} = \{\beta_{n+1}\}$ and $\alpha^{\mathsf{p}} = \{\beta_k : k = 1, 2, \ldots, n\}$.

Note that $\alpha \vee \alpha^{\mathsf{c}}$ is intuitionistically equivalent to α.

2.1 Sequent calculus $\mathsf{LJpm^c}$ for the intuitionistic propositional logic Ip

Axioms.
$$(T) \quad \Gamma \Rightarrow \Delta \text{ if } \Gamma \cap \Delta \cap Atm \neq \emptyset \qquad (\bot) \quad \Gamma \Rightarrow \Delta \text{ if } \bot \in \Gamma$$

Inference rules.
$$\frac{\alpha_0, \Gamma \Rightarrow \Delta \quad \alpha_1, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (L\vee) \qquad \frac{\alpha_0, \alpha_1, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (L\wedge)$$

where $(\alpha_0 \vee \alpha_1) \in \Gamma$ in $(L\vee)$, and $(\alpha_0 \wedge \alpha_1) \in \Gamma$ in $(L\wedge)$.

$$\frac{\Gamma \Rightarrow \Delta, \alpha_0, \alpha_0^{\mathsf{c}}, \alpha_1, \alpha_1^{\mathsf{c}}}{\Gamma \Rightarrow \Delta} \ (R\vee) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha_0, \alpha_0^{\mathsf{c}} \quad \Gamma \Rightarrow \Delta, \alpha_1, \alpha_1^{\mathsf{c}}}{\Gamma \Rightarrow \Delta} \ (R\wedge)$$

where $(\alpha_0 \vee \alpha_1) \in \Delta$ in $(R\vee)$, and $(\alpha_0 \wedge \alpha_1) \in \Delta$ in $(R\wedge)$.

$$\frac{\{\Gamma \Rightarrow \Delta, \beta, \beta^{\mathsf{c}} : \beta \in \alpha^{\mathsf{p}}\} \quad \alpha^{\mathsf{c}}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (L \supset) \qquad \frac{\alpha^{\mathsf{p}}, \Gamma \Rightarrow \alpha^{\mathsf{c}}}{\Gamma \Rightarrow \Delta} \ (R \supset)$$

where an implicational $\alpha \in \Gamma$ in $(L \supset)$, and an implicational $\alpha \in \Delta$ in $(R \supset)$.

Each sequent above the line is an *upper sequent*, and the sequent below the line is the *lower sequent* of an inference rule. For example in a $(L\vee)$, $\alpha_i, \Gamma \Rightarrow \Delta$ is an upper sequent for $i = 0, 1$, and $\Gamma \Rightarrow \Delta$ is the lower sequent. Observe that the antecedent of the lower sequent is a subset of the antecedent of each upper sequent in any inference rules, and the succedent of the lower sequent is a subset of the succcedent of each upper sequent in an inference rule other than $(R \supset)$.

Derivations in $\mathsf{LJpm^c}$ are defined as usual. These are labelled finite trees whose leaves are labelled axioms and which are locally correct with respect to inference rules. While *deductions* are labelled finite trees which are locally correct with respect to inference rules. In deductions labels of leaves may be any sequents.

A *Kripke frame* is a quasi order $\langle W, \preceq \rangle$. This means that $W \neq \emptyset$ and $\preceq$ is a reflexive and transitive relation on W. A *Kripke model* is a triple $\langle W, \preceq, V \rangle$, where $\langle W, \preceq \rangle$ is a Kripke frame, and $V : W \to \mathcal{P}(Atm)$ such that $V(w) \subset V(v)$ if $w \preceq v$.

For formulas α and $a \in W$, $a \models \alpha$ is defined recursively as follows.

1. $w \models p$ iff $p \in V(w)$. $w \not\models \bot$.

2. $w \models \alpha \vee \beta$ iff $w \models \alpha$ or $w \models \beta$. $w \models \alpha \wedge \beta$ iff $w \models \alpha$ and $w \models \beta$.

3. $w \models (\alpha \supset \beta)$ iff for any $v \succeq w$, if $v \models \alpha$, then $v \models \beta$.

A sequent $\Gamma \Rightarrow \Delta$ is *intuitionistically valid* if $w \models \bigwedge \Gamma$, then $w \models \bigvee \Delta$ for any Kripke model $\langle W, \preceq, V \rangle$ and any $w \in W$.

2.2 Proof search in LJpm$^{\text{c}}$

It is easy to define a terminating and bicomplete proof search procedure in a sequent calculus LKp for the classical propositional logic, cf. [10]. On the other side each $w \in W$ in a Kripke model $\langle W, \preceq, V \rangle$ determines a classical truth assignment which assigns a truth value to each atom and to each implicational formula. The truth value of an implicational formula is determined from ones of its components at $v \succeq w$. Let us apply the search procedure for LKp to a given sequent S in which we analyze antecedent implicational formulas, but leave the succedent implicational formulas. We obtain a deduction Tr_S, i.e., a tree of sequents which is locally correct with respect to inference rules. If every leaf sequent in the tree Tr_S is derivable, then so is the given sequent. Here each leaf sequent is of the form $\Gamma \Rightarrow \Delta, \Delta_\supset$ with the set $\Delta_\supset$ of succedent implicational formulas and it is saturated in the sense of Definition 2.3 below. By saturation we see that there exists a formula $(\alpha \supset \beta) \in \Delta_\supset$ such that $\alpha, \Gamma \Rightarrow \beta$ is intuitionistically derivable provided that $\Gamma \Rightarrow \Delta, \Delta_\supset$ is intuitionistically derivable, $\Gamma \Rightarrow \Delta$ is not an axiom and $\Delta_\supset \neq \emptyset$. We keep on examining sequents $S_{\alpha \supset \beta} = (\alpha, \Gamma \Rightarrow \beta)$ for each $(\alpha \supset \beta) \in \Delta_\supset$. This yields deductions $Tr_{S_{\alpha \supset \beta}}$, and so forth. We are constructing a tree $TR(S_0)$ of deductions for a given sequent S_0. Each node σ in the tree $TR(S_0)$ corresponds to a deduction $Tr_{S(\sigma)}$. From the whole tree $TR(S_0)$ one can extract either a derivation of S_0 or a Kripke model in which S_0 is false, cf. Theorem 2.11.

Given a sequent $S = (\Gamma \Rightarrow \Delta)$, Tr_S denotes the deduction of S constructed in a bottom-up manner. A formula is marked with a circle to indicate that the formula has not yet been analyzed. No implicational formula in succedents is marked. α° indicates that the formula α has not been analyzed. $\Gamma^\times$ denotes the set of formulas obtained from formulas in Γ by erasing the circle.

Definition 2.2 Let $S = (\Gamma \Rightarrow \Delta)$ be a sequent. $S_\supset$ denotes the set of implicational succedent formulas in Δ, while $S_\supset^r$ denotes the set of implicational formulas α in $S_\supset$ such that $\alpha^{\text{p}} \not\subset \Gamma$ or $\alpha^{\text{c}} \notin \Delta$.

The condition $\alpha^{\text{p}} \subset \Gamma$ & $\alpha^{\text{c}} \in \Delta$ means that the implicational succedent formula α has been already analyzed in the whole tree $TR(S_0)$ of deductions defined in Definition 2.7 below.

Definition 2.3 A sequent $\Gamma \Rightarrow \Delta$ is *saturated* if the following conditions are met:

1. If $(\alpha \vee \beta) \in \Gamma$, then $\{\alpha, \beta\} \cap \Gamma^\times \neq \emptyset$. If $(\alpha \wedge \beta) \in \Delta$, then $\{\alpha, \beta\} \cap \Delta^\times \neq \emptyset$.

2. If $(\alpha \vee \beta) \in \Delta$, then $\{\alpha, \beta\} \subset \Delta^\times$. If $(\alpha \wedge \beta) \in \Gamma$, then $\{\alpha, \beta\} \subset \Gamma^\times$.

3. If an implicational formula α is in Γ, then $\alpha^{\text{p}} \subset \Delta^\times$ or $\alpha^{\text{c}} \in \Gamma^\times$.

A saturated sequent $\Gamma \Rightarrow \Delta$ is *fully analyzed* if it is not an axiom, i.e., $\Gamma^\times \cap \Delta^\times \cap Atm = \emptyset$ and $\bot \notin \Gamma^\times$, and every marked formula α° in $\Gamma \cup \Delta$ is either an atom p° or $\bot^\circ$.

Note that these conditions for sequents to be saturated are only for unmarked formulas.

Definition 2.4 Put a saturated sequent S at the root of the tree Tr_S. Let us define inversion steps.

1. If an antecedent Γ contains a conjunction $(\alpha_0 \wedge \alpha_1)^\circ$ with a circle, then erase the circle and add marked conjuncts when these have not yet been analyzed: for $(\alpha_0 \wedge \alpha_1)^\circ \notin \Gamma_1$,

$$\frac{\{\alpha_i^\circ : \alpha_i \notin \Gamma_1, i = 0, 1\} \cup \Gamma_1 \cup \{\alpha_0 \wedge \alpha_1\} \Rightarrow \Delta}{(\alpha_0 \wedge \alpha_1)^\circ, \Gamma_1 \Rightarrow \Delta}$$

For simplicity let us denote it by

$$\frac{\alpha_0^\star, \alpha_1^\star, \Gamma_1, \alpha_0 \wedge \alpha_1 \Rightarrow \Delta}{(\alpha_0 \wedge \alpha_1)^\circ, \Gamma_1 \Rightarrow \Delta}$$

where $\alpha_i^\star = \alpha_i^\circ$ if $\alpha_i \notin \Gamma_1$, and $\alpha_i^\star$ is absent else.

2. If the succedent Δ contains a conjunction $(\alpha_0 \wedge \alpha_1)^\circ$, then erase the circle, add unmarked conjuncts and starred conclusions: for $(\alpha_0 \wedge \alpha_1)^\circ \notin \Delta_1$,

$$\frac{\Gamma \Rightarrow \alpha_0 \wedge \alpha_1, \Delta_1, \alpha_0, \alpha_0^{\text{c}\star} \quad \Gamma \Rightarrow \alpha_0 \wedge \alpha_1, \Delta_1, \alpha_1, \alpha_1^{\text{c}\star}}{\Gamma \Rightarrow \Delta_1, (\alpha_0 \wedge \alpha_1)^\circ}$$

where $\alpha_i^{\text{c}\star} \equiv (\alpha_i^{\text{c}})^\circ$ if $\alpha_i^{\text{c}} \notin \Delta_1$, and $\alpha_i^{\text{c}\star}$ is absent else.

3. If an antecedent Γ contains a disjunction $(\alpha_0 \vee \alpha_1)^\circ$ with a circle, then erase the circle and add starred disjuncts: for $(\alpha_0 \vee \alpha_1)^\circ \notin \Gamma_1$,

$$\frac{\alpha_0^\star, \Gamma_1, \alpha_0 \vee \alpha_1 \Rightarrow \Delta \quad \alpha_1^\star, \Gamma_1, \alpha_0 \vee \alpha_1 \Rightarrow \Delta}{(\alpha_0 \vee \alpha_1)^\circ, \Gamma_1 \Rightarrow \Delta}$$

4. If the succedent Δ contains a disjunction $(\alpha_0 \vee \alpha_1)^\circ$ with a circle, then erase the circle, add unmarked disjuncts and starred conclusions: for $(\alpha_0 \vee \alpha_1)^\circ \notin \Delta_1$,

$$\frac{\Gamma \Rightarrow \alpha_0 \vee \alpha_1, \Delta_1, \alpha_0, \alpha_1, \alpha_0^{\mathsf{c}\star}, \alpha_1^{\mathsf{c}\star}}{\Gamma \Rightarrow \Delta_1, (\alpha_0 \vee \alpha_1)^\circ}$$

5. Suppose the antecedent Γ contains an implication α° with a circle. Then erase the circle, add unmarked premises and starred conclusions of premises to the succedent, and add starred conclusion to the antecedent: for $\alpha^\circ \notin \Gamma_1$,

$$\frac{\{\alpha, \Gamma_1 \Rightarrow \Delta, \beta, \beta^{\mathsf{c}\star} : \beta \in \alpha^{\mathsf{p}}\} \quad \alpha^{\mathsf{c}\star}, \alpha, \Gamma_1 \Rightarrow \Delta}{\alpha^\circ, \Gamma_1 \Rightarrow \Delta}$$

Observe that if the lower sequent enjoys the the following condition (1), then so do the upper sequents in each inversion step.

$$\alpha^{\mathsf{c}} \in \Delta^\times \text{ for any implicational formula } \alpha \text{ in the succedent } \Delta. \tag{1}$$

As far as one of the inversions can be performed, continue it to construct the tree Tr_S.

Definition 2.5 Define numbers $b(\alpha)$ for marked or unmarked formulas α recursively.

1. $b(\alpha) = b(p^\circ) = b(\bot^\circ) = 0$ for any unmarked formulas α.

2. $b((\alpha \vee \beta)^\circ) = b((\alpha \wedge \beta)^\circ) = b(\alpha^\circ) + b(\beta^\circ) + 1$.

3. $b(\alpha^\circ) = \max(\{b(\beta^{\mathsf{c}\circ}) : \beta \in \alpha^{\mathsf{p}}\} \cup \{b(\alpha^{\mathsf{c}\circ})\}) + 1$ for implicational α.

For sequents $S = (\Gamma \Rightarrow \Delta)$, let $b(S) = \sum\{b(\alpha) : \alpha \in \Gamma \cup \Delta\}$.

Proposition 2.6 *The number decreases when we go up in the tree Tr_S. Namely in the above inversion steps $b(S_1) < b(S_0)$ holds for the lower sequent S_0 and an upper sequent S_1.*

Hence the process terminates, and Tr_S is a finite tree. Each leaf in Tr_S for a saturated sequent S is a saturated sequent, which is an axiom, or a fully analyzed sequent.

A fully analyzed sequent $S = (\Gamma, \Gamma_\supset \Rightarrow \Delta, \Delta_\supset^r)$ with $\Delta_\supset^r = S_\supset^r \neq \emptyset$ is extended by an 'inference rule' *branching* (br), where $\Gamma_\supset$ is the set of antecedent unmarked implicational formulas. $\Gamma_\supset^\circ$ is obtained from $\Gamma_\supset$ by marking each formula to analyze these again, $\Gamma_\supset^\circ = \{\alpha^\circ : \alpha \in \Gamma_\supset\}$. Let $\alpha^{\mathsf{p}\star} = \{\beta^\circ : \beta \in \alpha^{\mathsf{p}}, \beta \notin \Gamma\}$.

$$\frac{\{\alpha^{\mathsf{p}\star} \cup \Gamma \cup \Gamma_\supset^\circ \Rightarrow \alpha^{\mathsf{c}\circ} : \alpha \in \Delta_\supset^r\}}{\Gamma, \Gamma_\supset \Rightarrow \Delta, \Delta_\supset^r} \text{ (br)}$$

where each formula $\alpha \in \Delta_\supset^r$ is said to be a *major formula* of the rule (br).

This inference rule is a disjunctive one since if one of upper sequents is derivable, then so is the lower sequent using the inference rule $(R \supset)$. Note that each upper sequent $S_\alpha = (\alpha^{\mathsf{p}\star} \cup \Gamma \cup \Gamma_\supset^\circ \Rightarrow \alpha^{\mathsf{c}\circ})$ enjoys the condition (1) vacuously since $(S_\alpha)_\supset = (S_\alpha)_\supset^r = \emptyset$.

2.3 Construction of the tree of deductions

Let us define a tree $TR(S_0)$ of deductions for saturated sequents S_0. Each node in the tree $TR(S_0)$ corresponds to a deduction Tr_S.

$^{<\omega}\omega$ denotes the set of finite sequences of natural numbers $\sigma, \tau, \ldots$. The empty sequence is denoted $\emptyset$, and $(k_0, \ldots, k_{n-1}) * (j) = (k_0, \ldots, k_{n-1}, j)$.

Definition 2.7 Given a saturated sequent $S_0 = (\Gamma_0 \Rightarrow \Delta_0)$, let us define a tree $TR(S_0) \subset {}^{<\omega}\omega$, and a labeling function $(S(\sigma), g(\sigma), d(\sigma))$ for $\sigma \in TR(S_0)$, where $S(\sigma)$ is a saturated sequent, $g(\sigma) \in \{\vee, \wedge, 0, 1\}$ is a (logic) gate and $d(\sigma)$ is a deduction possibly with the branching rule such that

1. for each leaf σ in $TR(S_0)$, $d(\sigma)$ consists solely of the saturated sequent $S(\sigma)$, and either

    (a) $S(\sigma)$ is an axiom and $g(\sigma) = 1$ indicating that $S(\sigma)$ is derivable, or
    (b) $S(\sigma)$ is fully analyzed and $g(\sigma) = 0$ indicating that $S(\sigma)$ is underivable.

2. for each internal node σ in $TR(S_0)$, $g(\sigma) \in \{\vee, \wedge\}$ and

    (a) if $g(\sigma) = \wedge$, then $S(\sigma)$ is neither an axiom nor fully analyzed one, and $d(\sigma) = Tr_{S(\sigma)}$.
    (b) if $g(\sigma) = \vee$, then $S(\sigma)$ is a fully analyzed sequent such that $(S(\sigma))_{\supset}^r \neq \emptyset$, and $d(\sigma)$ is a deduction with a single inference rule (br) with its lower sequent $S(\sigma)$.

$TR(S_0)$ is a $(\wedge, \vee)$-tree, and it is constructed inductively according to $\wedge-$, $\vee-$**stage**.

initial. First the empty sequence $\emptyset \in TR(S_0)$ and $S(\emptyset) = S_0$. If S_0 is an axiom, then $g(\emptyset) = 1$. If S_0 is fully analyzed with $(S_0)_{\supset}^r = \emptyset$, then $g(\emptyset) = 0$. If $g(\emptyset) \in \{0, 1\}$, then $d(\emptyset)$ is the deduction consisting solely of S_0. If S_0 is fully analyzed with $(S_0)_{\supset}^r \neq \emptyset$, then $g(\emptyset) = \vee$ and the tree is extended according to the $\vee$-**stage**. Otherwise $g(\emptyset) = \wedge$ and the tree is extended according to the $\wedge$-**stage**.

$\wedge$-**stage**. Suppose $\sigma \in TR(S_0)$ and $g(\sigma) = \wedge$. Let $S(\sigma) = S$. Let $\{S_i\}_{i<I}$ ($I > 0$) be an enumeration of all leaves in $d(\sigma) = Tr_S$. For each $i < I$, let $\sigma * (i) \in TR(S_0)$ with $S(\sigma * (i)) = S_i$. If S_i is an axiom, then $g(\sigma * (i)) = 1$. Otherwise S_i is fully analyzed. If $(S_i)_{\supset}^r = \emptyset$, then $g(\sigma * (i)) = 0$. Otherwise let $g(\sigma * (i)) = \vee$ and the tree is extended according to the $\vee$-**stage**.

$\vee$-**stage**. Suppose $\sigma \in TR(S_0)$ and $g(\sigma) = \vee$. Let $S(\sigma) = S$ be a fully analyzed sequent $\Gamma, \Gamma_{\supset} \Rightarrow \Delta, \{\alpha_j\}_{j<J}$ ($J > 0$) where $\Gamma_{\supset}$ is the set of antecedent unmarked implicational formulas, and $\{\alpha_j\}_{j<J} = S_{\supset}^r$. Then $\sigma * (j) \in TR(S_0)$ for each $j < J$. Also let $S(\sigma * (j)) = (\alpha_j^{\mathtt{p}\star} \cup \Gamma \cup \Gamma_{\supset}^{\circ} \Rightarrow \alpha_j^{\mathtt{c}\circ})$, where each unmarked and implicational formula $\alpha \in \Gamma_{\supset}$ is marked in $\Gamma_{\supset}^{\circ}$ to be analyzed again. $d(\sigma)$ denotes the following deduction:

$$\frac{\{\alpha_j^{\mathtt{p}\star} \cup \Gamma \cup \Gamma_{\supset}^{\circ} \Rightarrow \alpha_j^{\mathtt{c}\circ}\}_{j<J}}{\Gamma, \Gamma_{\supset} \Rightarrow \Delta, \{\alpha_j\}_{j<J}} \text{ (br)}$$

Let $g(\sigma * (j)) = 1$ if $S(\sigma * (j))$ is an axiom, and $g(\sigma * (j)) = 0$ if it is fully analyzed with $(S(\sigma * (j)))_{\supset}^r = \emptyset$. Otherwise let $g(\sigma * (j)) = \wedge$ and the tree is extended according to the $\wedge$-**stage**.

Definition 2.8 For a formula α, let $Sbfml_{\supset}^{+}(\alpha)$ $[Sbfml_{\supset}^{-}(\alpha)]$ denote a set of positive [negative] implicational subformulas in α defined by simultaneous recursion as follows.

1. $Sbfml_{\supset}^{\pm}(p) = Sbfml_{\supset}^{\pm}(\bot) = \emptyset$.

2. $Sbfml_{\supset}^{\pm}(\alpha_0 \vee \alpha_1) = Sbfml_{\supset}^{\pm}(\alpha_0 \wedge \alpha_1) = Sbfml_{\supset}^{\pm}(\alpha_0) \cup Sbfml_{\supset}^{\pm}(\alpha_1)$.

3. Let α be an implicational formula.
   $Sbfml_{\supset}^{+}(\alpha) = \bigcup\{Sbfml_{\supset}^{-}(\beta) : \beta \in \alpha^{\mathtt{p}}\} \cup Sbfml_{\supset}^{+}(\alpha^{\mathtt{c}}) \cup \{\alpha\}$.
   $Sbfml_{\supset}^{-}(\alpha) = \bigcup\{Sbfml_{\supset}^{+}(\beta) : \beta \in \alpha^{\mathtt{p}}\} \cup Sbfml_{\supset}^{-}(\alpha^{\mathtt{c}})$.

For sequents $S = (\Gamma \Rightarrow \Delta)$, let $Sbfml_{\supset}^{\pm}(S) = \bigcup\{Sbfml_{\supset}^{\mp}(\alpha) : \alpha \in \Gamma\} \cup \bigcup\{Sbfml_{\supset}^{\pm}(\beta) : \beta \in \Delta\}$. $n_{\supset}^{+}(S)$ $[n_{\supset}^{-}(S)]$ denotes the cardinality of the finite set $Sbfml_{\supset}^{+}(S)$ $[Sbfml_{\supset}^{-}(S)]$, resp. and $n(S)$ the total number of occurrences of connectives $\vee, \wedge, \supset$ in S.

Lemma 2.9 *The whole process generating the tree $TR(S_0)$ terminates, and the number of $\vee$-gates along any branch in the tree $TR(S_0)$ is at most $n_{\supset}^{\pm}(S_0)$.*

Proof. Consider a branching rule in the tree $TR(S_0)$.

$$\frac{\overline{\quad}{\frac{}{S}} \text{ (br) } I}{\begin{array}{c} \vdots \ \pi \\ \cdots \quad \alpha_j^{\mathtt{p}\star} \cup \Gamma \cup \Gamma_{\supset}^{\circ} \Rightarrow \alpha_j^{\mathtt{c}\circ} \quad \cdots \\ \hline \Gamma, \Gamma_{\supset} \Rightarrow \Delta, \{\alpha_j\}_{j<J} \end{array}} \text{ (br)}$$

Each major formula α_j of the branching rule is in the set $Sbfml^+_\supset(S_0)$. In the branch π up to an upper sequent $S_j = (\alpha_j^{\text{p}\star} \cup \Gamma \cup \Gamma_\supset^\circ \Rightarrow \alpha_j^{\text{c}\circ})$, each formula in the set α_j^{p} is in the antecedents of sequents S in π. Moreover S enjoys the condition (1) since S_j enjoys it and it is preserved upward.

We claim that there occurs no branching rule I on π above the upper sequent S_j one of whose major formulas is the formula α_j. Suppose that there is such a branching rule I, and let S be the lower sequent of I. Then the formula α_j is in the succedent of S, and hence α_j^{c} is in the succedent by (1). Furthermore each formula in the set α_j^{p} is in the antecedent of sequent S. Hence $\alpha_j \notin S^r_\supset$, and α_j is not a major formula of the branching rule with the lower sequent S.

Therefore there are at most $n^+_\supset(S_0)$ applications of the branching rules along a branch in $TR(S_0)$. This means that the number of $\vee$-gates along any branch in the tree $TR(S_0)$ is at most $n^+_\supset(S_0)$. Since each deduction Tr_S is finite, the whole process generating the tree $TR(S_0)$ terminates. □

Let us compute the value of the $(\wedge, \vee)$-tree $TR(S_0)$ with gates $g(\sigma)$. Let $v(\sigma)$ denote the value of $\sigma \in TR(S_0)$. If the value $v(\emptyset)$ is 1, then S_0 is derivable where a derivation of S_0 is obtained by putting the deduction $Tr_{S(\sigma)}$ of $S(\sigma)$ from $\{S_i\}_{i<I}$

$$
\begin{array}{ccccc}
\cdots & & S_i & & \cdots \\
& \searrow & \downarrow & \swarrow & \\
& & (S(\sigma), \wedge) & &
\end{array}
$$

to the $\wedge$-node σ, and choosing one of upper sequents $S(\sigma * (i, j_i))$ such that $v(\sigma * (i, j_i)) = 1$ for each lower sequent $S(\sigma * (i))$ of (br), i.e., $g(\sigma * (i)) = \vee$.

In what follows consider the case when the value $v(\emptyset)$ of $TR(S_0)$ is 0. In a bottom-up manner let us shrink the tree $TR(S_0)$ to a tree $T \subset TR(S_0)$ as follows. Simultaneously a set $V_T(\sigma)$ of atoms is assigned. For each node $\sigma \in T$ $g(\sigma) \neq \vee$. First $\emptyset \in T$.

Suppose $v(\sigma) = 0$ for a node $\sigma \in T$ with $g(\sigma) = \wedge$. Pick a son $\sigma * (i)$ such that $v(\sigma * (i)) = 0$, and identify the node $\sigma * (i)$ with σ. This means that we have chosen an underivable sequent $S(\sigma * (i))$, which is a non-axiom leaf in the deduction $Tr_{S(\sigma)}$. Let $V_T(\sigma) = \Gamma(\sigma)^\times \cap Atm$ where $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ denotes the leaf sequent $S(\sigma * (i))$ chosen from $Tr_{S(\sigma)}$. If $g(\sigma * (i)) = 0$, then σ will be a leaf in T. Otherwise $g(\sigma * (i)) = \vee$, and keep its sons in a shrunken tree, i.e., $\sigma * (i, j) \in T$.

Thus we have defined a Kripke model $\langle T, \subset_e, V_T \rangle$ where $\sigma \subset_e \tau$ iff σ is an initial segment of τ. In this case we say that τ is an extension of σ. By definition, $\sigma \subset_e \sigma$.

For each $\sigma \in T$, $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ denotes the leaf sequent $S(\sigma * (i))$ chosen from $Tr_{S(\sigma)}$.

From the construction we see readily the followings for any $\sigma, \tau \in T$.

1. $\sigma \subset_e \tau \Rightarrow \Gamma(\sigma)^\times \subset \Gamma(\tau)^\times$. $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ is saturated. $\Gamma(\sigma)^\times \cap \Delta(\sigma)^\times$ has no common atom, and $\perp \notin \Gamma(\sigma)^\times$.

2. if an implicational formula $\alpha \in \Delta(\sigma)$, then there exists an extension $\tau \in T$ of σ such that $\alpha^{\text{p}} \subset \Gamma(\tau)^\times$ and $\alpha^{\text{c}} \in \Delta(\tau)^\times$.

Proposition 2.10 *If $\alpha \in \Gamma(\sigma)^\times$ $[\alpha \in \Delta(\sigma)^\times]$, then $\sigma \models \alpha$ $[\sigma \not\models \alpha]$, resp. in the Kripke model $\langle T, \subset_e, V_T \rangle$. Hence $\sigma \models \bigwedge \Gamma(\sigma)$ and $\sigma \not\models \bigvee \Delta(\sigma)$.*

Proof. By simultaneous induction on α using the above facts. □

Theorem 2.11 (Schütte's dichotomy)
For any saturated sequent S_0, $v(\emptyset) = 1$ iff $\mathsf{LJpm^c} \vdash S_0$.

Specifically if $v(\emptyset) = 0$, then each Kripke model $\langle T, \subset_e, V_T \rangle$ falsifies the given saturated sequent S_0, no matter which non-axiom leaves are chosen from Tr_{S_i}.

On the contrary, if $v(\emptyset) = 1$, then we can extract a (cut-free) derivation of S_0 by choosing a derivable sequent from each (br).

Hence $\mathsf{LJpm^c}$ is intuitionistically complete in the sense that any intuitionistically valid sequent is derivable in $\mathsf{LJpm^c}$. Moreover $\mathsf{LJpm^c}$ admits the Hauptsatz, i.e., the cut rule is admiissible: if both of the sequents $\Gamma_0 \Rightarrow \Delta_0, \alpha$ and $\alpha, \Gamma_1 \Rightarrow \Delta_1$ are derivable in $\mathsf{LJpm^c}$, then so is the sequent $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1$.

Proof. Let $S = (\Gamma \Rightarrow \Delta, \Delta_\supset)$ be a sequent without circles, where $\Delta_\supset = S_\supset$ is the set of implicational formulas in the succedent. Then $S_0 = (\Gamma^\circ \Rightarrow \Delta^\circ, \Delta_\supset)$ is saturated. If a derivable sequent is chosen from upper sequents of a (br), then the rule (br) turns to a $(R \supset)$. □

Definition 2.12 For a sequent $S_0 = (\Gamma_0 \Rightarrow \Delta_0)$, let $S_0^\circ = (\Gamma_0^\circ \Rightarrow \Delta_1^\circ, \Delta_\supset)$ be the saturated sequent, where $\Delta_0 = \Delta_1 \cup \Delta_\supset$ with $\Delta_\supset = (S_0)_\supset$. Then $De(S_0)$ denotes the deduction with the branching rule (br), obtained from $TR(S_0^\circ)$ by carrying out the intermediate deductions $Tr_{S(\sigma)}$ for $\sigma \in TR(S_0^\circ)$ with $g(\sigma) = \wedge$.

Proposition 2.13 *Let S_0 be a sequent and π be a branch in the deduction $De(S_0)$.*

1. *For each implicational formula α in $Sbfml_\supset^+(S_0^\circ)$, there is at most one application of a rule* (br) *with the major formula α along the branch π. Hence the number of applications of the rule* (br) *along the branch π is at most $n_\supset^+(S_0^\circ)$.*

2. *The number of applications of the left rule $(L \supset)$ along the branch π is at most $n_\supset^-(S_0^\circ)\left(1 + n_\supset^+(S_0^\circ)\right)$.*

3. *The depth of the deduction $De(S_0)$ is bounded by $n(S_0)\left(1 + n_\supset^+(S_0^\circ)\right)$.*

If the sequent S_0 is intuitionistically valid, then we can extract a derivation D of S_0 in $\mathsf{LJpm^c}$ from $De(S_0)$ enjoying the above conditions. In D the number of applications of the rule $(R \supset)$ along each branch is at most $n_\supset^+(S_0^\circ)$.

Proof. These are seen from the proofs of Proposition 2.6 and Lemma 2.9 using the fact $b(S) \le n(S)$. □

3 The intuitionistic propositional logic is in PSPACE

In this section we describe a PSPACE-algorithm deciding the deducibility of the given sequent S_0 in $\mathsf{LJpm^c}$. This is a result due to R. Ladner [7], who shows that a modal logic $\mathsf{S4}$ is in PSPACE, and it is well known that $\mathsf{S4}$ interprets the intuitionistic propositional logic Ip linearly. J. Hudelmaier [4] sharpens the result by giving an $O(n \log n)$-space decision procedure for Ip.

Corollary 3.1 (Ladner [7])
The intuitionistic propositional logic Ip is in PSPACE.

Recall that $De(S_0)$ denotes the deduction obtained from $TR(S_0^\circ)$ by carrying out the intermediate deductions $Tr_{S(\sigma)}$ for $\sigma \in TR(S_0^\circ)$ with $g(\sigma) = \wedge$. Let $\#S$ be the *size* of the sequents S, which is the total number of occurrences of symbols in S.

Proposition 3.2 *Let $\vec{S} = S_0, S_1, \ldots, S_{n-1}$ be a branch in $De(S_0)$, where S_{i+1} is an upper sequent of an inference rule with its lower sequent S_i. Then $\#\vec{S} := \sum_{i<n} \#S_i$ is bounded by a (quartic) polynomial in the size $\#S_0$ of the given sequent S_0.*

Proof. From Proposition 2.13 and $n(S_0) \le \#S_0$ we see that the length n of branches $\vec{S}$ is at most $(1 + k^+(S_0)) \cdot \#S_0$, which is bounded by a quadratic polynomial in $\#S_0$. On the other side the maximal size $\#S_i$ of sequents S_i is bounded by a quadratic polynomial, too, since each S_i is essentially a sequence of subformulas of formulas in S_0. □

Let us traverse sequents in the tree $De(S_0)$ starting from the root S_0 as follows. Let S be the current sequent.
Case 1 If S is a lower sequent of an inference rule in $De(S_0)$, then visit the leftmost upper sequent next.
Case 2 Otherwise S is a leaf in $De(S_0)$, and in $TR(S_0)$. Let $\sigma \in TR(S_0)$ be the node such that $S(\sigma) = S$. $S(\sigma)$ is a leaf in a deduction $Tr_{S(\tau)}$ for a $\tau \in TR(S_0)$ with $g(\tau) = \wedge$.
Case 2.1 First consider the case when $g(\sigma) = 1$.
Case 2.1.1 Let T be the uppermost sequent below S in $Tr_{S(\tau)}$ such that there is an upper sequent S' of the inference rule with its lower sequent T, which we have not yet visited. Next let us visit the leftmost such sequent S' if such a sequent exists.

$$Tr_{S(\tau)} = \quad \begin{array}{c} S \\ \vdots \\ \cdots \quad \overline{ S' \quad \cdots} \\ T \\ \vdots \end{array}$$

Case 2.1.2 Suppose that there is no such sequent. This means that $S(\tau)$ is derivable. If $\tau = \emptyset$, then we are done. Otherwise $S(\tau)$ is an upper sequent of a (br), and we see that the lower sequent $S(\rho)$ of the (br) is

derivable. Let us change the gate $g(\rho) = \vee$ to $g(\rho) = 1$, and continue the search for the next visiting sequent in the deduction $Tr_{S(\kappa)}$, where $S(\rho)$ is a leaf in the deduction $Tr_{S(\kappa)}$.

Case 2.2 Second consider the case when $g(\sigma) = 0$. This means that $S(\tau)$ is underivable. If $\tau = \emptyset$, then we are done. Otherwise $S(\tau)$ is an upper sequent of a (br).

Case 2.2.1 If $S(\tau)$ is not the rightmost upper sequent, then visit the next right one.

Case 2.2.2 Otherwise the lower sequent $S(\rho)$ of the (br) is underivable. Let us change the gate $g(\rho) = \vee$ to $g(\rho) = 0$, and continue the search for the next visiting sequent in the deduction $Tr_{S(\kappa)}$, where $S(\rho)$ is a leaf in the deduction $Tr_{S(\kappa)}$.

In the PSPACE-algorithm, we record sequences $\vec{S} = S_0, S_1, \ldots, S_{n-1}$ of sequents on a tape, where $\vec{S}$ is an initial segment of a branch in $De(S_0)$. The next sequence $\vec{S}'$ is recursively computed as follows. If the tail S_{n-1} is a lower sequent of an inference rule in $De(S_0)$, then $\vec{S}' = \vec{S} * (S_n)$, i.e., extend the sequence $\vec{S}$ by adding the leftmost upper sequent S_n as a tail, cf. **Case 1** in the traversal. Suppose $S_{n-1} = S(\sigma)$ is a leaf in a deduction $Tr_{S(\tau)}$.

First consider the case when $g(\sigma) = 1$. If there is an $S_i\,(i < n-1)$ such that S_{i+1} is not the rightmost upper sequent, then break the sequence $\vec{S}$ at S_i and put the next right upper sequent S', $\vec{S}' = S_0, \ldots, S_i, S'$ for the maximal such i, cf. **Case 2.1.1**.

$$\frac{\cdots \quad S_{i+1} \quad S' \quad \cdots}{S_i}$$

Suppose there is no such S_i. If $\tau = \emptyset$, then halt and print 'DERIVABLE'. Otherwise $S(\tau)$ is an upper sequent of a (br) with the lower sequent $S(\rho)$. Continue the computation of the next sequence $\vec{S}'$ in the deduction $Tr_{S(\kappa)}$, where $S(\rho)$ is a leaf in the deduction $Tr_{S(\kappa)}$, cf. **Case 2.1.2**.

Second consider the case when $g(\sigma) = 0$. If $\tau = \emptyset$, then halt and print 'UNDERIVABLE'. Otherwise $S(\tau)$ is an upper sequent of a (br). If $S(\tau)$ is not the rightmost upper sequent, then break the sequence $\vec{S}$ at S_i and put the next right upper sequent S', $\vec{S}' = S_0, \ldots, S_i, S'$, where $S_{i+1} = S(\tau)$ and S_i is the lower sequent of the (br), cf. **Case 2.2.1**.

$$\frac{\cdots \quad S_{i+1}(= S(\tau)) \quad S' \quad \cdots}{S_i} \text{ (br)}$$

Otherwise continue the computation of the next sequence $\vec{S}'$ in the deduction $Tr_{S(\kappa)}$, where the lower sequent S_i of the (br) is a leaf in the deduction $Tr_{S(\kappa)}$.

In each moment, we see from Proposition 3.2 that the size $\#\vec{S}$ of the recorded sequence $\vec{S}$ is bounded by a (quartic) polynomial in the size $\#S_0$ of the given sequent S_0. Therefore the algorithm runs in a polynomial space in the size $\#S_0$ of the input S_0.

When we restrict the number of positive implicational subformulas to a constant, the decision problem is in the polytime hierarchy. Let $\supset_c^+$ denote the class of sequents S such that $n_\supset^+(S) \leq c\,(c \in \omega)$.

Corollary 3.3 *The decision problem of the intuitionistic validity for formulas in the class $\supset_c^+$ is solvable in Π_{1+c}^p of the polytime hierarchy.*

Proof. This is seen from Proposition 2.13. $\square$

R. Statman [16] showed that lp is PSPACE-complete. Is the decision problem for the class $\supset_c^+$ of formulas Π_{1+c}^p-complete for each $c \in \omega$?

4   A decidable fragment of lq

A *relational* language $\mathcal{L}$ of the predicate logic (without equality) consists of *propositional connectives* $\bot, \vee, \wedge, \supset$, *quantifiers* $\exists, \forall$, (finite) list of *predicate symbols* $R, \ldots$, *individual contants* $c, \ldots$ *Free variables* $a_i\,(i \in \omega)$ are denoted $a, \ldots$, and *bound variables* $x_i, (i \in \omega)$ are denoted x, y, z. $FV = \{a_i : i \in \omega\}$ denotes the set of free variables. Each predicate symbol receives a fixed positive integer, its *arity*. In this section Atm denotes the set of atomic formulas $R(t_1, \ldots, t_n)$, where t_i is a term , i.e., either an individual constant or a free variable. A formula is said to be *relational* if it is a formula in a relational language. lq denotes the intuitionistic predicate logic over a relational language.

4.1 Sequent calculus $\mathsf{LJm^c}$ for lq

$\mathsf{LJm^c}$ is obtained from $\mathsf{LJpm^c}$ by adding the following inference rules for quantifiers with their *eigenvariables* a and non-empty lists $\{t_i\}_i$ of terms:

$$\frac{\alpha(a), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (L\exists) \quad \frac{\Gamma \Rightarrow \alpha(a), \alpha(a)^c}{\Gamma \Rightarrow \Delta} \ (R\forall) \quad \frac{\{\alpha(t_i)\}_i \cup \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (L\forall) \quad \frac{\Gamma \Rightarrow \Delta \cup \{\alpha(t_i), \alpha(t_i)^c\}_i}{\Gamma \Rightarrow \Delta} \ (R\exists)$$

where $\exists x\, \alpha(x) \in \Gamma$ in $(L\exists)$, $\forall x\, \alpha(x) \in \Delta$ in $(R\forall)$, $\forall x\, \alpha(x) \in \Gamma$ in $(L\forall)$, and $\exists x\, \alpha(x) \in \Delta$ in $(R\exists)$.

A *Kripke model* for a relational language $\mathcal{L}$ is a quadruple $\langle W, \preceq, D, I \rangle$, where $\langle W, \preceq \rangle$ is a Kripke frame. This has to enjoy the following for $a \preceq b$. $D : W \to \mathcal{P}(X)$ for a set X such that $\emptyset \neq D(a) \subset D(b)$, and for each $a \in W$, $I(a)$ is an $\mathcal{L}$-structure with the universe $D(a)$ and relations $R^a \subset D(a)^n$ for n-ary predicate symbols $R \in \mathcal{L}$ and elements $c^a \in D(a)$ for individual contants $c \in \mathcal{L}$ such that $R^a \subset R^b$, and $c^a = c^b$.

For closed formulas $\alpha \in \mathcal{L}(X)$ and $a \in W$, $a \models \alpha$ is defined recursively.

1. $a \models R(c_1, \ldots, c_n)$ iff $(c_1, \ldots, c_n) \in R^a$. $a \not\models \bot$.

2. $a \models \alpha \lor \beta$ iff $a \models \alpha$ or $a \models \beta$. $a \models \alpha \land \beta$ iff $a \models \alpha$ and $a \models \beta$.

3. $a \models (\alpha \supset \beta)$ iff for any $b \succeq a$, if $b \models \alpha$, then $b \models \beta$.

4. $a \models \exists x\, \alpha(x)$ iff there exists a $c \in D(a)$ such that $a \models \alpha(c)$.

5. $a \models \forall x\, \alpha(x)$ iff for any $b \succeq a$ and any $c \in D(b)$, $b \models \alpha(c)$.

4.2 Proof search for a fragment

A formula is said to be *positive* (with respect to quantifiers) iff any universal quantifier [existential quantifier] occurs only positively [negatively] in it, resp. Here positive/negative occurrence of quantifiers is meant in the usual classical sense.

G. Mints [9] showed that it is decidable whether or not a given positive formula is intuitionisitically derivable, cf. [2] for an alternative proof. Let us introduce a class $\forall^{(++,-)}\exists^{(+,-)}$ of formulas.

Definition 4.1 A relational formula is defined to be in the class $\forall^{(++,-)}\exists^{(+,-)}$ iff each positively occurring universal quantifier occurs strictly positive. Universal quantifiers may occur negatively, and existential quantifiers may occur positively and/or negatively.

In this subsection we show that there is an algorithm, which decides the intuitionistic derivability of formulas in the class $\forall^{(++,-)}\exists^{(+,-)}$.

A sequent is said to be in the class $\forall^{(++,-)}\exists^{(+,-)}$ iff any succedent formula in it is in $\forall^{(++,-)}\exists^{(+,-)}$, universal quantifiers occur only positively in its antecedent formulas.

A formula of the form $\forall x\, \alpha(x)$ is a *universal formula*. In a proof-search of sequents in $\forall^{(++,-)}\exists^{(+,-)}$ only sequents in $\forall^{(++,-)}\exists^{(+,-)}$ are produced.

For a sequent S, let $VC(S)$ denote the set of free variables and individual contents occurring in a formula in S. If there is no such free variable nor individual constant in S, let $VC(S) = \{c\}$ for an individual constant c. $S = (\Gamma \Rightarrow \Delta)$ is *saturated* iff it enjoys the following conditions besides ones in Definition 2.3:

1. If $(\exists y\, \beta(y)) \in \Gamma$, then $(\beta(b)) \in \Gamma^\times$ for a free variable $b \in FV$.

2. If $(\exists y\, \beta(y)) \in \Delta$, then $(\beta(a)) \in \Delta^\times$ for every $a \in VC(S)$.

3. If $(\forall x\, \alpha(x)) \in \Gamma$, then $(\alpha(a)) \in \Gamma^\times$ for every $a \in VC(S)$.

A saturated sequent $\Gamma \Rightarrow \Delta$ is *fully analyzed* if it is not an axiom, any marked formula α° in $\Gamma \cup \Delta$ is one of atomic formulas $R(t_1, \ldots, t_n)^\circ$, $\bot^\circ$ ($\bot^\circ \notin \Gamma$).

Inversion steps for $\land, \lor, \supset$ are as in Definition 2.4. For quantifiers define as follows.

1. If an antecedent Γ of a sequent $S = (\Gamma \Rightarrow \Delta)$ contains a universal formula $(\forall y\, \beta(y))^\circ$ with a circle, then erase the circle and add starred instances for $VC(S)$: for $\Gamma = \{(\forall y\, \beta(y))^\circ\} \cup \Gamma_1$ with $(\forall y\, \beta(y))^\circ \notin \Gamma_1$,

$$\frac{\{\beta(a)^\star : a \in VC(S)\} \cup \Gamma_1 \cup \{\forall y\, \beta(y)\} \Rightarrow \Delta}{(\forall y\, \beta(y))^\circ, \Gamma_1 \Rightarrow \Delta}$$

where $\{\beta(a)^\star : a \in VC(S)\} = \{\beta(a)^\circ : a \in VC(S), \beta(a) \notin \Gamma_1\}$.

2. If a succedent Δ of a sequent $S = (\Gamma \Rightarrow \Delta)$ contains a universal formula $(\forall y\, \beta(y))^\circ$ with a circle, then erase the circle: for $\Delta = \Delta_1 \cup \{(\forall y\, \beta(y))^\circ\}$ with $(\forall y\, \beta(y))^\circ \notin \Delta_1$,

$$\frac{\Gamma \Rightarrow \Delta_1, \forall y\, \beta(y)}{\Gamma \Rightarrow \Delta_1, (\forall y\, \beta(y))^\circ}$$

3. If a succedent Δ of a sequent $S = (\Gamma \Rightarrow \Delta)$ contains an existential formula $(\exists y\, \beta(y))^\circ$ with a circle, then erase the circle and add instances for $VC(S)$ together with starred conclusions: for $\Delta = \Delta_1 \cup \{(\exists y\, \beta(y))^\circ\}$ with $(\exists y\, \beta(y))^\circ \notin \Delta_1$,

$$\frac{\Gamma \Rightarrow \{\exists y\, \beta(y)\} \cup \Delta_1 \cup \{\beta(a), \beta(a)^{\mathsf{c}\star} : a \in VC(S)\}}{\Gamma \Rightarrow \Delta_1, (\exists y\, \beta(y))^\circ}$$

   where $\{\beta(a), \beta(a)^{\mathsf{c}\star} : a \in VC(S)\} = \{\beta(a) : a \in VC(S)\} \cup \{\beta(a)^{\mathsf{c}\circ} : a \in VC(S), \beta(a)^{\mathsf{c}} \notin \Delta_1\}$.

4. Suppose that an antecedent Γ contains an existential formula $(\exists y\, \beta(y))^\circ$ with a circle, and Γ contains an instance $\beta(a)$ of $\exists y\, \beta(y)$ for a free variable a. Then erase the circle: Let $\Gamma = \{(\exists y\, \beta(y))^\circ, \beta(a)\} \cup \Gamma_1$ with $(\exists y\, \beta(y))^\circ \notin \Gamma_1$

$$\frac{\Gamma_1, \exists y\, \beta(y), \beta(a) \Rightarrow \Delta}{(\exists y\, \beta(y))^\circ, \beta(a), \Gamma_1 \Rightarrow \Delta}$$

5. Suppose that an antecedent Γ contains an existential formula $(\exists y\, \beta(y))^\circ$ with a circle, and Γ contains no instance $\beta(a)$ of $\exists y\, \beta(y)$ for free variables a. Then erase the circle, add a starred instance with an eigenvariable and instances for the eigenvariable together with starred conclusions: Let $\Gamma_\forall$ denote the set of universal unmarked formulas in Γ, and $\Delta_\exists$ the set of existential unmarked formulas in Δ. Then $\Gamma = \{(\exists y\, \beta(y))^\circ\} \cup \Gamma_1 \cup \Gamma_\forall$ with $(\exists y\, \beta(y))^\circ \notin \Gamma_1$ and $\Delta = \Delta_1 \cup \Delta_\exists$

$$\frac{\beta(b)^\circ, \Gamma_1, \exists y\, \beta(y), \Gamma_\forall(b)^\star \Rightarrow \Delta_1, \Delta_\exists(b), \Delta_\exists(b)^{\mathsf{c}\star}}{(\exists y\, \beta(y))^\circ, \Gamma_1, \Gamma_\forall \Rightarrow \Delta_1, \Delta_\exists}$$

   where the variable b does not occur in the lower sequent, $\Gamma_\forall(b)^\star = \{\alpha(b)^\circ : \forall x\, \alpha(x) \in \Gamma_\forall, \alpha(b) \notin \Gamma\}$, $\Delta_\exists(b) = \{\gamma(b) : \exists z\, \gamma(z) \in \Delta_\exists\}$ and $\Delta_\exists(b)^{\mathsf{c}\star} = \{\gamma(b)^{\mathsf{c}\circ} : \exists z\, \gamma(z) \in \Delta_\exists, \gamma(b)^{\mathsf{c}} \notin \Delta\}$.

   This inversion step for existential antecedent formulas is a condensed one. The lower sequent is derivable from the upper one using some $(L\forall)$ and some $(R\exists)$ followed by an $(L\exists)$.

Observe that if the lower sequent enjoys the the condition (1), then so do the upper sequents in each inversion step.

As far as one of the inversions can be performed, continue it to construct a tree Tr_S. As in Proposition 2.6 we see that the process terminates, and Tr_S is a finite tree. Its depth is bounded by the size $\#S$ of the sequent S. Each leaf in Tr_S for a saturated sequent S is seen to be a saturated sequent, which is either an axiom or a fully analyzed sequent.

Let $S = (\Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta, \Delta^r_\supset, \Delta_\forall)$ be a sequent in $\forall^{(++,-)}\exists^{(+,-)}$, where $\Gamma_\supset$ denotes the set of unmarked implicational formulas in the antecedent, $\Gamma_\forall$ the set of unmarked universal formulas in the antecedent, $\Delta^r_\supset = S^r_\supset$ defined in Definition 2.2, and $\Delta_\forall$ the set of (unmarked) universal formulas in the succedent, resp. $\Gamma \cup \Delta$ is the remainders.

Assume that $\Delta^r_\supset \cup \Delta_\forall \neq \emptyset$. The sequent follows from several sequents. The branching rule (br) here is of the form:

$$\frac{\{\alpha^{\mathsf{p}\star} \cup \Gamma \cup \Gamma^\circ_\supset, \Gamma^\circ_\forall \Rightarrow \alpha^{\mathsf{c}\circ} : \alpha \in \Delta^r_\supset\} \quad \{\Gamma, \Gamma^\circ_\supset, \Gamma^\circ_\forall \Rightarrow \gamma(a), \gamma(a)^{\mathsf{c}\circ} : (\forall y\, \gamma(y)) \in \Delta_\forall\}}{\Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta, \Delta^r_\supset, \Delta_\forall} \text{ (br)}$$

where a is an eigenvariable of the (br) and does not to occur in the sequent $\Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \forall y\, \gamma(y)$. Implicational formulas and universal formulas in the antecedent are marked to be analyzed again. Each formula $\alpha \in \Delta^r_\supset$ and each $(\forall y\, \gamma(y)) \in \Delta_\forall$ is a *major formula* of the rule (br), and each $\gamma(a)$ is a *minor formula*.

Let S_0 be a given saturated sequent in the class $\forall^{(++,-)}\exists^{(+,-)}$. As in the propositional case, Definition 2.7, a $(\wedge, \vee)$-tree of deductions $TR(S_0)$ is constructed from deductions $d(\sigma) = Tr_{S(\sigma)}$ for $\sigma \in TR(S_0)$ with $g(\sigma) = \wedge$, where $S(\sigma)$ denotes the sequent at the node σ in $TR(S_0)$.

Definition 4.2 Let α be a formula. $n_\forall^{++}(\alpha)$ $[n_\exists^\pm(\alpha)]$ denotes the number of nesting of strictly positive universal quantifiers in α [the number of positive/negative existential quantifiers in α], resp. $q(\alpha)$ denote the maximal number of bound variables occurring in a positive existential subformula, and of bound variables occurring in a negative universal subformula in α. These are defined recursively as follows. $n_\forall^{++}(\alpha) = n_\exists^\pm(\alpha) = q^\pm(\alpha) = 0$ if α is an atomic formula. $n_\forall^{++}(\alpha_0 \circ \alpha_1) = \max\{n_\forall^{++}(\alpha_i) : i = 0, 1\}$, $n_\exists^\pm(\alpha_0 \circ \alpha_1) = n_\exists^\pm(\alpha_0) + n_\exists^\pm(\alpha_1)$, and $q^\pm(\alpha_0 \circ \alpha_1) = \max\{q^\pm(\alpha_i) : i = 0, 1\}$ for $\circ \in \{\vee, \wedge\}$. $n_\forall^{++}(\alpha \supset \beta) = n_\forall^{++}(\beta)$. $n_\exists^\pm(\alpha \supset \beta) = n_\exists^\mp(\alpha) + n_\exists^\pm(\beta)$. $q^\pm(\alpha \supset \beta) = \max\{q^\mp(\alpha), q^\pm(\beta)\}$. $n_\forall^{++}(\exists x\, \alpha) = n_\forall^{++}(\alpha)$. $n_\forall^{++}(\forall x\, \alpha) = 1 + n_\forall^{++}(\alpha)$. $n_\exists^+(\exists x\, \alpha) = 1 + n_\exists^+(\alpha)$. $n_\exists^-(\exists x\, \alpha) = n_\exists^-(\alpha)$. $n_\exists^\pm(\forall x\, \alpha) = n_\exists^\pm(\alpha)$. $q^+(\forall x\, \alpha) = q^+(\alpha)$. $q^-(\exists x\, \alpha) = q^-(\alpha)$. $q^-(\forall x\, \alpha) = 1 + q^-(\alpha)$ and $q^+(\exists x\, \alpha) = 1 + q^+(\alpha)$ if x occurs in α. $q^-(\forall x\, \alpha) = q^-(\alpha)$ and $q^+(\exists x\, \alpha) = q^+(\alpha)$ otherwise.

For sequents $S_0 = (\Gamma \Rightarrow \Delta)$, let $n_\forall^{++}(S_0) = \max\{n_\forall^{++}(\alpha) : \alpha \in \Delta\}$, $n_\exists^-(S_0) = n_\exists^-(\bigwedge \Gamma \supset \bigvee \Delta)$, and $q^+(S_0) = q^+(\bigwedge \Gamma \supset \bigvee \Delta)$.

Lemma 4.3 *The whole process generating the tree $TR(S_0)$ for $\forall^{(++,-)}\exists^{(+,-)}$-sequents terminates, and the number of $\vee$-gates along any branch in the tree $TR(S_0)$ is at most $n_\forall^{++}(S_0) + n_\supset^+(S_0) \cdot d^q$, where $d = n_\forall^{++}(S_0) + n_\exists^-(S_0) + \#VC(S_0)$ and $q = q^+(S_0)$.*

Proof. We say that a formula α is an instance of a formula β if α is obtained from β by replacing some variables by variables and individual constants.

Each major formula of a branching rule in the tree $TR(S_0)$ is an instance of a positive subformula of S_0, which is either implicational or universal.

Consider a branching rule in the tree $TR(S_0)$ one of whose major formulas is a universal formula $\forall y\, \gamma(y)$.

$$\frac{\cfrac{\overline{S}}{\vdots}^{\,(\mathrm{br})\, I} \qquad \cdots \quad \Gamma, \Gamma_\supset^\circ \Rightarrow \gamma(a), \gamma(a)^{\mathrm{c}\circ} \quad \cdots}{\Gamma, \Gamma_\supset \Rightarrow \Delta, \Delta_\supset^r, \Delta_\forall}\ (\mathrm{br})$$

where $(\forall y\, \gamma(y)) \in \Delta_\forall$ and a is the eigenvariable not occurring in $\Gamma, \Gamma_\supset \Rightarrow \forall y\, \gamma(y)$.

Suppose that there occurs a branching rule I above an upper sequent $S_\gamma = (\Gamma, \Gamma_\supset^\circ \Rightarrow \gamma(a), \gamma(a)^{\mathrm{c}\circ})$ one of whose major formulas is a universal formula $\forall z\, \delta(z)$. Then the formula $\forall z\, \delta(z)$ is an instance of a subformula of $\gamma(a)^{\mathrm{c}}$ since $(L \supset)$ is the only rule where formulas are added to the succedent of the upper sequent, which may not be a subformula of the succedent of the lower sequent, and no universal quantifier occurs in premises α^{p} of antecedent implicational formulas α in $\forall^{(++,-)}\exists^{(+,-)}$-sequents.

Therefore each universal positive subformula in S_0 is analyzed at most once along any branch in $TR(S_0)$, and there are at most $n_\forall^{++}(S_0)$ applications of the branching rules along any branch whose minor formula is an instance of a universal major formula. This means that there occurs at most one instance of the matrix $\gamma(y)$ of a universal subformula $\forall y\, \gamma(y)$ in S_0 along any branch in $TR(S_0)$.

Next consider a negative existential formula $\exists y\, \beta(y)$. Such a formula is analyzed in an inversion step

$$\frac{\beta(b)^\circ, \Gamma_1, \exists y\, \beta(y) \Rightarrow \Delta_1, \Delta_\exists(b), \Delta_\exists(b)^{\mathrm{c}\star}}{(\exists y\, \beta(y))^\circ, \Gamma \Rightarrow \Delta_1, \Delta_\exists}$$

Once it is analyzed, it will be never analyzed again, since the formula $\beta(b)$ is in the antecedents in sequents occurring above the upper sequent.

Since a new variable is introduced only when either a universal formula in succedents or an existential formula in antecedents is analyzed, each bound variable is replaced by at most one free variable along any branch. For each formula α occurring in S_0, only one instance of α occurs in the whole tree $TR(S_0)$ of deductions. Therefore the number of terms (free variables and individual constants) occurring in $De(S_0)$ is bounded by d. Furthermore along any branch, there is at most one instance α' of a positive implicational subformula α which is one of major formulas of a branching rule. As in Lemma 2.9 we see that there is at most one branching rule along any branch one of whose major formula is the instance α'.

Therefore the number of $\vee$-gates along any branch in the tree $TR(S_0)$ is at most $n_\forall^{++}(S_0) + n_\supset^+(S_0) \cdot d^q$. Therefore the whole process generating the tree $TR(S_0)$ terminates. $\qquad \square$

As for the propositional case, let us compute the value $v(\sigma)$ of the node σ in the $(\wedge, \vee)$-tree $TR(S_0)$. If the value $v(\emptyset)$ is 1, then S_0 is derivable. Otherwise shrink the tree $TR(S_0)$ to a tree $T \subset TR(S_0)$ in a bottom-up

manner. Simultaneously a structure $I_T(\sigma)$ with a universe $D_T(\sigma)$ is assigned. Suppose $v(\sigma) = 0$ for a node $\sigma \in T$ with $g(\sigma) = \wedge$. Pick a son $\sigma * (i)$ such that $v(\sigma * (i)) = 0$, identify the node $\sigma * (i)$ with σ, and choose the underivable sequent $S(\sigma * (i))$, which is a non-axiom leaf in the deduction $Tr_{S(\sigma)}$. Let $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ be the leaf sequent $S(\sigma * (i))$ chosen from $Tr_{S(\sigma)}$. Then let $D_T(\sigma) = VC(S(\sigma * (i)))$ and for $a_1, \ldots, a_n \in D_T(\sigma)$ and n-ary predicate symbol R, $R(a_1, \ldots, a_n)$ is true in the structure $I_T(\sigma)$ iff $R(a_1, \ldots, a_n) \in \Gamma(\sigma)^{\times}$. If $g(\sigma * (i)) = 0$, then σ will be a leaf in T. Otherwise $g(\sigma * (i)) = \vee$, and keep its sons in a shrunken tree, i.e., $\sigma * (i, j) \in T$.

Thus we have defined a Kripke model $\langle T, \subset_e, D_T, I_T \rangle$.

For each $\sigma \in T$, $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ denotes the leaf sequent $S(\sigma * (i))$ chosen from $Tr_{S(\sigma)}$.

From the construction we see readily the followings for any $\sigma, \tau \in T$.

1. $\sigma \subset_e \tau \Rightarrow \Gamma(\sigma)^{\times} \subset \Gamma(\tau)^{\times}$.

2. $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ is saturated.

3. if an implicational formula $\alpha \in \Delta(\sigma)$, then there exists an extension $\tau \in T$ of σ such that $\alpha^{\mathsf{p}} \subset \Gamma(\tau)^{\times}$ and $\alpha^{\mathsf{c}} \in \Delta(\tau)^{\times}$.

4. if a universal formula $\forall x\, \alpha(x) \in \Delta(\sigma)$, then there exists an extension $\tau \in T$ of σ such that $\alpha(a) \in \Delta(\tau)^{\times}$ for a variable a.

5. $\Gamma(\sigma)^{\times} \cap \Delta(\sigma)^{\times}$ has no common atom, and $\perp \notin \Gamma(\sigma)^{\times}$.

Proposition 4.4 *If $\alpha \in \Gamma(\sigma)^{\times}$ [$\alpha \in \Delta(\sigma)^{\times}$], then $\sigma \models \alpha$ [$\sigma \not\models \alpha$], resp. in the Kripke model $\langle T, \subset_e, D_T, I_T \rangle$. Hence $\sigma \models \bigwedge \Gamma(\sigma)$ and $\sigma \not\models \bigvee \Delta(\sigma)$.*

Proof. By simultaneous induction on α using the above facts. □

Theorem 4.5 (Schütte's dichotomy)
For any saturated sequent S_0 in $\forall^{(++,-)} \exists^{(+,-)}$, $v(\emptyset) = 1$ iff $\mathsf{LJm^c} \vdash S_0$.

Corollary 4.6 *There exists a constant $c > 0$ for which the following hold.*
Each intuitionistically valid sequent S_0 in the class $\forall^{(++,-)} \exists^{(+,-)}$ has a derivation $\mathcal{D}$ in $\mathsf{LJm^c}$ such that $\mathcal{D}$ is a binary tree and the depth of the tree is bounded by cn^n for the size $n = \#(S_0)$ of the sequent S_0. Also the size of the sequents occurring in $\mathcal{D}$ is bounded by cn^n.

Corollary 4.7 *The decision problem of the intuitionistic validity for formulas in the class $\forall^{(++,-)} \exists^{(+,-)}$ is solvable.*

5 Proof search in LJm

In this section we consider a proof search procedure for the full intuitionistic predicate logic Iq over a finite language possibly with function symbols. Here a search tree may be infinite.

LJm denotes a standard sequent calculus $\mathsf{m\text{-}G3i}$ for the intuitionistic predicate logic Iq in [17]. For example the right rule for existential formula is of the form.

$$\frac{\Delta \Rightarrow \Gamma, \alpha(t)}{\Delta \Rightarrow \Gamma} \; (R\exists)$$

where $\exists x\, \alpha(x) \in \Gamma$ and the minor formula $\alpha(t)$ need not to be accompanied with its conclusion $\alpha(t)^{\mathsf{c}}$.

For sequents S, the search tree Tr_S is in general infinite due to the presence of universal formulas $\forall x\, \alpha(x)$ in antecedents and existential formulas $\exists y\, \beta(y)$ in succedents. A formula is *non-invertible* if it is either an implicational formula or a universal formula. It is desirable for us that each stage in constructing the tree of deductions is executed in a finite number of steps. In order to do so, each stage tests only a finite number of free variables for universal formulas in antecedents and for existential formulas in succedents. Let $\{t_i\}_i$ be an enumeration of all terms. $Tm(A)$ denotes the set of all terms over a set $A \subset FV$ of free variables, and let $Tm(A){\upharpoonright}n := \{t_i \in Tm(A) : i < n\}$.

Let $n < \omega$ and A a set of free variables. A sequent $\Gamma \Rightarrow \Delta$ is (n, A)-*saturated* iff it is saturated with respect to propositional connectives $\vee, \wedge$ as in Definition 2.3 , and it enjoys the following conditions:

1. If $\alpha \supset \beta \in \Gamma$, then $\alpha \in \Gamma^\times$ and $\beta \in \Delta^\times$.

2. If $(\exists y\, \beta(y)) \in \Gamma$, then $(\beta(a)) \in \Gamma^\times$ for a free variable $a \in FV$.

3. If $(\exists y\, \beta(y)) \in \Delta$, then $(\beta(t)) \in \Delta^\times$, and if $(\forall x\, \alpha(x)) \in \Gamma$, then $(\alpha(t)) \in \Gamma^\times$ for every $t \in Tm(A)\!\restriction\! n$.

An (n, A)-saturated sequent $\Gamma \Rightarrow \Delta$ is (n, A)-*analyzed* if it is not an axiom, any marked formula α° in $\Gamma \cup \Delta$ is one of atomic formulas $R(t_1, \ldots, t_n)^\circ$, $\bot^\circ$ ($\bot^\circ \notin \Gamma$), or non-invertible formulas in Δ. An (n, A)-analyzed sequent is *fully analyzed* if there is no existential succedent formula [no universal antecedent formula], resp. Note that each fully analyzed sequent is $(0, A)$-analyzed for every A.

A deduction $Tr_S^{(n,A)}$ is constructed in a finite number of steps as for propositional case by leaving any non-invertible succedent formulas and applying $(L\exists), (R\exists), (L\forall)$ up to the n-th terms in $Tm(A)$. Put the given sequent S at the root of the tree $Tr_S^{(n,A)}$. The inversion steps for quantifiers are as follows.

$$\frac{\alpha(a)^\circ, \Gamma, \exists x\, \alpha(x) \Rightarrow \Delta}{\exists x\, \alpha(x)^\circ, \Gamma \Rightarrow \Delta}\ (L\exists)$$

where the eigenvariable a does not occur in the lower sequent nor in the finite set A of free variables. Moreover a is chosen so that the condition ($\dagger$) in the next subsubsection 5.1 is met.

$$\frac{\Delta \Rightarrow \exists x\, \alpha(x), \Gamma, \{\alpha(t)^\circ\}_{t \in Tm(A)\restriction n}}{\Delta \Rightarrow \Gamma, \exists x\, \alpha(x)^\circ}\ (R\exists) \qquad \frac{\{\alpha(t)^\circ\}_{t \in Tm(A)\restriction n}, \Gamma, \forall x\, \alpha(x) \Rightarrow \Delta}{\forall x\, \alpha(x)^\circ, \Gamma \Rightarrow \Delta}\ (L\forall)$$

All of terms in the finite set $Tm(A)\restriction n$ are tested for existential formulas in succedent and for universal formulas in antecedent.

Each leaf in $Tr_S^{(n,A)}$ is (n, A)-saturated, which is either an axiom or an (n, A)-analyzed sequent if S is (n, A)-saturated.

5.1 Extensions for non-invertible succedent formulas and postponed instantiations

In a $\vee$-**stage** of our proof search for the predicate logic we examine all possibilities with succedent non-invertible formulas by introducing a branching rule (br) as for the propositional case. Consider an (n, A)-saturated sequent

$$\Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ$$

where $\Gamma_\supset$ denotes the set of unmarked implicational formulas, and $\Gamma_\forall$ the set of unmarked universal formulas in the antecedent, resp. $\Delta_\exists$ denotes the set of unmarked existential formulas, $\Delta_\supset^\circ$ the set of marked implicational formulas, and and $\Delta_\forall^\circ$ the set of marked universal formulas in the succedent, resp. $\Gamma \cup \Delta$ is the remainders. Each marked formula in $\Gamma \cup \Delta$ is an atomic formula $R(t_1, \ldots, t_n)^\circ$ or $\bot^\circ$ with $\bot^\circ \notin \Gamma$. Each unmarked formula in Γ is one of a disjunctive formula, a conjunctive formula and an existential formula. Each unmarked formula in Δ is either disjunctive or conjunctive.

Assume that $\Gamma_\forall \cup \Delta_\exists \cup \Delta_\supset^\circ \cup \Delta_\forall^\circ \neq \emptyset$. The sequent may follow from several sequents. Let us depict the several possibilities as an 'inference rule' as follows.

$$\frac{\Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \Delta_\exists^\circ, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ \quad \{\gamma^\circ, \Gamma, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \delta^\circ : (\gamma \supset \delta)^\circ \in \Delta_\supset^\circ\} \quad \{\Gamma, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \gamma(a)^\circ : (\forall y\, \gamma(y)) \in \Delta_\forall^\circ\}}{\Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ}\ (\text{br})$$

where the sequent $\Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \Delta_\exists^\circ, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ$ is absent when $\Gamma_\forall \cup \Delta_\exists = \emptyset$, and a is an eigenvariable distinct each other for universal formulas $(\forall y\, \gamma(y)) \in \Delta_\forall$ and chosen such that the condition ($\dagger$) below is met.

If one of upper sequents of (br) is derivable, then so is the lower sequent possibly using one of non-invertible inference rules $(R \supset)$ and $(R\forall)$. Each upper sequent $\gamma^\circ, \Gamma, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \delta^\circ$ with $(\gamma \supset \delta) \in \Delta_\supset^\circ$ and each $\Gamma, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \gamma(a)^\circ$ with $(\forall y\, \gamma(y)) \in \Delta_\forall^\circ$ is said to be a *non-invertible upper sequent* of the inference rule (br). While the leftmost upper sequent $\Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \Delta_\exists^\circ, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ$ is the *continued* sequent. Here is the condition on eigenvariables. First each eigenvariable is distinct each other, and

> ($\dagger$) each eigenvariable occurs only either above the inference rule or in the right part of the inference where the variable is introduced.

This means that if σ is the node of the upper sequent of the inference rule where an eigenvariable a is introduced, and a occurs in the sequent at a node τ, then either $\sigma \subset_e \tau$ (, i.e., τ is above σ) or $\rho * (i) \subset_e \sigma$ and $\rho * (j) \subset_e \tau$ for some ρ and $i < j$ (, i.e., τ is right to σ).

As in the propositional case of Definition 2.7, let us construct a tree $TR(S_0) \subset {}^{<\omega}\omega$ for a given sequent S_0. The tree $TR(S_0)$ is constructed in ω-steps. $TR(S_0)_n$ denotes the piece of $TR(S_0)$ in the nth step such that for each $\sigma \in TR(S_0)_n$, the *length* $lh(\sigma) \leq n$. The labeling function $(S(\sigma), d(\sigma), g(\sigma))$ for $\sigma \in \bigcup_{n \in \omega} TR(S_0)_n$ is defined simultaneously in the construction of $TR(S_0)$.

Let $S(\tau)$ be the sequent at the node τ in $TR(S_0)$, and $FV(\tau)$ the set of free variables occurring in the sequent $S(\tau)$. A finite set $FV_{\subset_e}(\sigma)$ of free variables is assigned to sequences $\sigma \in TR(S_0)$ as follows. The set is finite since the tree $TR(S_0)_n \subset {}^{<n+1}\omega$ is finitely branching.

$$FV_{\subset_e}(\sigma) = \bigcup\{FV(\tau) : \exists\rho(\rho \subset_e \sigma \;\&\; \rho \subset_e^0 \tau \in TR(S_0)_{lh(\sigma)})\}$$

where

$$\rho \subset_e^0 \tau :\Leftrightarrow \rho \subset_e \tau \;\&\; \forall\kappa[\rho \subset_e \kappa \subsetneq_e \tau \;\&\; g(\kappa) = \vee \Rightarrow \tau(lh(\kappa)) = 0]$$

with the i-th *component* $\tau(i)$ of sequences τ for $i < lh(\tau)$. $\sigma \subset_e^0 \tau$ means that τ continues to substitute terms for $\forall y$ in antecedents and $\exists x$ in succedent for any $\vee$-**stage** after σ.

Let us denote

$$Tr_\sigma := Tr_{S(\sigma)}^{(lh(\sigma), FV_{\subset_e}(\sigma))}.$$

$De(S_0)$ denotes the whole tree of deductions obtained from $TR(S_0)$ by fulfilling intermediate deductions, and is constructed recursively. Each $\wedge$-**stage** analyzes the current leaves parallel as in the propositional case. After the $\wedge$-**stage**, we extend the tree by non-invertible (br) inference rules in $\vee$-**stage**. In each moment $De(S_0)$ is constructed so that the condition ($\dagger$) on eigenvariables is met.

Definition 5.1 Given a sequent $S_0 = (\Gamma_0 \Rightarrow \Delta_0)$, let us define trees $TR(S_0)_n \subset {}^{<n+1}\omega$, and a labeling function $(S(\sigma), d(\sigma), g(\sigma))$ for $\sigma \in TR(S_0)_n$, where $S(\sigma)$ is a sequent, $g(\sigma) \in \{\vee, \wedge, 0, 1\}$ is a gate and $d(\sigma)$ is a deduction possibly with the branching rule.

First the empty sequence $\emptyset \in TR(S_0)_0 = \{\emptyset\}$ and $S(\emptyset) = S_0$ where each formula in S_0 is marked. Let $FV(\emptyset)$ be the set of free variables occurring in S_0 if the set is non-empty. Otherwise $FV(\emptyset) = \{a_0\}$. If S_0 is an axiom, then $g(\emptyset) = 1$. If S_0 is fully analyzed, then $g(\emptyset) = 0$. If $g(\emptyset) \in \{0, 1\}$, then $d(\emptyset)$ is the deduction consisting solely of S_0. Otherwise $g(\emptyset) = \wedge$ and the tree is extended according to the $\wedge$-**stage**.

Suppose that $TR(S_0)_n$ has been constructed, and there exists a leaf $\sigma \in TR(S_0)_n$ such that $g(\sigma) \in \{\wedge, \vee\}$. (Otherwise we are done, and $TR(S_0)_{n+1}$ is not defined.) If n is even [odd], the tree is extended according to the $\wedge$-**stage** [according to the $\vee$-**stage**], resp.

$\wedge$-**stage**. Consider each leaf $\sigma \in TR(S_0)_n$ with $g(\sigma) = \wedge$. Extend the tree $De(S_0)$ by putting the deduction $d(\sigma) = Tr_\sigma$ for each such σ. Let $\{S_i\}_{i<I}$ be an enumeration of all leaves in $d(\sigma)$. For each $i < I$, let $\sigma * (i) \in TR(S_0)_{n+1}$ with $S(\sigma * (i)) = S_i$. If S_i is an axiom, then $g(\sigma * (i)) = 1$. If S_i is fully analyzed, then $g(\sigma * (i)) = 0$. Otherwise S_i is not fully analyzed, but $(lh(\sigma), FV_{\subset_e}(\sigma))$-analyzed. This means that either its antecedent contains a universal formula, or its succedent contains either an existential formula or a non-invertible formula. Let $g(\sigma * (i)) = \vee$, and the tree is extended according to the $\vee$-**stage**.

$TR(S_0)_{n+1}$ is defined to be the union of $TR(S_0)_n$ and nodes $\sigma * (i)$ for each leaf $\sigma \in TR(S_0)_n$ such that $g(\sigma) = \wedge$ and $i < I$.

$\vee$-**stage**. Consider each leaf $\sigma \in TR(S_0)_n$ with $g(\sigma) = \vee$. Extend the tree $De(S_0)$ by the inference rule (br) parallel for each such σ.

Let $S(\sigma) = S$. S is a non-invertible sequent $\Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ$ where cedents $\Gamma, \Gamma_\supset, \Gamma_\forall, \Delta_\exists, \Delta, \Delta_\supset^\circ$ and $\Delta_\forall^\circ$ are defined as in the beginning of this subsection, and $\Gamma_\forall \cup \Delta_\exists \cup \Delta_\supset^\circ \cup \Delta_\forall^\circ \neq \emptyset$. Let $S(\sigma * (0))$ be the sequent $\Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \Delta_\exists^\circ, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ$. Let $\Delta_\supset^\circ = \{\beta_j^\circ\}_{0<j\leq J_\supset}$, and $\Delta_\forall^\circ = \{\beta_j^\circ\}_{J_\supset<j\leq J_\forall}$. Then $\sigma * (j) \in TR(S_0)_{n+1}$ for each j with $0 \leq j \leq J_\forall$. For $j > 0$ the sequent $S(\sigma * (j))$ is defined by analyzing the j-th non-invertible formula β_j. Namely if $\beta_j \equiv (\gamma \supset \delta)$, then $S(\sigma * (j)) = (\gamma^\circ, \Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \delta^\circ)$. If $\beta_j \equiv (\forall y \, \gamma(y))$, then $S(\sigma * (j)) = (\Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \gamma(a)^\circ)$ where the eigenvariables a are fresh, i.e., do not occur in any $S(\tau)$ for $\tau \in TR(S_0)_n$, and distinct each other for $(\forall y \, \gamma(y))^\circ \in \Delta_\forall$. $d(\sigma)$ denotes the deduction consisting of a (br) with its lower sequent $S = S(\sigma)$.

Let $g(\sigma * (j)) = 1$ if $S(\sigma * (j))$ is an axiom, and $g(\sigma * (j)) = 0$ if it is fully analyzed. Otherwise let $g(\sigma * (j)) = \wedge$ and the tree is extended according to the $\wedge$-**stage**.

$TR(S_0)_{n+1}$ is defined to be the union of $TR(S_0)_n$ and nodes $\sigma * (j)$ for each leaf $\sigma \in TR(S_0)_n$ such that $g(\sigma) = \vee$ and $j \leq J_\forall$.

Finally let $TR(S_0) = \bigcup_{n\in\omega} TR(S_0)_n$.

Let $\emptyset \neq T \subset TR(S_0)$ be a subtree of $TR(S_0)$ such that $g(\sigma) \neq 1$ for any $\sigma \in T$. Let us construct a Kripke model $\langle T, \subset_e, D_T, I_T \rangle$ as follows.

Definition 5.2   1. For $S(\sigma) = (\Gamma(\sigma) \Rightarrow \Delta(\sigma))$, let $\Gamma^\infty(\tau; T) = \bigcup\{\Gamma(\rho)^\times : \tau \subset_e^0 \rho \in T\}$, $\Gamma_{\subset_e}^\infty(\sigma; T) = \bigcup\{\Gamma^\infty(\tau; T) : \tau \subset_e \sigma\}$, and $\Delta^\infty(\tau; T) = \bigcup\{\Delta(\rho)^\times : \tau \subset_e^0 \rho \in T\}$.

   Note that $\Gamma(\sigma)^\times \subset \Gamma^\infty(\sigma; T) \subset \Gamma_{\subset_e}^\infty(\sigma; T)$ and $\Delta(\sigma)^\times \subset \Delta^\infty(\sigma; T)$ for $\sigma \in T$.

   2. Let $D_T(\sigma) = Tm(FV_{\subset_e}^\infty(\sigma; T))$, where $FV^\infty(\tau; T) = \bigcup\{FV(\rho) : \tau \subset_e^0 \rho \in T\}$ and $FV_{\subset_e}^\infty(\sigma; T) = \bigcup\{FV^\infty(\tau; T) : \tau \subset_e \sigma\}$.

   3. For $\sigma \in T$ and an n-ary predicate symbol R and function symbol f, let $R^\sigma = \{(t_1, \ldots, t_n) : t_1, \ldots, t_n \in D_T(\sigma) \,\&\, R(t_1, \ldots, t_n) \in \Gamma_{\subset_e}^\infty(\sigma; T)\}$ and $f^\sigma(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for $t_1, \ldots, t_n \in D_T(\sigma)$.

   4. $I_T(\sigma)$ is a structure with its universe $D_T(\sigma)$ and relations R^σ and functions f^σ.

Proposition 5.3 *Let $\emptyset \neq T \subset TR(S_0)$ be a subtree of $TR(S_0)$ such that $g(\sigma) \neq 1$ for any $\sigma \in T$. Then $\langle T, \subset_e, D_T, I_T \rangle$ is a Kripke model.*

Proof. Let $\sigma \subset_e \tau$ for $\sigma, \tau \in T$. Then $FV_{\subset_e}^\infty(\sigma; T) \subset FV_{\subset_e}^\infty(\tau; T)$, and hence $D_T(\sigma) \subset D_T(\tau)$. Moreover we have $\Gamma_{\subset_e}^\infty(\sigma; T) \subset \Gamma_{\subset_e}^\infty(\tau; T)$, and $R^\sigma \subset R^\tau$. □

A pair $\Gamma \Rightarrow \Delta$ of (possibly infinite) sets Γ, Δ of formulas is *A-saturated* for a set A of free variables iff it is (n, A)-saturated for any n. This means besides the saturation with respect to propositional connectives and existential formulas in Γ that if $(\exists y\, \beta(y)) \in \Delta$, then $(\beta(t)) \in \Delta^\times$, and if $(\forall x\, \alpha(x)) \in \Gamma$, then $(\alpha(t)) \in \Gamma^\times$ for any $t \in Tm(A)$. A-saturated pair $\Gamma \Rightarrow \Delta$ is *A-analyzed* if $\bot \notin \Gamma^\times$, Γ and Δ has no common atomic formula.

Proposition 5.4 *Let $\emptyset \neq T \subset TR(S_0)$ be a subtree of $TR(S_0)$ such that $g(\sigma) \neq 1$ for any $\sigma \in T$. Suppose that T enjoys the following conditions for any $\sigma \in T$.*

   1. *$\Gamma_{\subset_e}^\infty(\sigma; T) \Rightarrow \Delta^\infty(\sigma; T)$ is $FV_{\subset_e}^\infty(\sigma; T)$-analyzed.*

   2. *(a) If $(\alpha \supset \beta) \in \Delta^\infty(\sigma; T)$, then there exists an extension $\tau \in T$ of σ such that $\alpha \in \Gamma_{\subset_e}^\infty(\tau; T)$ and $\beta \in \Delta^\infty(\tau; T)$.*

      (b) If $(\forall x\, \alpha(x)) \in \Delta^\infty(\sigma; T)$, then there exist an extension $\tau \in T$ of σ and an $a \in FV_{\subset_e}^\infty(\tau; T)$ such that $(\alpha(a)) \in \Delta^\infty(\tau; T)$.

      (c) $\Gamma_{\subset_e}^\infty(\sigma; T) \cap \Delta^\infty(\sigma; T)$ has no common atomic formula.

   Let $\sigma \in T$ and α be a formula all of whose free variables are in the set $D_T(\sigma)$. In the Kripke model $\langle T, \subset_e, D_T, I_T \rangle$, if $\alpha \in \Gamma_{\subset_e}^\infty(\sigma; T)$, then $\sigma \models \alpha$, and if $\alpha \in \Delta^\infty(\sigma; T)$, then $\sigma \not\models \alpha$. Hence $\sigma \models \bigwedge \Gamma(\sigma)$ and $\sigma \not\models \bigvee \Delta(\sigma)$.

Proof. This is shown by simultaneous induction on α. Consider the case when α is an atomic formula $R(t_1, \ldots, t_n)$. By the assumption we have $\alpha \notin \Gamma_{\subset_e}^\infty(\sigma; T) \cap \Delta^\infty(\sigma; T)$. Hence if $\alpha \in \Delta^\infty(\sigma; T)$, then $\alpha \notin \Gamma_{\subset_e}^\infty(\sigma; T)$, i.e., $\sigma \not\models \alpha$. On the other side if $\alpha \in \Gamma_{\subset_e}^\infty(\sigma; T)$ and $t_1, \ldots, t_n \in D_T(\sigma)$ by the assumption, then $\sigma \models \alpha$.

Next consider the case when $\alpha \equiv (\forall x\, \beta(x))$. Suppose $\alpha \in \Gamma_{\subset_e}^\infty(\sigma; T)$. For any extension τ of σ in T, i.e., $\sigma \subset_e \tau \in T$, $\alpha \in \Gamma_{\subset_e}^\infty(\sigma; T) \subset \Gamma_{\subset_e}^\infty(\tau; T)$. $\beta(t) \in \Gamma_{\subset_e}^\infty(\tau; T)$ for any $t \in D_T(\tau)$ by the supposition. By IH $\tau \models \beta(t)$. Hence $\sigma \models \alpha$. Next suppose $\alpha \in \Delta^\infty(\sigma; T)$. Then by the supposition, for an extension $\tau \in T$ of σ and a free variable $a \in FV_{\subset_e}^\infty(\tau; T)$, we have $\beta(a) \in \Delta^\infty(\tau; T)$. Hence $a \in D_T(\tau)$ as long as a occurs in $\beta(a)$, and $\tau \not\models \beta(a)$ by IH. Thus $\sigma \not\models \alpha$. Other cases are seen easily. □

The first condition (1) and the second (2a) and the third one (2b) in Proposition 5.4 are easily enjoyed when T has sufficiently many nodes, i.e., when nodes are prolonged in T unlimitedly. It is hard to meet the fourth one (2c). Obviously the tree $TR(S_0)$ of the deductions enjoys the conditions besides the fourth one. But $\Gamma_{\subset_e}^\infty(\sigma; TR(S_0)) \cap \Delta^\infty(\sigma; TR(S_0))$ may have a common atomic formula for a $\sigma \in TR(S_0)$ with $g(\sigma) = \vee$.

Proposition 5.5 *Let $\emptyset \neq T \subset TR(S_0)$ be a subtree of $TR(S_0)$ such that $g(\sigma) \neq 1$ for any $\sigma \in T$. Suppose that each $\wedge$-gate has a unique $\vee$-gate son in T. Then $\Gamma^\infty(\sigma; T) \cap \Delta^\infty(\sigma; T)$ has no common atomic formula for any $\sigma \in T$.*

Proof. Suppose that $\Gamma^\infty(\sigma;T) \cap \Delta^\infty(\sigma;T)$ has a common atomic formula α. Let $\sigma \subset_e^0 \rho_a, \rho_s \in T$ be such that $g(\rho_a) \neq 1, g(\rho_s) \neq 1$ and $\alpha \in \Gamma^\times(\rho_a) \cap \Delta^\times(\rho_s)$. We see that ρ_a and ρ_s are comparable in the order $\subset_e^0$, since each $\wedge$-gate has a unique $\vee$-gate son in T, and each $\vee$-gate has a unique son, i.e., the leftmost continued one in $\subset_e^0$. Then $\alpha \in \Gamma^\times(\rho) \cap \Delta^\times(\rho)$ for a common extension $\rho \in \{\rho_a, \rho_s\}$, and $S(\rho)$ is an axiom. This is a contradiction $1 = g(\rho) \neq 1$. $\qquad\square$

5.2 Completeness

In order to have the Schütte's dichotomy, we need to transform the tree $TR(S_0)$ of deductions, cf. subsection 5.3. However when we need only to show the completeness of LJm with the cut rule (cut) denoted $\mathsf{LJm}+(cut)$, one can extract a consistent tree T from $TR(S_0)$ by which S_0 is refuted provided that S_0 is not derivable in $\mathsf{LJm} + (cut)$. In this subsection we consider the tree $TR(S_0)$. For formulas α, $\alpha^\forall$ denotes ambiguously formulas obtained from α by binding some (possibly none) free variables by universal quantifiers.

For formulas α and β, $\oplus(\alpha, \beta)$ denotes the formula $\forall \vec{x}(\gamma(\vec{x}) \supset (\delta(\vec{x}) \vee \beta))$ if α is a formula $\forall \vec{x}(\gamma(\vec{x}) \supset \delta(\vec{x}))$ for a (possibly empty) list $\vec{x}$ of bound variables, where $\vec{x}$ does not occur in β. Otherwise $\oplus(\alpha, \beta) :\equiv (\alpha \vee \beta)$. For sequences $\vec{\alpha} = \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ of formulas α_i, let $\oplus(\vec{\alpha}, \beta) :\equiv \oplus(\alpha_0, \oplus(\alpha_1, \cdots, \oplus(\alpha_{n-1}, \beta) \cdots))$, and $\oplus^\forall(\vec{\alpha}, \beta)$ denotes formulas $\oplus^\forall(\alpha_0, \oplus^\forall(\alpha_1, \cdots, \oplus^\forall(\alpha_{n-1}, \beta) \cdots))$.

For the sequent $S(\sigma) = (\Gamma(\sigma) \Rightarrow \Delta(\sigma))$ let $\chi(\sigma) :\Leftrightarrow (\bigwedge \Gamma(\sigma) \supset \bigvee \Delta(\sigma))$.

Let $T \subset TR(S_0)$ be a *finite* subtree such that for each $\sigma \in T$, if $g(\sigma) = \wedge$, then σ has a single son $\sigma * (i)$ in T. And if $g(\sigma) = \vee$, then either σ is a leaf in T, or σ has all of sons $\sigma * (j)$ in T, i.e., $\forall j[\sigma * (j) \in T \leftrightarrow \sigma * (j) \in TR(S_0)]$. Moreover $g(\sigma) \in \{0, \vee\}$ for any leaves σ in T. Such a tree T is called a *selected* tree. Following [6] let us introduce *characteristic formulas* $\chi(\sigma; T)$ for nodes σ of such a tree T recursively as follows. Recall that $FV(\sigma)$ denotes the set of free variables occurring in the sequent $S(\sigma)$ for $\sigma \in TR(S_0)$.

For leaves σ in T, $\chi(\sigma; T) \equiv \chi(\sigma)$. Let $\sigma \in T$ be an internal node with $g(\sigma) = \wedge$, and $\sigma * (i)$ the unique son of σ in T. Also let $\chi(\sigma * (i); T) \equiv \alpha(\vec{a})$ where $\vec{a}$ is the set of eigenvariables of inference rules $(L\exists)$ occurring between the leaf $S(\sigma * (i))$ and the root σ of the deduction Tr_σ. Then let $\chi(\sigma; T) :\equiv \chi^\forall(\sigma * (i); T) :\equiv \forall \vec{x} \, \alpha(\vec{x})$.

Let $\sigma \in T$ be an internal node with $g(\sigma) = \vee$, and $\sigma * (j) \, (j \leq J_\forall)$ all of sons of σ in T, where $\Delta_\forall^\circ = \{\beta_j\}_{J_\supset < j \leq J_\forall}$ for the set $\Delta_\forall^\circ$ of marked universal formulas in the succedent of $S(\sigma)$. For each j with $J_\supset < j \leq J_\forall$ let $\chi(\sigma * (j); T) \equiv \alpha(a)$ for the eigenvariable a introduced at the j-th upper sequent $S(\sigma * (j))$. Then for $\chi^\forall(\sigma * (j); T) \equiv \forall x \, \alpha(x)$, let

$$\chi(\sigma; T) \equiv \oplus \left(\chi(\sigma * (0); T), \bigvee_{0 < j \leq J_\supset} \chi(\sigma * (j); T) \vee \bigvee_{J_\supset < j \leq J_\forall} \chi^\forall(\sigma * (j); T) \right).$$

Finally let $\chi(T) :\equiv \chi(\sigma; T)$ for the root σ in T.

Proposition 5.6   *1.* $\mathsf{LJm} + (cut) \vdash \alpha \vee \beta \supset \oplus(\alpha, \beta)$. $\mathsf{LJm} + (cut) \vdash \oplus(\alpha, \bot) \leftrightarrow \alpha \leftrightarrow \oplus(\bot, \alpha)$ *and* $\mathsf{LJm} + (cut) \vdash \oplus^\forall(\oplus^\forall(\alpha, \beta), \gamma) \leftrightarrow \oplus^\forall(\alpha, \beta \vee \gamma)$.

2. *Let $T \subset TR(S_0)$ be a selected tree, and σ be a leaf in T. Then there exist formulas $\vec{\alpha}$ such that $\mathsf{LJm} + (cut) \vdash \chi(T) \leftrightarrow \oplus^\forall(\vec{\alpha}, \chi(\sigma))$.*

3. *Let $g(\sigma) = \wedge$ and $\mathsf{LJm} + (cut) \not\vdash \oplus^\forall(\vec{\alpha}, \oplus(\chi^\forall(\sigma), \beta))$ for some formulas $\vec{\alpha}, \beta$. Then there exists a leaf $\sigma * (i)$ in the deduction Tr_σ such that $\mathsf{LJm} + (cut) \not\vdash \oplus^\forall(\vec{\alpha}, \oplus(\chi^\forall(\sigma * (i)), \beta))$.*

4. *Let $g(\sigma) = \vee$ and $\mathsf{LJm} + (cut) \not\vdash \oplus^\forall(\vec{\alpha}, \oplus(\chi(\sigma), \beta))$ for formulas $\vec{\alpha}, \beta$. Then for each j there exists a leaf $\sigma * (j, i_j)$ in the deduction $Tr_{\sigma * (j)}$ such that $\mathsf{LJm} + (cut) \not\vdash \oplus(\vec{\alpha}, \oplus(\oplus(\chi(\sigma * (0, i_0)), \bigvee_{0 < j \leq J_\supset} \chi(\sigma * (j, i_j)) \vee \bigvee_{J_\supset < j \leq J_\forall} \chi^\forall(\sigma * (j, i_j))), \beta))$.*

5. *Let $T \subset TR(S_0)$ be a selected tree. Assume $\mathsf{LJm} + (cut) \not\vdash \chi(T)$. Then for each leaf $\sigma \in T$ and each j there exists a leaf $\sigma * (j, i_j)$ in the deduction $Tr_{\sigma * (j)}$ such that $\mathsf{LJm} + (cut) \not\vdash \chi(T')$ for the tree T' obtained from T by extending each leaf σ to $\sigma * (j), \sigma * (j, i_j)$.*

Proof. 5.6.2 is seen by induction on the size of the tree T using Proposition 5.6.1.
5.6.3 is seen by inspection to inference rules in LJm except non-invertible ones $(R \supset)$ and $(R\forall)$.
5.6.4. Assume $g(\sigma) = \vee$ and $\mathsf{LJm} + (cut) \not\vdash \oplus(\vec{\alpha}, \oplus(\chi(\sigma), \beta))$. Then

$$\mathsf{LJm} + (cut) \not\vdash \oplus(\vec{\alpha}, \oplus(\oplus(\chi(\sigma * (0)), \bigvee_{j > 0} \chi^\forall(\sigma * (j)))), \beta))$$

by the definition of the rule (br). In other words,

$$\mathsf{LJm} + (cut) \nvdash \oplus(\vec{\alpha}, \oplus(\chi(\sigma * (0)), \bigvee_{j>0} \chi^{\forall}(\sigma * (j)) \vee \beta)).$$

By Proposition 5.6.3 pick an i_0 such that

$$\mathsf{LJm} + (cut) \nvdash \oplus(\vec{\alpha}, \oplus(\chi^{\forall}(\sigma * (0, i_0)), \bigvee_{j>0} \chi^{\forall}(\sigma * (j)) \vee \beta)).$$

Hence

$$\mathsf{LJm} + (cut) \nvdash \oplus(\vec{\alpha}, \oplus(\oplus(\chi^{\forall}(\sigma * (0, i_0)), \bigvee_{j>1} \chi^{\forall}(\sigma * (j))) \vee \beta, \chi(\sigma * (1)))).$$

Then again by Proposition 5.6.3 pick an i_1 such that

$$\mathsf{LJm} + (cut) \nvdash \oplus(\vec{\alpha}, \oplus(\oplus(\chi^{\forall}(\sigma * (0, i_0)), \bigvee_{j>1} \chi^{\forall}(\sigma * (j))) \vee \beta, \chi^{\forall}(\sigma * (1, i_1)))).$$

In this way we can pick numbers i_j so that

$$\mathsf{LJm} + (cut) \nvdash \oplus(\vec{\alpha}, \oplus(\oplus(\chi^{\forall}(\sigma * (0, i_0)), \bigvee_{j>0} \chi^{\forall}(\sigma * (j, i_j))), \beta)).$$

5.6.5. Let σ be a leaf in T. By Proposition 5.6.2 we have $\mathsf{LJm} + (cut) \vdash \chi(T) \leftrightarrow \oplus^{\forall}(\vec{\alpha}, \chi(\sigma)) \leftrightarrow \oplus^{\forall}(\vec{\alpha}, \oplus(\chi(\sigma), \bot))$ for some formulas $\vec{\alpha}$. On the other side the formula $\chi(T')$ is obtained from $\chi(T)$ by replacing $\chi(\sigma)$ by $\oplus(\chi^{\forall}(\sigma * (0, i_0); T), \bigvee_{0 < j \le J_\supset} \chi^{\forall}(\sigma * (j, i_j); T) \vee \bigvee_{J_\supset < j \le J_\forall} \chi^{\forall}(\sigma * (j, i_j); T))$ for the leaf σ in T. Thus the proposition is seen from Proposition 5.6.4. □

Supposing the given sequent S_0 is underivable in $\mathsf{LJm} + (cut)$, let us pick a tree $T \subset TR(S_0)$ for which the following holds. Let $T_n = \{\sigma \in T : lh(\sigma) \le 2n + 1\}$.

1. for any $\sigma \in T$, $g(\sigma) \in \{0, \wedge, \vee\}$,

2. $\emptyset \in T$, and there exists a unique son (i) of $\emptyset$ in T such that $\mathsf{LJm} + (cut) \nvdash S((i))$. Namely $T_0 = \{\emptyset, (i)\}$.
   Let $\chi(T_0) :\equiv \chi((i)) \equiv (\bigwedge \Gamma((i)) \supset \bigvee \Delta((i)))$. Then $\mathsf{LJm} + (cut) \nvdash \chi(T_0)$.

3. for any $\sigma \in T_n$ with $g(\sigma) = \vee$, every son $\sigma * (j) \in TR(S_0)$ is in T_{n+1}, and there exists a unique son $\sigma * (j, i_j)$ for each j. Namely $T_{n+1} = T_n \cup \{\sigma * (j), \sigma * (j, i_j) : \sigma \in T_n, lh(\sigma) = 2n + 1\}$.
   Let $lh(\sigma) = 2n + 1$ and assume $\mathsf{LJm} + (cut) \nvdash \chi(T_n)$. The sons (j, i_j) are chosen so that $\mathsf{LJm} + (cut) \nvdash \chi(T_{n+1})$. Such an extension is possible by Proposition 5.6.5.

It is clear that T is a subtree of $TR(S_0)$ such that $g(\sigma) \ne 1$ for any $\sigma \in T$, and each $\wedge$-gate has a unique $\vee$-gate son in T.

Lemma 5.7 *For $\sigma \in T$, $\Gamma^{\infty}_{\subset_e}(\sigma; T) \Rightarrow \Delta^{\infty}(\sigma; T)$ is $FV^{\infty}_{\subset_e}(\sigma; T)$-analyzed, and $\Gamma^{\infty}_{\subset_e}(\sigma; T) \cap \Delta^{\infty}(\sigma; T)$ has no common atomic formula*

Proof. It is easy to see that for any $\tau \in T$, $\bot \notin \Gamma^{\times}(\tau)$, and hence $\bot \notin \Gamma^{\infty}_{\subset_e}(\sigma; T)$.

Since each term over the set $FV^{\infty}_{\subset_e}(\sigma; T)$ is eventually tested for universal antecedent formula and existential succedent formula in the extensions τ of σ with $\sigma \subset^0_e \tau$, $\Gamma^{\infty}_{\subset_e}(\sigma; T) \Rightarrow \Delta^{\infty}(\sigma; T)$ is $FV^{\infty}_{\subset_e}(\sigma)$-saturated.

Suppose that α is a common atomic formula in $\Gamma^{\infty}_{\subset_e}(\sigma; T)$ and $\Delta^{\infty}(\sigma; T)$. Let $\sigma_1 \in T$ be such that $g(\sigma_1) \ne 1$, $\sigma \subset^0_e \sigma_1$ and $\alpha \in \Delta(\sigma_1)^{\times}$. Also let $\rho, \sigma_0 \in T$ be such that $g(\sigma_0) \ne 1$, $\rho \subset_e \sigma$, $\rho \subset^0_e \sigma_0$ and $\alpha \in \Gamma(\sigma_0)^{\times}$. We see $\rho \subsetneqq_e \sigma$ from Proposition 5.5, $\alpha \notin \Gamma^{\infty}(\sigma; T) \cap \Delta^{\infty}(\sigma; T)$.

We see that $\rho \not\subset^0_e \sigma$, otherwise σ_0 and σ_1 are comparable in the order $\subset^0_e$, and one of sequents $S(\sigma_0)$ and $S(\sigma_1)$ would be an axiom with $1 \in \{g(\sigma_0), g(\sigma_1)\}$. Therefore there exist a μ and an $i \ne 0$ such that $\rho \subset_e \mu$, $\mu * (i) \subset_e \sigma$ and $g(\mu) = \vee$. Let μ be the lowest, i.e., the shortest such sequence. Then $\rho \subset^0_e \mu$ and $\mu * (0) \subset^0_e \sigma_0$. We can assume that $\rho = \mu$. In other words ρ is the infimum of σ_0 and σ_1.

$$\frac{\begin{array}{ccccc} S(\sigma_0) : \alpha, \Pi_0 \Rightarrow \Lambda_0 & & & S(\sigma_1) : \Pi_1 \Rightarrow \Lambda_1, \alpha & \\ \vdots\, d_0 & & & \vdots\, d_1 & \\ \Gamma_0 \Rightarrow \Delta_0 & \cdots & & \Gamma_1 \Rightarrow \Delta_1 & \cdots \end{array}}{S(\rho) : \Gamma \Rightarrow \Delta} \text{ (br)}$$

Let $2n + 1 := \max\{lh(\sigma_0), lh(\sigma_1)\}$. Let a be an eigenvariable of an inference rule occurring between σ_1 and ρ. If the variable a occurs in the formula α, then so does in the sequent $S(\sigma_0) : \alpha, \Pi_0 \Rightarrow \Lambda_0$, and this contradicts (†). Hence the variable a does not occur in the formula α.

There are formulas β and $\varphi(\alpha)$ such that $\mathsf{LJm} + (cut) \vdash \chi(\rho; T_n) \leftrightarrow ((\alpha \wedge \beta) \supset \varphi(\alpha))^\forall$, where α occurs in $\varphi(\alpha)$ possibly in the scopes of $\vee, \forall$ and in the succedents of $\supset$, but no universal quantifier binds free variables in α. Namely $\varphi(\alpha)$ is in the class $\Phi(\alpha)$ such that $\alpha \in \Phi(\alpha), \varphi(\alpha) \in \Phi(\alpha) \Rightarrow \{\beta \vee \varphi(\alpha), (\beta \supset \varphi(\alpha))^\forall\} \subset \Phi(\alpha)$ for any β. No universal quantifier in $(\beta \supset \varphi(\alpha))^\forall$ binds free variables in α since a universal quantifier in $(\beta \supset \varphi(\alpha))^\forall$ binds an eigenvariable of an inference rule occurring between σ_1 and ρ.

We see inductively that $\mathsf{LJm} + (cut) \vdash \alpha \supset \varphi(\alpha)$ for any $\varphi(\alpha) \in \Phi(\alpha)$. Hence $\mathsf{LJm} + (cut) \vdash \chi(\rho; T_n)$, and $\mathsf{LJm} + (cut) \vdash \chi(T_n)$ by Proposition 5.6.1. On the other side $\mathsf{LJm} + (cut) \nvdash \chi(T_n)$. This is a contradiction.

□

Theorem 5.8 *Suppose that* $\mathsf{LJm} + (cut) \nvdash S_0$. *Each Kripke model* $\langle T, \subset_e, D_T, I_T \rangle$ *falsifies the given sequent* S_0, *no matter which tree T is chosen. Hence* $\mathsf{LJm} + (cut)$ *is intuitionistically complete.*

Proof. T enjoys the conditions in Proposition 5.4. The fourth condition (2c) follows from Lemma 5.7. Hence for $S_0 = S(\emptyset) = (\Gamma(\emptyset) \Rightarrow \Delta(\emptyset))$, $\emptyset \models \bigwedge \Gamma(\emptyset)$ and $\emptyset \nvDash \bigvee \Delta(\emptyset)$ in the Kripke model $\langle T, \subset_e, D_T, I_T \rangle$ defined from the tree T.

□

5.3 Transfer

It may be the case that for a $\sigma \in TR(S_0)$, $\Gamma_{\subset_e}^\infty(\sigma; TR(S_0)) \cap \Delta^\infty(\sigma; TR(S_0))$ has a common atomic formula, and we need to transform $TR(S_0)$.

We consider a transformation of deductions inspired by the transfer rule in [6]. The tree $TR(S_0)$ of deductions is transformed to another tree $TTR(S_0)$ in which there is no σ such that $\Gamma_{\subset_e}^\infty(\sigma; T) \cap \Delta^\infty(\sigma; T)$ has a common atomic formula, where T is a subtree of $TTR(S_0)$ such that each $\wedge$-gate has a unique $\vee$-gate son in T.

Let us modify the inference rule (br) by combining a weakening as follows.

$$\frac{\Pi, \Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \Lambda, \Delta_\exists^\circ, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ \quad \{\gamma^\circ, \Gamma, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \delta^\circ : (\gamma \supset \delta)^\circ \in \Delta_\supset^\circ\} \quad \{\Gamma, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \gamma(a)^\circ : (\forall y \, \gamma(y))^\circ \in \Delta_\forall^\circ\}}{\Pi, \Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Lambda, \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ} \text{ (wbr)}$$

where Π and Λ are arbitrary cedents.

Moreover let us introduce the following inference rule.

$$\frac{\Gamma, \Gamma_\forall, \Gamma_\forall^\circ \Rightarrow \Delta_\exists^\circ, \Delta_\exists, \Delta}{\Gamma, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta} \text{ (∘)}$$

where $\Gamma_\forall$ is the set of unmarked universal formulas in the antecedent of lower sequent, $\Delta_\exists$ the set of unmarked existential formulas in the succedent.

Let T be a (finite or infinite) labelled tree of deductions in which the inference rules (wbr), (∘) may occur besides inference rules in LJm, and $S(\sigma)$ is a sequent for $\sigma \in T$. Suppose that the tree T of deductions enjoys the condition (†) on the eigenvariables. For $\sigma_0, \sigma_1 \in T$, we say that a pair (σ_0, σ_1) is a *transferable pair* if the following conditions are met, cf. the proof of Lemma 5.7:

1. The antecedent of $S(\sigma_0)$ and the succedent of $S(\sigma_1)$ have a common marked atomic formula α°. This means that $S(\sigma_0)$ is a sequent $\alpha^\circ, \Pi_0 \Rightarrow \Lambda_0$, and $S(\sigma_1)$ is $\Pi_1 \Rightarrow \Lambda_1, \alpha^\circ$ for some cedents Π_i, Λ_i.

2. Each inference rule at σ_0 and at the infimum ρ of σ_0, σ_1 is a (wbr).

3. No non-invertible upper sequent occurs from $S(\sigma_0)$ to $S(\rho * (0))$ in the deduction, i.e., $\rho \subset_e^0 \sigma_0 = \rho * \kappa_0$ for some $\kappa_0 = (0) * \kappa_0'$, and $\rho \subsetneq_e \sigma_1 = \rho * \kappa_1$ with $\rho \not\subset_e^0 \sigma_1$ for some $\kappa_1 = (i) * \kappa_1'$ and $i \neq 0$.

All of transferrable pairs have to be removed from the constructed deduction in proof search to avoid an inconsistency in the definition of Kripke models, cf. Proposition 5.4 and Lemma 5.9.

For this, let us transform the tree $TR(S_0)$ of deductions to another tree $TTR(S_0)$. The gate $g(\sigma)$ is changed simultaneously in the transformations.

Let $TTR(S_0, 0) = TR(S_0)$ with the same labeling functions $S(\sigma, 0) = S(\sigma)$, $d(\sigma, 0) = d(\sigma)$ and $g(\sigma, 0) = g(\sigma)$. Suppose that $TTR(S_0, n)$ has been defined. Pick a minimal transferable pair (σ_0, σ_1) in $TTR(S_0, n)$,

where the minimality means that the length $lh(\rho)$ of the infimum ρ of σ_0 and σ_1 is minimal. Let $\sigma_i = \rho * \kappa_i$ for $i = 0, 1$.

$$
\begin{array}{cc}
\vdots & \vdots \\
\rho * \kappa_0 : \alpha^\circ, \Pi_0 \Rightarrow \Lambda_0 & \rho * \kappa_1 : \Pi_1 \Rightarrow \Lambda_1, \alpha^\circ \\
\vdots\; d_0 & \vdots\; d_1 \\
\end{array}
$$

$$
\cfrac{\Pi, \Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \Delta_\exists^\circ, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ, \Lambda \quad \cdots \quad \Gamma_1, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \delta_1}{\rho : \Pi, \Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ, \Lambda} \;\text{(wbr)}
$$

where d_0 denotes a deduction of the leftmost continued sequent up to $\rho * \kappa_0$, and d_1 a deduction of a non-invertible upper sequent $\Gamma, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \delta_1$ up to $\rho * \kappa_1$.

Then let us combine these two deductions, and transfer the current tree $TTR(S_0, n)$ by the pair (σ_0, σ_1) to get $TTR(S_0, n+1)$.

$$
\begin{array}{cc}
\vdots & \\
\rho * \kappa_0 * \kappa_0 : \alpha^\circ, \Pi_0, \alpha'^\circ, \Pi_0' \Rightarrow \Lambda_0', \Lambda_0 & \rho * \kappa_0 * \kappa_1 : \alpha^\circ, \Pi_1 \Rightarrow \Lambda_1, \alpha^\circ \\
\vdots\; \alpha^\circ \Pi_0 * d_0' * \Lambda_0 & \vdots\; \alpha^\circ * d_1 \\
\end{array}
$$

$$
\cfrac{\alpha^\circ, \Pi_0, \Pi, \Gamma, \Gamma_\supset, \Gamma_\forall^\circ \Rightarrow \Delta_\exists^\circ, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ, \Lambda, \Lambda_0 \quad \cdots \quad \alpha^\circ, \Gamma_1, \Gamma_\supset^\circ, \Gamma_\forall^\circ \Rightarrow \delta_1}{\rho * \kappa_0 : \alpha^\circ, \Pi_0, \Pi, \Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ, \Lambda, \Lambda_0} \;\text{(wbr)}
$$

$$
\vdots\; \Gamma_2 * d_0 * \Delta_2
$$

$$
\cfrac{\Pi, \Gamma, \Gamma_\supset, \Gamma_\forall^\circ, \Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta_\exists^\circ, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ, \Lambda}{\rho : \Pi, \Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ, \Lambda} \;(\circ)
$$

where $\rho * \kappa_0 * \kappa_1 : \alpha^\circ, \Pi_1 \Rightarrow \Lambda_1, \alpha^\circ$ is an axiom, $\alpha^\circ * d_1$ is a deduction obtained from d_1 by appending α° to antecedents. $\Gamma_2 * d_0 * \Delta_2$ is a deduction obtained from d_0 by appending $\Gamma_2 = \Pi \cup \Gamma \cup \Gamma_\supset \cup \Gamma_\forall$ to antecedents and $\Delta_2 = \Delta_\exists \cup \Delta \cup \Delta_\supset^\circ \cup \Delta_\forall^\circ \cup \Lambda$ to succedents. $\alpha^\circ \Pi_0 * d_0' * \Lambda_0$ is a deduction obtained from d_0 by appending $\alpha^\circ \cup \Pi_0$ to the antecedents, Λ_0 to the succedents, and renaming eigenvariables from ones in d_0 for the condition (†) on eigenvariables. α'° (Π_0') [Λ_0'] is obtained from α° (Π_0) [Λ_0] by renaming free variables which are introduced as eigenvariables of $(L\exists)$ in d_0, resp. The above deduction is said to be *transferred* by the pair (σ_0, σ_1) (and the common atomic formula α°).

Let $g(\sigma, n+1) \in \{0, 1, \vee, \wedge\}$ be the gate labeling function for the transferred deduction. In the transferred deduction, each sequent up to $\rho * \kappa_0 : \alpha^\circ, \Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ$ receives the same label μ and the same gate $g(\mu, n+1) = g(\mu, n)$ as in the original deduction, e.g., $g(\rho, n+1) = \vee = g(\rho, n)$ and $g(\rho * (0), n+1) = \wedge = g(\rho * (0), n)$. On the other side in the above part of $\rho * \kappa_0 : \alpha^\circ, \Gamma, \Gamma_\supset, \Gamma_\forall \Rightarrow \Delta_\exists, \Delta, \Delta_\supset^\circ, \Delta_\forall^\circ$, insert κ_0 to each label where $\sigma_0 = \rho * \kappa_0$. $g(\rho * \kappa_0 * \kappa_0, n+1) = g(\rho * \kappa_0, n+1) = \vee = g(\rho * \kappa_0, n)$. In the transferred deduction $TTR(S_0, n+1)$ the gate of the node $\rho * \kappa_0 * \kappa_1$ becomes 1, $g(\rho * \kappa_0 * \kappa_1, n+1) = 1$, since $S(\rho * \kappa_0 * \kappa_1, n+1)$ is an axiom. Below the axiom $\rho * \kappa_0 * \kappa_1$, modify the values of gates as follows. Let $\tau' = \rho * \kappa_0 * \mu$ for $\tau = \rho * \mu$ with $\emptyset \neq \mu \subsetneq_e \kappa_1$. Consider the case when $g(\tau, n) = \vee$. If there exists an i such that $g(\tau * (i), n) = 1$, then $g(\tau', n+1) = 1$. If $g(\tau * (i), n) = 0$ for any i, then $g(\tau', n+1) = 0$. Second consider the case when $g(\tau, n) = \wedge$. If there exists an i such that $g(\tau * (i), n) = 0$, then $g(\tau', n+1) = 0$. If $g(\tau * (i), n) = 1$ for any i, then $g(\tau', n+1) = 1$. In all other cases the gate is unchanged, $g(\tau', n+1) = g(\tau, n)$. Some gates might receive the value 1 by this modifications.

Let us check the conditions on multi-succedents and eigenvariables. Since there is no non-invertible upper sequent from ρ to $\rho * \kappa_0$, we can append formulas in Λ_0 and formulas in $\Delta_2 = \Delta_\exists \cup \Delta \cup \Delta_\supset^\circ \cup \Delta_\forall^\circ$ to the succedents. Next consider a free variable a occurring in α. Assume that a is introduced as an eigenvariable of an $(L\exists)$ or a (wbr)(, which subsumes $(R\forall)$) between ρ and $\rho * \kappa_1$ in d_1. But the variable a occurs in $\rho * \kappa_0$, and this is not the case by (†) since $\rho * \kappa_0$ is not above the inference rule nor to the right of it. Furthermore free variables occurring in $\Gamma_2 \cup \Delta_2$ are distinct from eigenvariables of $(L\exists)$ occurring in d_0 since ρ is below d_0.

Iterate the transformations to yield a tree of deductions $TTR(S_0)$, in which there is no transferable pair. This ends the construction of $TTR(S_0) \subset {}^{<\omega}\omega$, where

$$
TTR(S_0) = \liminf_{n \to \infty} TTR(S_0, n) = \bigcup_n \bigcap_{m \geq n} TTR(S_0, m).
$$

$S(\sigma, n), d(\sigma, n), g(\sigma, n)$ are labeling functions of sequents, deductions and gates for the nodes $\sigma \in TTR(S_0, n)$. Let for $\sigma \in TTR(S_0)$

$$
\bar{S}(\sigma) \quad = \quad \lim_{n \to \infty} S(\sigma, n)
$$

$$\bar{d}(\sigma) \quad = \quad \lim_{n \to \infty} d(\sigma, n)$$

$$\bar{g}(\sigma) \quad = \quad \lim_{n \to \infty} g(\sigma, n)$$

These limits exist since for any k there exists an n_k such that for any m and any transferable pair (σ_0, σ_1) in $TTR(S_0, n_k + m)$, $lh(\rho) > k$ holds for the infimum ρ of σ_0, σ_1. Then $S(\sigma, n_k + m) = S(\sigma, n_k)$ and $d(\sigma, n_k + m) = d(\sigma, n_k)$ for any m and any σ with $lh(\sigma) \leq k$. Furthermore for such a σ, the series $\{g(\sigma, n_k + m)\}_m$ changes the values monotonically, $g(\sigma, n_k + m) \leq g(\sigma, n_k + m + l)$ in the order $\vee, \wedge < 0 < 1$ on $\{0, 1, \vee, \wedge\}$. Hence the limit $\bar{g}(\sigma)$ exists, too.

Lemma 5.9 *Let $\emptyset \neq T \subset TTR(S_0)$ be a subtree such that $\forall \sigma \in T[\bar{g}(\sigma) \neq 1]$ and each $\wedge$-gate has a unique $\vee$-gate son in T, i.e., $\forall \sigma \in T \exists! i[\bar{g}(\sigma) = \wedge \Rightarrow \bar{g}(\sigma * (i)) = \vee \& \sigma * (i) \in T]$. Then for $\sigma \in T$, there is no common atomic formula in $\Gamma^\infty_{\subset_e}(\sigma; T) \cap \Delta^\infty(\sigma; T)$.*

Proof. Suppose that α is a common atomic formula in $\Gamma^\infty_{\subset_e}(\sigma; T)$ and $\Delta^\infty(\sigma; T)$. Let $\sigma_1 \in T$ be such that $\alpha \in \Delta(\sigma_1)^\times$ and $\sigma \subset^0_e \sigma_1$.

First consider the case when $\alpha \in \Gamma(\tau_0)^\times$ for some $\tau_0 \subset_e \sigma$. Then $\alpha \in \Gamma(\sigma_1)^\times$. This means that $S(\sigma_1)$ is an axiom, and $\bar{g}(\sigma_1) = 1$. Let $\rho, \sigma_0 \in T$ be such that $\bar{g}(\sigma_0, lh(\sigma_0)) = \vee$, $\rho \subset_e \sigma$, $\rho \subset^0_e \sigma_0$ and $\alpha \in \Gamma(\sigma_0)^\times$. We see $\alpha \notin \Gamma^\infty(\sigma; T) \cap \Delta^\infty(\sigma; T)$ from Proposition 5.5, and $\rho \subsetneq_e \sigma$. Furthermore $\rho \not\subset^0_e \sigma$ otherwise σ_0 and σ_1 are comparable in the order $\subset^0_e$, and one of sequents $S(\sigma_0)$ and $S(\sigma_1)$ would be an axiom with $1 \in \{\bar{g}(\sigma_0, lh(\sigma_0)), \bar{g}(\sigma_1, lh(\sigma_1))\}$. Therefore there exist a τ and an $i \neq 0$ such that $\rho \subset_e \tau$, $\tau * (i) \subset_e \sigma$ and $\bar{g}(\tau, lh(\tau)) = \vee$. Let τ be the lowest, i.e., the shortest such sequence. Then $\rho \subset^0_e \tau$ and $\tau * (0) \subset^0_e \sigma_0$. This means that (σ_0, σ_1) is a transferable pair. Such a pair has been removed from $TTR(S_0, n)$ for an n, and from $TTR(S_0)$. Hence this is not the case. $\square$

Theorem 5.10 (Schütte's dichotomy)
In $TTR(S_0)$, $\bar{g}(\emptyset) = 1$ iff $\mathsf{LJm} \vdash S_0$.

Proof. If $\bar{g}(\emptyset) = 1$, then it is plain to see that $\mathsf{LJm} \vdash S_0$.

In what follows assume $\bar{g}(\emptyset) \neq 1$. Then $\bar{g}(\emptyset) \in \{0, \wedge\}$. Extract a subtree $T \subset TTR(S_0)$ as follows. First $\emptyset \in T$. The nodes $\sigma \in T$ with $\bar{g}(\sigma) = 0$ are leaves in T. Suppose $\sigma \in T$ has been chosen so that $\bar{g}(\sigma) = \wedge$. Then in the deduction $d(\sigma)$, pick a leaf $\sigma * (j)$ such that $\bar{g}(\sigma * (j)) \neq 1$. If $\bar{g}(\sigma * (j)) = 0$, then we would have $\bar{g}(\sigma) = 0$. Hence $\bar{g}(\sigma * (j)) = \vee$, and $\bar{g}(\sigma * (j, i)) \neq 1$ for any son $\sigma * (j, i)$. Moreover there exists an i such that $\bar{g}(\sigma * (j, i)) \neq 0$. Otherwise we would have $\bar{g}(\sigma * (j)) = 0$. Let $\sigma * (i), \sigma * (j, i) \in T$ for any i such that $\bar{g}(\sigma * (j, i)) = \wedge$.

Then T enjoys the conditions in Proposition 5.4. The fourth condition (2c) follows from Lemma 5.9. Hence for $S_0 = S(\emptyset) = (\Gamma(\emptyset) \Rightarrow \Delta(\emptyset))$, $\emptyset \models \bigwedge \Gamma(\emptyset)$ and $\emptyset \not\models \bigvee \Delta(\emptyset)$ in the Kripke model $\langle T, \subset_e, D_T, I_T \rangle$ defined from the tree T. Therefore $\mathsf{LJm} \not\vdash S_0$. $\square$

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