A Remark on Lattice Models of Second-Order Intuitionistic Propositional Logic[1]

Toshihiko Kurata[2]
Faculty of Business Administration, Hosei University

Ken-etsu Fujita[3]
Department of Computer Science, Gunma University

Abstract

We summarize a topological representation of the completely distributive algebraic lattices. This together with the discussion in [KF19] gives a definition of lattice models from an abstract viewpoint, with respect to which the completeness of the basic system of second-order intuitionistic propositional logic is ensured.

1. Introduction

A framework of Kripke models have been studied to give a mathematical semantics of the basic formal system $\mathbf{IPC_2}$ of second-order intuitionistic propositional logic, for which we can find some aspects on structure of domains of the models. Actually, when we restrict to the class of principal Kripke models, that is, every member of the class is endowed with a constant domain consisting of all upsets of worlds, the logic defined by such models is known to be non-recursively formalizable, as is shown in [Skv97, Kre97]. Furthermore, even if we generalize the definition of constant domain by adopting the class of secondary Kripke models satisfying Gabbay's completeness property, the logic characterized by such models is known to be formalized not by the system $\mathbf{IPC_2}$ but by a variant system with the Grzegorczyk scheme, as is shown in [Gab74]. These results suggest that another structure of domains is indispensable to ensure the completeness of the system $\mathbf{IPC_2}$ unless we maintain the framework of Kripke models. In this respect, a nested structure of domains satisfying Sobolev's completeness property is incorporated into the Kripke models by Sobolev [Sob77], by which the system $\mathbf{IPC_2}$ is ensured to be complete.

Directing our attention to the semantics of intuitionistic propositional logic, we can also find a framework of lattice models based on Heyting algebras as an alternative approach, which correlates to the framework of Kripke models. However, the correspondence between them is not very strict and Heyting algebras might give more general aspects of semantics in a sense. Actually, every Kripke model can be regarded

---

[1]This research was supported by JSPS KAKENHI Grant Number JP20K03711 and also by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

[2]Faculty of Business Administration, Hosei University, 2-17-1 Fujimi, Chiyoda-ku, Tokyo 102-8160, Japan. e-mail:kurata@hosei.ac.jp

[3]Department of Computer Science, Gunma University, 1-5-1 Tenjin-cho, Kiryu-shi, Gunma 376-8515, Japan. e-mail:fujita@gunma-u.ac.jp

as a neighbourhood model especially based on the Alexandrov topology, in which an order structure of complete Heyting algebra is inherent. In contrast, existence of arbitrary joins is not guaranteed in Heyting algebras in general, which would be one of the obstacles to giving a successful interpretation of the second order quantification from an algebraic point of view.

To overcome the deficiency above, in [KF19] the category of completely distributive algebraic lattices is applied, into which we incorporate the structure of nested domains satisfying Sobolev's completeness property for giving an algebraic semantics of $\mathbf{IPC_2}$. Indeed, this category is ensured to be dually equivalent to the category of Alexandrov spaces by a version of Stone duality, and it enables us to present a definition of lattice models in which the interpretation of second-order quantification in [Sob77] is simulated exactly. We briefly review this result in Section 2.

In every lattice model introduced in [KF19], the interpretation of a proposition is presented not directly as an element of the lattice but indirectly as an open set on its topological representation. This presentation of lattice models is mainly for the sake of simplifying the correspondence with the Kripke models by Sobolev and we note that the complication associated with our definition would be not inevitable to establish the completeness theorem. This is because of a representation theorem that every completely distributive algebraic lattice is prime algebraic [Win83, Win09] and so order isomorphic to its topological representation. Actually, this isomorphism enables us to simulate the interpretation in [KF19] exactly in terms of the elements of the underlying completely distributive algebraic lattice. As an additional remark on lattice models of the system $\mathbf{IPC_2}$, we demonstrate it in Section 3.

2. Lattice Models on Topological Representation

We fix a syntax of the formal system $\mathbf{IPC_2}$,[4] and the set $\mathbf{Prop_2}$ of the propositions of $\mathbf{IPC_2}$ is generated by the following abstract grammar:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \forall p.A \mid \exists p.A$$

where p ranges over the set $\mathbf{Vars}$ of propositional variables. We use letters $p, q, r, \ldots$ to denote propositional variables and $A, B, C, \ldots$ to denote propositions. We also use letters $\mathit{\Gamma}$, $\mathit{\Delta}$, ... to denote sets of propositions. If a proposition A is derivable from assumptions in $\mathit{\Gamma}$ by means of the deduction rules of $\mathbf{IPC_2}$, then we write $\mathit{\Gamma} \vdash_{\mathbf{IPC_2}} A$.

In the lattice models of the system $\mathbf{IPC_2}$ presented in [KF19], the interpretation of each proposition is given as an open set on a topological space generated from the order structure of a completely distributive algebraic lattice. Here we briefly review some basic definitions and the completeness theorem with respect to this interpretation.

Let $\langle L, \sqsubseteq \rangle$ be a poset. Then we say that a subset X of L is directed if every finite subset of X has an upper bound in X. An element $x \in L$ is said to be compact if $x \sqsubseteq \bigsqcup^{\uparrow} X$ implies $\uparrow x \cap X \neq \emptyset$ for every directed subset X of L. Here we denote by $\uparrow x$ the smallest upset containing x, namely $\uparrow x = \{y \in L \mid x \sqsubseteq y\}$. We define $\mathsf{K}L$ to be the set of compact elements of L and $\mathsf{K}L(x) = \{y \in \mathsf{K}L \mid y \sqsubseteq x\}$ for every $x \in L$.

[4] We can refer to the system in [SU06, Definition 11.1.2], a Hilbert-style counterpart of which we can also find as the system $C2J$ in [Sob77] and the system H_2 in [Skv97].

Then a complete lattice $\langle L, \sqsubseteq\rangle$ is said to be algebraic if it satisfies $x = \bigsqcup^{\uparrow} \mathsf{K}L(x)$ for every $x \in L$, and completely distributive if it satisfies

$$\bigsqcap_{\lambda\in\Lambda} \bigsqcup_{x\in X} x_\lambda = \bigsqcup_{x\in X} \bigsqcap_{\lambda\in\Lambda} x_\lambda$$

for every cartesian product $X = \prod_{\lambda\in\Lambda} X_\lambda$ of subsets of L. We write **CDA** for the class of completely distributive algebraic lattices.

With this specified class of lattices, we can associate a class of topological spaces according to a version of Stone duality. Suppose $L \in \mathbf{CDA}$.[5] Then an upward closed subset of L is said to be a filter on L if it is also closed under finite meets. For instance, it is clear for every $x \in L$ that the set $\uparrow x$ is a filter, which is called the principal filter generated from x. Then a filter F on L is said to be completely prime if $\bigsqcup X \in F$ implies $F \cap X \neq \emptyset$ for every subset X of L. An element of L is said to be completely prime if the principal filter generated from it is completely prime. We use letters a, b, $c, \ldots$ to designate completely prime elements of L, and define $\mathsf{C}L$ to be the set of completely prime elements of L and $\mathsf{C}L(x) = \{a \in \mathsf{C}L \mid a \sqsubseteq x\}$ for every $x \in L$.

The set of completely prime filters on L is regarded as a poset together with the order of set inclusion, for which we denote the set of compact elements by pt^*L. Further we introduce a topology $\tau(L)$ on the set pt^*L which consists of the open sets of the form $\{F \in \mathsf{pt}^*L \mid x \in F\}$ for some $x \in L$. We note that this topology is identical with the Alexandrov topology on pt^*L, as is shown in Corollary 3. Therefore it follows that every meet and join in $\tau(L)$ are given by the intersection and the union of open sets, respectively.

Let $L \in \mathbf{CDA}$. Then the topological space $\langle \mathsf{pt}^*L, \tau(L)\rangle$ generated from L underlies the lattice models introduced in [KF19]. In this framework, we suppose that we have a mapping d which associates with every $F \in \mathsf{pt}^*L$ a domain $d(F) \subseteq \tau(L)$, and a nested structure that for every $F \in \mathsf{pt}^*L$ there exists an open neighbourhood $U \in \tau(L)$ of F satisfying

$$\forall G \in U \;\; d(F) \subseteq d(G).$$

Then we call the triple $\mathscr{C} = \langle \tau(L), \subseteq, d\rangle$ a *concrete model* of $\mathbf{IPC_2}$, in which we can interpret every proposition as a member of the topology $\tau(L)$. More precisely, to give an interpretation of the second-order propositions in the concrete model $\mathscr{C}$, we set an environment ξ as a mapping from **Vars** to $\tau(L)$. Then, for every proposition $A \in \mathbf{Prop_2}$ and environment ξ, we define the interpretation $[\![A]\!]_\xi \in \tau(L)$ of A under ξ in the model $\mathscr{C}$ by induction on the structure of A, as follows:

$$[\![\bot]\!]_\xi = \emptyset,$$

$$[\![p]\!]_\xi = \xi(p),$$

$$[\![A \wedge B]\!]_\xi = [\![A]\!]_\xi \cap [\![B]\!]_\xi,$$

$$[\![A \vee B]\!]_\xi = [\![A]\!]_\xi \cup [\![B]\!]_\xi,$$

$$[\![A \to B]\!]_\xi = \bigcup \{U \in \tau L \mid [\![A]\!]_\xi \cap U \subseteq [\![B]\!]_\xi\},$$

[5] Denoting a member $\langle L, \sqsubseteq\rangle$ of **CDA**, we often omit to indicate the order relation $\sqsubseteq$ when no confusion can arise.

$$[\![\forall p.A]\!]_\xi = \bigcup \{U \in \tau(L) \mid \forall F \in U \;\; \forall V \in d(F) \;\; F \in [\![A]\!]_{\xi(p:V)}\},$$
$$[\![\exists p.A]\!]_\xi = \bigcup \{U \in \tau(L) \mid \forall F \in U \;\; \exists V \in d(F) \;\; F \in [\![A]\!]_{\xi(p:V)}\}.$$

Here we write $\xi(p : V)$ for the environment ξ with the value of p updated to V, that is, $\xi(p : V)(p) = V$ and $\xi(p : V)(q) = \xi(q)$ for every $q \in \mathbf{Vars} \setminus \{p\}$. We also define $[\![\Gamma]\!]_\xi = \bigcap_{A\in\Gamma}[\![A]\!]_\xi$ for every set Γ of propositions.

Furthermore, we confine our attention to a specific sort of model structures in which every domain contains an approximate interpretation of every proposition. We say a concrete model $\mathscr{C} = \langle L, \sqsubseteq, d\rangle$ is full if, for every $A \in \mathbf{Prop_2}$, $F \in \mathsf{pt}^*L$ and environment ξ such that $\xi(\mathrm{FV}(A)) \subseteq d(F)$, we can find $U \in d(F)$ and $V \in \tau(L)$ which satisfy $F \in V$ and $U \cap V = [\![A]\!]_\xi \cap V$. Then, we write $\Gamma \models_{\mathbf{con}} A$ if

$$F \in [\![\Gamma]\!]_\xi \Longrightarrow F \in [\![A]\!]_\xi$$

holds for every full concrete model $\mathscr{C} = \langle\tau(L), \subseteq, d\rangle$, $F \in \mathsf{pt}^*L$ and environment ξ such that $\xi(\mathrm{FV}(\Gamma, A)) \subseteq d(F)$.[6]

Then it is verified in [KF19] that our concrete models are shown to be comparable with the Kripke models introduced by Sobolev [Sob77], with respect to which the derivation of the formal system $\mathbf{IPC_2}$ is complete.

Theorem 1 ([KF19]**).** $\Gamma \vdash_{\mathbf{IPC_2}} A$ if and only if $\Gamma \models_{\mathbf{con}} A$.

3. Abstract Version of Lattice Models

In every concrete model considered in the preceding section, an interpretation of every proposition is given as a member of a complete Heyting algebra $\langle\tau(L), \subseteq\rangle$ in the form of a topological representation. Here, in case where $L \in \mathbf{CDA}$, this representation is especially shown to be order isomorphic to the underlying lattice L. This is confirmed because of the prime algebraicity of completely distributive algebraic lattice; that is,

$$x = \bigsqcup \mathsf{C}L(x)$$

holds for every $x \in L$, as is proved in [Win83, Corollary 8] and [Win09, Corollary 5]. To see the isomorphism, we begin by showing that the members of pt^*L are characterized exactly in terms of the completely prime elements of L.

Lemma 2. Let $L \in \mathbf{CDA}$. Then we have $F \in \mathsf{pt}^*L$ if and only if $F = \uparrow a$ for some $a \in \mathsf{C}L$.

Proof. For the "if" part, it is immediate from the complete primeness of a that $\uparrow a$ is a completely prime filter on L. Furthermore, let $\uparrow a \subseteq \bigcup^{\uparrow}_{\lambda\in\Lambda} F_\lambda$ for a directed set $\{F_\lambda \mid \lambda \in \Lambda\}$ of completely prime filters. Then we can find $\lambda \in \Lambda$ such that $a \in F_\lambda$, and so $\uparrow a \subseteq F_\lambda$, from which the compactness of $\uparrow a$ follows.

For the "only-if" part, we note that there exists $a \in F \cap \mathsf{C}L(x)$ for every $x \in F$ because of the prime algebraicity of L and the complete primeness of F. Thus, when $a, b \in F \cap \mathsf{C}L$, we can find a completely prime element $c \in F \cap \mathsf{C}L(a \sqcap b)$ since we have $a \sqcap b \in F$. This entails that $\{\uparrow a \in \mathsf{pt}^*L \mid a \in F \cap \mathsf{C}L\}$ is a directed subset of pt^*L, for which the equality $F = \bigcup^{\uparrow}\{\uparrow a \in \mathsf{pt}^*L \mid a \in F \cap \mathsf{C}L\}$ is clear. Therefore, by the

[6] This property is also refereed to as Sobolev's completeness property.

compactness of F, we can find $a \in F \cap \mathsf{C}L$ such that $F \subseteq \uparrow a$, from which $F = \uparrow a$ is immediate. □

This lemma allows us to have an order-reversing isomorphism $k : \mathsf{C}L \to \mathsf{pt}^* L$ such that $k(a) = \uparrow a$ for every $a \in \mathsf{C}L$, the inverse of which is presented by $k^{-1}(F) = \min F$ for every $F \in \mathsf{pt}^* L$. This fact makes it easy to see that the Alexandrov topology underlies our topological representation of the completely distributive algebraic lattices.

Corollary 3. Let $L \in \mathbf{CDA}$. Then the topology $\tau(L)$ is identical with the Alexandrov topology on $\mathsf{pt}^* L$.

Proof. The upward closedness of every $U \in \tau(L)$ is trivial from the definition. Conversely, for every upset U of $\mathsf{pt}^* L$, we can take $x_U = \bigsqcup \{ a \in \mathsf{C}L \mid k(a) \in U \}$ in L. Now let us suppose $G \in \mathsf{pt}^* L$. Then the definition of completely prime filter implies $x_U \in G$ if and only if there exists $a \in G \cap \mathsf{C}L$ such that $k(a) \in U$. Therefore we can conclude $U = \{G \in \mathsf{pt}^* L \mid x_U \in G\}$. □

Turning our attention to the correspondence between L and $\tau(L)$, we are also allowed to have an isomorphism $\varphi : L \to \tau(L)$ such that $\varphi(x) = \{F \in \mathsf{pt}^* L \mid k^{-1}(F) \in \mathsf{C}L(x)\}$ for every $x \in L$. This is verified as follows.

Lemma 4. Let $L \in \mathbf{CDA}$. Then the functions φ is an order isomorphism between L and $\tau(L)$, the inverse of which is given by $\varphi^{-1}(U) = \bigsqcup\{a \in \mathsf{C}L \mid k(a) \in U\}$ for every $U \in \tau(L)$.

Proof. We first note that $\varphi(x)$ is upward closed and so $\varphi(x) \in \tau(L)$ by Corollary 3. It is clear that both φ and φ^{-1} are order preserving. Furthermore, in regard to their composition, we have

$$\begin{aligned} \varphi \circ \varphi^{-1}(U) &= \{ F \in \mathsf{pt}^* L \mid k^{-1}(F) \sqsubseteq \textstyle\bigsqcup\{a \in \mathsf{C}L \mid k(a) \in U\} \} \\ &= \{ F \in \mathsf{pt}^* L \mid \exists a \in \mathsf{C}L \ (k(a) \in U \ \& \ k^{-1}(F) \sqsubseteq a) \} \\ &= U \end{aligned}$$

for every $U \in \tau(L)$. On the other hand, we have $\varphi^{-1} \circ \varphi(x) = \bigsqcup \mathsf{C}L(x) = x$ for every $x \in L$ because of the prime algebraicity of completely distributive algebraic lattices. □

It is clear from this lemma that φ preserves all meets and joins on $\langle L, \sqsubseteq\rangle$ and that φ^{-1} preserves all intersections and unions on $\langle \tau(L), \subseteq\rangle$. By means of these isomorphisms, we know that every completely prime element of $\langle \tau(L), \subseteq\rangle$ is characterized in terms of an element F of $\mathsf{pt}^* L$ as $\varphi \circ k^{-1}(F) = \{G \in \mathsf{pt}^* L \mid F \subseteq G\}$. So we can characterize the relation that $a \in \mathsf{C}L(x)$ in $L \in \mathbf{CDA}$ in terms of the topological representation $\langle \tau(L), \subseteq\rangle$, as follows:

$$k(a) \in \varphi(x) \iff \varphi(a) \subseteq \varphi(x) \iff a \sqsubseteq x. \tag{3.1}$$

Taking the inverse translations into account, this is equivalent to the condition that

$$F \in U \iff \varphi \circ k^{-1}(F) \subseteq U \iff k^{-1}(F) \sqsubseteq \varphi^{-1}(U). \tag{3.2}$$

Based on the observation above, we now give an interpretation of the propositions of $\mathbf{IPC_2}$ directly as elements of a completely distributive algebraic lattice, which we proceed by analogy with the definition of the interpretation in Section 2. Suppose $L \in \mathbf{CDA}$. We also suppose that d is a mapping which associate with every $a \in \mathsf{C}L$ a domain $d(a) \subseteq L$ and which have a nested structure that for every $a \in \mathsf{C}L$ there exists an element $x \in \uparrow a$ satisfying

$$\forall b \in \mathsf{C}L(x) \;\; d(a) \subseteq d(b).$$

Then we call the triple $\mathscr{A} = \langle L, \sqsubseteq, d\rangle$ an *abstract model* of $\mathbf{IPC_2}$. We set an environment ξ on $\mathscr{A}$ as a mapping from **Vars** to L. Then, for every proposition $A \in \mathbf{Prop_2}$ and environment ξ, we define the interpretation $[\![A]\!]_\xi \in L$ by induction on the structure of A, as follows:

$$\begin{aligned}
&[\![\bot]\!]_\xi = \bot,\\
&[\![p]\!]_\xi = \xi(p),\\
&[\![A \wedge B]\!]_\xi = [\![A]\!]_\xi \sqcap [\![B]\!]_\xi,\\
&[\![A \vee B]\!]_\xi = [\![A]\!]_\xi \sqcup [\![B]\!]_\xi,\\
&[\![A \to B]\!]_\xi = \bigsqcup\{x \in L \mid [\![A]\!]_\xi \sqcap x \sqsubseteq [\![B]\!]_\xi\},\\
&[\![\forall p.A]\!]_\xi = \bigsqcup\{x \in L \mid \forall a \in \mathsf{C}L(x) \;\; \forall y \in d(a) \;\; a \sqsubseteq [\![A]\!]_{\xi(p:y)}\},\\
&[\![\exists p.A]\!]_\xi = \bigsqcup\{x \in L \mid \forall a \in \mathsf{C}L(x) \;\; \exists y \in d(a) \;\; a \sqsubseteq [\![A]\!]_{\xi(p:y)}\}.
\end{aligned}$$

We also define $[\![\Gamma]\!]_\xi = \prod_{A\in\Gamma}[\![A]\!]_\xi$ for every set Γ of propositions. We say that an abstract model $\mathscr{L}$ is full if, for every $A \in \mathbf{Prop_2}$, $a \in \mathsf{C}L$ and environment ξ such that $\xi(\mathrm{FV}(A)) \subseteq d(a)$, we can find $x \in d(a)$ and $y \in\uparrow a$ satisfying $x \sqcap y = [\![A]\!]_\xi \sqcap y$. A judgement $\Gamma \vdash_{\mathbf{IPC_2}} A$ is said to be valid with respect to a full abstract model $\mathscr{A}$ if

$$a \sqsubseteq [\![\Gamma]\!]_\xi \Rightarrow a \sqsubseteq [\![A]\!]_\xi$$

holds for the model $\mathscr{A}$. Then we write $\Gamma \models_{\mathbf{abs}} A$ if the judgement $\Gamma \vdash_{\mathbf{IPC_2}} A$ is valid with respect to every full abstract model.

Now we show the validity based on the definition above is equivalent to that based on the concrete models defined in Section 2, which entails the completeness of $\mathbf{IPC_2}$ with respect to the abstract models.

Let $L \in \mathbf{CDA}$, which is endowed with isomorphisms $k: \mathsf{C}L \to \mathsf{pt}^*L$ and $\varphi: L \to \tau(L)$ by itself. Then, with an abstract model $\mathscr{A} = \langle L, \sqsubseteq, d\rangle$ where d assigns a subset of L to every element of $\mathsf{C}L$, we associate a triple $\langle \tau(L), \subseteq, \varphi \circ d \circ k^{-1}\rangle$ and denote it by $\mathscr{A}_*$. On the other hand, with a concrete model $\mathscr{C} = \langle \tau(L), \subseteq, d\rangle$ where d assigns a subset of $\tau(L)$ to every element of pt^*L, we associate a triple $\langle L, \sqsubseteq, \varphi^{-1} \circ d \circ k\rangle$ and denote it by $\mathscr{C}^*$. We first verify that the results of these two translations also satisfy the requirement to be the models of $\mathbf{IPC_2}$.

Lemma 5. (1) For every abstract model $\mathscr{A}$, the structure $\mathscr{A}_*$ is a concrete model.
(2) For every concrete model $\mathscr{C}$, the structure $\mathscr{C}^*$ is an abstract model.

Proof. (1) It suffices to verify the condition of nested domain for $\varphi \circ d \circ k^{-1}$. Suppose $F \in \mathsf{pt}^*L$. Then we have $k^{-1}(F) \in \mathsf{C}L$, which together with the assumption concerning d implies the existence of an element $x \in\uparrow k^{-1}(F)$ such that $d \circ k^{-1}(F) \subseteq d(b)$ holds for every $b \in \mathsf{C}L(x)$. So we can take $\varphi(x) \in \tau(L)$ as an open neighbourhood of F since

$F \in \varphi \circ k^{-1}(F) \subseteq \varphi(x)$ is clear. Then for every $G \in \varphi(x)$, we have $k^{-1}(G) \in \mathsf{C}L(x)$. Thus we obtain $d \circ k^{-1}(F) \subseteq d \circ k^{-1}(G)$ by means of the assumption concerning d, and so $\varphi \circ d \circ k^{-1}(F) \subseteq \varphi \circ d \circ k^{-1}(G)$ holds.

(2) It can be verified analogously based on a dual aspect of the translations. □

Besides the structure of nested domain, the property of fullness is also shown to be preserved under the translations for models. This is a straightforward consequence of the fact that the interpretations of a proposition are interchangeable for every abstract model and its corresponding concrete model. It is demonstrated in the following, in which we specify the lattice model in the expression of interpretation and denote by $[\![A]\!]^{\mathscr{L}}_{\xi}$ the interpretation of A in a lattice model $\mathscr{L}$.

Lemma 6. Suppose $\mathscr{A} = \langle L, \sqsubseteq, d\rangle$ is an abstract model.
(1) We have $\varphi([\![A]\!]^{\mathscr{A}}_{\xi}) = [\![A]\!]^{\mathscr{A}_*}_{\varphi \circ \xi}$ for every $A \in \mathbf{Prop_2}$ and $\xi : \mathbf{Vars} \to L$.
(2) If $\mathscr{A}$ is full, then so is the concrete model $\mathscr{A}_*$.

Proof. (1) We verify the statement by induction on the structure of A. Here we focus on the case where $A \equiv \forall p.B$. Proofs for the other cases are similar or easier, which we omit.

For the proof of this case, it suffices to verify the equivalence of the following two conditions on $x \in L$:

$$\forall a \in \mathsf{C}L(x)\ \forall y \in d(a)\ \ a \sqsubseteq [\![B]\!]^{\mathscr{A}}_{\xi(p:y)}, \tag{3.3}$$

$$\forall F \in \varphi(x)\ \forall V \in \varphi \circ d \circ k^{-1}(F)\ \ F \in [\![B]\!]^{\mathscr{A}_*}_{(\varphi \circ \xi)(p:V)}. \tag{3.4}$$

To see that (3.3) implies (3.4), we let $F \in \varphi(x)$ and $V \in \varphi \circ d \circ k^{-1}(F)$ for $F \in \mathsf{pt}^*L$ and $V \in \tau(L)$. Then we can take $k^{-1}(F) \in \mathsf{C}L$, for which $k^{-1}(F) \sqsubseteq x$ holds since $x \in F$. Furthermore, taking $\varphi^{-1}(V) \in d \circ k^{-1}(F)$, we are allowed to have $k^{-1}(F) \sqsubseteq [\![B]\!]^{\mathscr{A}}_{\xi(p:\varphi^{-1}(V))}$ by (3.3). This implies $[\![B]\!]^{\mathscr{A}}_{\xi(p:\varphi^{-1}(V))} \in F$, from which $F \in \varphi([\![B]\!]^{\mathscr{A}}_{\xi(p:\varphi^{-1}(V))}) = [\![B]\!]^{\mathscr{A}_*}_{(\varphi \circ \xi)(p:V)}$ follows by the induction hypothesis.

To see the reverse direction, we let $a \in \mathsf{C}L(x)$ and $y \in d(a)$ for $a \in \mathsf{C}L$ and $y \in L$. Then we can take $k(a) \in \mathsf{pt}^*L$, for which $k(a) \in \varphi(x)$ holds since $a \sqsubseteq x$. Furthermore, taking $\varphi(y) \in \varphi \circ d \circ k^{-1}(k(a))$, we are allowed to have $k(a) \in [\![B]\!]^{\mathscr{A}_*}_{(\varphi \circ \xi)(p:\varphi(y))} = \varphi([\![B]\!]^{\mathscr{A}}_{\xi(p:y)})$ by (3.4) and the induction hypothesis. This implies $[\![B]\!]^{\mathscr{A}}_{\xi(p:y)} \in k(a)$, from which $a \sqsubseteq [\![B]\!]^{\mathscr{A}}_{\xi(p:y)}$ follows.

(2) Assume that we have $A \in \mathbf{Prop_2}$, $\xi : \mathbf{Vars} \to \tau(L)$ and $F \in \mathsf{pt}^*L$ which satisfy $\xi(\mathrm{FV}(A)) \subseteq \varphi \circ d \circ k^{-1}(F)$. Then it is clear that $k^{-1}(F) \in \mathsf{C}L$ and $\varphi^{-1} \circ \xi : \mathbf{Vars} \to L$ satisfy $\varphi^{-1} \circ \xi(\mathrm{FV}(A)) \subseteq d \circ k^{-1}(F)$. So, because of the fullness of the model $\mathscr{A}$, we can find $x \in d \circ k^{-1}(F)$ and $y \in \uparrow k^{-1}(F)$ such that $x \sqcap y = [\![A]\!]^{\mathscr{A}}_{\varphi^{-1} \circ \xi} \sqcap y$. Hence, $\varphi(x) \in \varphi \circ d \circ k^{-1}(F)$ and $\varphi(y) \in \tau(L)$ ensure the fullness of $\mathscr{A}_*$. Indeed, we have $F \in \varphi(y)$ since $y \in F$, and

$$\varphi(x) \cap \varphi(y) = \varphi(x \sqcap y) = \varphi([\![A]\!]^{\mathscr{A}}_{\varphi^{-1} \circ \xi} \sqcap y) = \varphi([\![A]\!]^{\mathscr{A}}_{\varphi^{-1} \circ \xi}) \cap \varphi(y) = [\![A]\!]^{\mathscr{A}_*}_{\xi} \cap \varphi(y)$$

by (1) of this lemma. □

Lemma 7. Suppose $\mathscr{C} = \langle \tau(L), \subseteq, d \rangle$ is a concrete model.
(1) We have $\varphi^{-1}([\![A]\!]^{\mathscr{C}}_{\xi}) = [\![A]\!]^{\mathscr{C}^*}_{\varphi^{-1} \circ \xi}$ for every $A \in \mathbf{Prop_2}$ and $\xi : \mathbf{Vars} \to \tau(L)$.
(2) If $\mathscr{C}$ is full, then so is the abstract model $\mathscr{C}^*$.

Proof. To show (1) in the case where $A \equiv \forall p.B$, we only need to ensure the equivalence of the following two conditions on $U \in \tau(L)$:

$$\forall F \in U \;\; \forall V \in d(F) \;\;\; F \in [\![B]\!]^{\mathscr{C}}_{\xi(p:V)},$$

$$\forall a \in \mathsf{C}L(\varphi^{-1}(U)) \;\; \forall y \in \varphi^{-1} \circ d \circ k(a) \;\;\; a \sqsubseteq [\![B]\!]^{\mathscr{C}^*}_{(\varphi^{-1} \circ \xi)(p:y)}.$$

We can give a proof by analogy with the proof of Lemma 6 based on a dual aspect of the translations between abstract and concrete models. □

By the lemmas above, the notion of validity with respect to the concrete models is shown to be equivalent to that with respect to the abstract models. Consequently, the completeness theorem with respect to the abstract models is obtained.

Lemma 8. $\Gamma \models_{\mathbf{con}} A$ if and only if $\Gamma \models_{\mathbf{abs}} A$.

Proof. To show the "if" part, suppose $\mathscr{C} = \langle \tau(L), \subseteq, d \rangle$ is a full concrete model in which $F \in \mathsf{pt}^* L$ and an environment $\xi : \mathbf{Vars} \to \tau(L)$ satisfy $\xi(\mathrm{FV}(\Gamma, A)) \subseteq d(F)$. Let us also suppose $F \in [\![\Gamma]\!]^{\mathscr{C}}_{\xi}$. Then we can take $k^{-1}(F) \in \mathsf{C}L$ and $\varphi^{-1} \circ \xi : \mathbf{Vars} \to L$ in the full abstract model $\mathscr{C}^*$ which satisfy $\varphi^{-1} \circ \xi(\mathrm{FV}(\Gamma, A)) \subseteq \varphi^{-1} \circ d \circ k(k^{-1}(F))$. Then we obtain $k^{-1}(F) \sqsubseteq \varphi^{-1}([\![\Gamma]\!]^{\mathscr{C}}_{\xi}) = [\![\Gamma]\!]^{\mathscr{C}^*}_{\varphi^{-1} \circ \xi}$ by (3.2) and Lemma 7 (1), from which $k^{-1}(F) \sqsubseteq [\![A]\!]^{\mathscr{C}^*}_{\varphi^{-1} \circ \xi} = \varphi^{-1}([\![A]\!]^{\mathscr{C}}_{\xi})$ follows by the assumption and Lemma 7 (1). Therefore, we obtain $F \in [\![A]\!]^{\mathscr{C}}_{\xi}$ by (3.2).

The "only-if" part is shown analogously for the reverse translation. Suppose $\mathscr{A} = \langle L, \sqsubseteq, d \rangle$ is a full abstract model, in which $a \in \mathsf{C}L$ and an environment $\xi : \mathbf{Vars} \to L$ satisfy $\xi(\mathrm{FV}(\Gamma, A)) \subseteq d(a)$. Let us also suppose $a \sqsubseteq [\![\Gamma]\!]^{\mathscr{A}}_{\xi}$. Then we can take $k(a) \in \mathsf{pt}^* L$ and $\varphi \circ \xi : \mathbf{Vars} \to \tau(L)$ in the full concrete model $\mathscr{A}_*$ which satisfy $\varphi \circ \xi(\mathrm{FV}(\Gamma, A)) \subseteq \varphi \circ d \circ k^{-1}(k(a))$. Then we obtain $k(a) \in \varphi([\![\Gamma]\!]^{\mathscr{A}}_{\xi}) = [\![\Gamma]\!]^{\mathscr{A}_*}_{\varphi \circ \xi}$ by (3.1) and Lemma 6 (1), from which $k(a) \in [\![A]\!]^{\mathscr{A}_*}_{\varphi \circ \xi} = \varphi([\![A]\!]^{\mathscr{A}}_{\xi})$ follows by the assumption and Lemma 6 (1). Therefore, we obtain $a \sqsubseteq [\![A]\!]^{\mathscr{A}}_{\xi}$ by (3.1). □

Theorem 9. $\Gamma \vdash_{\mathbf{IPC_2}} A$ if and only if $\Gamma \models_{\mathbf{abs}} A$.

Proof. It is immediate from Theorem 1 and Lemma 8. □

References

[Gab74] D. M. Gabbay. On 2nd order intuitionistic propositional calculus with full comprehension. *Archiv für mathematische Logik und Grundlagenforschung*, 16:177–186, 1974.

[KF19] T. Kurata and K. Fujita. Neighbourhood and lattice models of second order intuitionistic propositional logic. *Fundamenta Informaticae*, 170:223–240, 2019.

[Kre97] P. Kremer. On the complexity of propositional quantification in intuitionistic logic. *Journal of Symbolic Logic*, 62(2):529–544, 1997.

[Skv97] D. Skvortsov. Non-axiomatizable second order intuitionistic propositional logic. *Annals of Pure and Applied Logic*, 86(1):33–46, 1997.

[Sob77] S. K. Sobolev. The intuitionistic propositional calculus with quantifiers. *Matematicheskie Zametki*, 22(1):69–76, 1977. English translation in *Mathematical notes of the Academy of Sciences of the USSR*, 1977. **22**(1): 528-532. doi:10.1007/BF01147694.

[SU06] M. H. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard Isomorphism*, volume 149 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2006.

[Win83] G. Winskel. A representation of completely distributive algebraic lattices. Technical Report CMU-CS-83-154, Carnegie-Mellon University SCS, 1983.

[Win09] G. Winskel. Prime algebraicity. *Theoretical Computer Science*, 410(41):4160–4168, 2009.