On Weierstrass numerical semigroups generated by four elements [1]

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Abstract

We study numerical semigroups H generated by four elements. If H is almost symmetric and the minimum odd integer in H is sufficiently large, we show that it is Weierstrass. Otherwise, applying Herzog-Watanabe's result [2] we obtain that almost symmetric numerical semigroups satisfying some property are Weierstrass.

1 Terminologies and introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0\backslash H$ is finite. The cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, denoted by $g(H)$. In this paper H always stands for a numerical semigroup. We set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteqq H\},$$

which is called the *conductor* of H. It is well-known that $c(H) \leqq 2g(H)$. H is said to be *symmetric* if $c(H) = 2g(H)$. H is said to be *quasi-symmetric* if $c(H) = 2g(H) - 1$. We have $(c(H) - 1) + h \in H$ for any $h \in H$ with $h > 0$. The number $c(H) - 1$ is called the *Frobenius number* of H. An element $f \in \mathbb{N}_0\backslash H$ is called a *pseudo-Frobenius number* of H if $f + h \in H$ for any $h \in H$ with $h > 0$. We denote by $PF(H)$ the set of pseudo-Frobenius numbers. The cardinality of the set $PF(H)$ is denoted by $t(H)$, which is called the *type* of H. It is known that $c(H)+t(H) \leqq 2g(H)+1$. H is said to be *almost symmetric* if the equality $c(H) + t(H) = 2g(H) + 1$ holds. A symmetric numerical semigroup and a quasi-symmeric numerical semigroup are almost symmetric. There exists a numerical semigroup H with $c(H) = 2g(H) - 2$ which is not almost symmetric.

A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0. For a pointed curve (C, P) we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = \alpha P\},$$

where $k(C)$ is the field of rational functions on C. Then $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) with $H(P) = H$. It is well-known that

[1] This paper is an extended abstract and the details will be published (see [4])
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every numerical semigroup generated by two elements is Weierstrass. Using [6] we can show that a numerical semigroup generated by three elements is Weierstrass. Moreover, Bresinsky [1] proved that any symmetric numerical semigroup generated by four elements is Weierstrass. Every quasi-symmetric numerical semigroup generated by four elements is also Weierstrass by [3].

A numerical semigroup H is said to be of *double covering type* , which is abbreviated to *DC*, if there exists a double covering of curves with a ramification point P with $H(P) = H$. Hence, if a numerical semigroup is DC, then it is Weierstrass. For a numerical semigroup H we set

$$d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\},$$

which is a numerical semigroup. Let $\pi : C \longrightarrow C'$ be a double covering of curves with a ramification point P. Then we have $d_2(H(P)) = H(\pi(P))$.

2 Almost symmetric numerical semigroups generated by four elements

For a numerical semigroup H we denote by $M(H)$ the minimal set of generators of H. Moscariello [5] gave a characterization of an almost symmetric numerical semigroup H with $\sharp M(H) = 4$ using the conductor $c(H)$ as follows.

Remark 2.1. Let H be an almost symmetric numerical semigroup which is neither symmetric nor quasi-symmetric. Then we have $c(H) = 2g(H) - 2$.

Let H be a numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$. For $f \in PF(H)$ we define an (n, n) matrix $RF(f) = (\beta_{ij})$ where $\beta_{ii} = -1$ and $\sum_{j=1}^{n} \beta_{ij} a_j = f$, because $f \in PF(H)$ implies that $f + a_i$ belongs to the monoid generated by $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$. We call $RF(f)$ an *RF-matrix* of f. We note that an RF-matrix of f is not uniquely determined by f. Nevertheless, $RF(f)$ will be the notation for one of the possible RF-matrices of f.

Herzog-Watanabe [2] showed the following:

Theorem 2.2. Let H be an almost symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$. Assume that for some $f \in PF^*(H) := PF(H)\backslash\{c(H) - 1\}$ a matrix $RF(f)$ has only one positive entry in each row, which is called the *RF condition*. For any $i \in \{1, 2, 3, 4\}$ we set $\alpha_i = \min\{\alpha > 0 \mid \alpha a_i \in \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_4\rangle\}$. Then renumbering a_1, a_2, a_3 and a_4, we have $\alpha_1 a_1 = (\alpha_2 - 1)a_2 + a_4$, $\alpha_2 a_2 = a_1 + (\alpha_2 - 1)a_3$, $\alpha_3 a_3 = a_2 + (\alpha_4 - 1)a_4$, $\alpha_4 a_4 = (\alpha_1 - 1)a_1 + a_3$ and

$$a_4 = \begin{vmatrix} \alpha_1 & -(\alpha_2 - 1) & 0 \\ -1 & \alpha_2 & -(\alpha_3 - 1) \\ 0 & -1 & \alpha_3 \end{vmatrix}.$$

Let H be a numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$. Let α_i be as in Theorem 2.3. We set $\alpha_i a_i = \sum_{j=1, j \neq i}^{4} \alpha_{ij} a_j$. H is said to be *1-neat* if the following three conditions are satisfied by renumbering a_1, a_2, a_3 and a_4:

(1) $0 \leqq \alpha_{ij} < \alpha_j$ for any i and j.

(2) $\alpha_i = \sum_{k \neq i} \alpha_{ki}$ for any i.

(3) $a_4 = \begin{vmatrix} \alpha_1 & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_2 & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_3 \end{vmatrix}$.

Theorem 4.11 in [3] proved the following:

Theorem 2.3. Let H be a numerical semigroup with $\sharp M(H) = 4$. If H is 1-neat, then it is Weierstrass.

Combining Theorem 2.2 with Theorem 2.3 we see the main result in this section as follows.

Corollary 2.4. Let H be an almost symmetric numerical semigroup with $\sharp M(H) = 4$. Assume that H satisfies the RF condition. Then it is Weierstrss.

3 Weierstrss numerical semigroups generated by four elements

To describe a numerical semigroup we use the following notations: For any non-negative integers $a_1, a_2, \cdots, a_n$ we denote by $\langle a_1, a_2, \cdots, a_n \rangle$ the additive monoid generated by $a_1, a_2, \cdots, a_n$. For a numerical semigroup H the minimum positive integer in H is denoted by $m(H)$, which is called the *multiplicity* of H. We set $s_i = \min\{h \in H \mid h \equiv i \bmod m(H)\}$ for $i = 1, \ldots, m(H) - 1$. The set $S(H) = \{m(H), s_1, \ldots, s_{m(H)-1}\}$ is called the *standard basis* for H. We set

$$n(H) = \min\{h \in H \mid h \text{ is odd}\}.$$

From now on we always assume that $\sharp M(H) = 4$. In a forthcoming article [4] we are going to give the proofs of the results in this section.

Lemma 3.1. Assume that $n(H) \geqq c(d_2(H)) + m(d_2(H))$. Then we obtain $\sharp M(d_2(H)) = 2$ or 3.

Theorem 3.2. Assume that $\sharp M(d_2(H)) = 3$ and $n(H) \geqq c(d_2(H)) + m(d_2(H)) - 1$. We have the following:

(1) $H = 2d_2(H) + n(H)\mathbb{N}_0$.

(2) $g(H) = 2g(d_2(H)) + \dfrac{n(H)-1}{2}$.

(3) We set $c(d_2(H)) = 2g(d_2(H)) - r$ with $r \geqq 0$. Then $c(H) = 2g(H) - 2r$.

(4) If $c(H) = 2g(H) - 2$, then H is not almost symmetric.

(5) H is DC, hence it is Weierstrass.

Theorem 3.3. Let a and b be positive integers with $2 \leqq a < b$ satisfying $(a, b) = 1$. Let n be an odd integer with $n \geqq (a-1)(b-1) + a - 1$. We set $H = 2\langle a, b\rangle + \langle n, n + 2(b - ar)\rangle$. where r is a positive integer with $b - ar > 0$. Then we have the following:

(1) $d_2(H) = \langle a, b\rangle$.

(2) $g(H) = 2g(\langle a, b\rangle) + \dfrac{n-1}{2} - (a-1)r$.

(3) $c(H) = 2g(H) - 2r$.

(4) If $n \geqq (a-1)(b-1) + 2r(a-1) + 1$, then H is DC, hence it is Weierstrass.

Theorem 3.4. Assume that $\sharp M(d_2(H)) = 2$, $n(H) \geqq c(d_2(H)) + m(d_2(H))$ and $c(H) = 2g(H) - 2$. Then we have

(1) $H = 2\langle a, b\rangle + \langle n(H), n(H) + 2(b - a)\rangle$, where we set $d_2(H) = \langle a, b\rangle$ with $2 \leqq a < b$.

(2) $g(H) = 2g(d_2(H)) + \dfrac{n(H)-1}{2} - (a-1)$.

(3) H is almost symmetric.

(4) If $n(H) \geqq (a-1)(b-1) + 2a - 1$, then H is DC, hence it is Weierstrass.

(5) If $n(H) \geqq (a-1)(b-1) + 2a$, then for any $f \in PF^*(H)$ a matrix $RF(f)$ has at least two positive entries in some row, i.e., the RF condition is not satisfied.

(6) If $n(H) = (a-1)(b-1) + 2a - 1$, then the matrix $RF(n - 2a)$ has only one positive entry in each row, i.e., the RF condition is satisfied.

Main Theorem 3.5. Le H be an almost symmetric numerical semigroup with $\sharp M(H) = 4$ which is neither symmetric nor quasi-symmetric. Then we have the following:

(1) If $n(H) \geqq c(d_2(H)) + 2m(d_2(H)) - 1$, then it is DC, hence Weierstrass.

(2) If H satisfies the RF condition, then it is Weierstrass.

Example 3.6. Let H be a numerical semigroup with $M(H) = \{10, 14, 35, 39\}$. Hence, $d_2(H) = \langle 5, 7\rangle$, $m(d_2(H)) = 5$, $g(d_2(H)) = 12$ and $c(d_2(H)) = 24$. Then we obtain

$$S(H) = \{10, 14, 28, 35, 39, 42, 53, 56, 67, 81\}.$$

We have $g(H) = 37$ and $c(H) = 81 - 10 + 1 = 72 = 2g(H) - 2$. Moreover, we obtain $PF(H) = \{35 - 10, 56 - 10, 81 - 10\} = \{25, 46, 71\}$. Hence, we get $t(H) = 3$, which implies that H is almost symmetric. An RF-matrix of 25 is

$$RF(25) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 6 & 0 & -1 & 0 \\ 5 & 1 & 0 & -1 \end{pmatrix}.$$

An RF-matrix of 46 is

$$RF(46) = \begin{pmatrix} -1 & 4 & 0 & 0 \\ 6 & -1 & 0 & 0 \\ 0 & 3 & -1 & 1 \\ 5 & 0 & 1 & -1 \end{pmatrix}.$$

Thus, H does not satisfy the RF condition. But we have

$$n(H) = 35 > c(d_2(H)) + 2m(d_2(H)) = 24 + 10 = 34.$$

Hence, H is DC, which implies that it is Weierstrass.

Example 3.7. Let H be a numerical semigroup with $M(H) = \{7, 8, 17, 26\}$. Then $S(H) = \{7, 8, 17, 26, 16, 25, 34\}$. We have $g(H) = 15$ and $c(H) = 34 - 7 + 1 = 28 = 2g(H) - 2$. Moreover, we obtain $PF(H) = \{16-7, 25-7, 34-7\} = \{9, 18, 27\}$. Hence, we get $t(H) = 3$, which implies that H is almost symmetric. On the other hand, we have $d_2(H) = \langle 4, 7, 13\rangle$, $g(d_2(H)) = 7$ and $c(d_2(H)) = 14 - 4 + 1 = 11$. Hence, we obtain

$$n(H) = 7 < c(d_2(H)) + 2m(d_2(H)) - 2 = 11 + 8 - 2 = 17.$$

But an RF-matrix of 9 is

$$RF(9) = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 5 & 0 & 0 & -1 \end{pmatrix}.$$

Hence, H satisfies the RF condition, which implies that it is Weierstrass.

References

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