On a formula of parametric version for Heronian triangles

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1 Introduction

Pythagorean triangles are right triangles whose side lengths are all integers. For example, the triangles with side lengths $3, 4, 5$, and with side lengths $5, 12, 13$, and others. The area of the Pythagorean triangle is known to be an integer.

Heronian triangles are triangles whose side lengths and area are all integers. For example, the triangles with side lengths $3, 4, 5$, area 6, and with side lengths $5, 5, 6$, area 12, and others. Any Pythagorean triangle is a Heronian triangle, that is, Heronian triangle is a generalization of this Pythagorean triangle. We are interested in combining Pythagorean triangles to create Heronian triangles and have been conducting research. This study is in the category of elementary geometry and elementary number theory.

In section 2, we describe the classical theory of the Pythagorean triples. The Pythagorean triple is a well-known topic in elementary number theory. Here, we introduce the formula that determines the Pythagorean triple. It also shows all Pythagorean triples less than 100.

In section 3, we explain the definition of the Heronian triangles. Next, we explain the relationship between the Heronian triangles and the Pythagorean triangles. We explain the formulas that make up the Heronian triangle explained by Carmichael. It is believed that this formula was not discovered by Carmichael, but explained by him. This formula expresses the ratio of the three side lengths of the Heronian triangle. Therefore, we do not give the three side lengths directly. We study the relationship between the three side lengths of the Pythagorean triangle and the three side lengths of the Heronian triangle.

2 Pythagorean triples

The facts mentioned in this chapter are classic and famous, so no reference is given.

A *Pythagorean triple* is a triple a, b, c of natural numbers that satisfy the equation $a^2 + b^2 = c^2$. A triangle whose side lengths form a Pythagorean triple is a Pythagorean triangle, and is necessarily a right triangle.

Proposition 2.1 *Let a, b, c be a solution of the equation*

$$a^2 + b^2 = c^2$$

in relatively prime integers. Then a or b must be even. On the assumption that a is even, a, b, c has the form

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2,$$

where u and v are relatively prime natural numbers with $u+v$ odd and $u > v$. Every such pair u, v corresponds to exactly one primitive Pythagorean triple a, b, c, that is, greatest common divisor of a, b, c is equal to 1.

Example 2.2 *In Proposition 2.1, by setting $u = 2$, $v = 1$, we get $a = 4$, $b = 3$, $c = 5$. And by setting $u = 3$, $v = 2$, we get $a = 12$, $b = 5$, $c = 13$, and by setting $u = 4$, $v = 3$, we get $a = 24$, $b = 7$, $c = 25$.*

Example 2.3 *Examples of Primitive Pythagorean triples are as follows:*

Table 1: Primitive Pythagorean triples ($a \leq b \leq c < 100$)

No.	a	b	c	S	No.	a	b	c	S
P1	3	4	5	6	P9	16	63	65	504
P2	5	12	13	30	P10	20	21	29	210
P3	7	24	25	84	P11	28	45	53	630
P4	8	15	17	60	P12	33	56	65	924
P5	9	40	41	180	P13	36	77	85	1386
P6	11	60	61	330	P14	39	80	89	1560
P7	12	35	37	210	P15	48	55	73	1320
P8	13	84	85	546	P16	65	72	97	2340

3 Heronian triangles

3.1 Definition of Heronian Triangles

Definition 3.1 *Heronian triangles are triangles whose side lengths and area are all integers.*

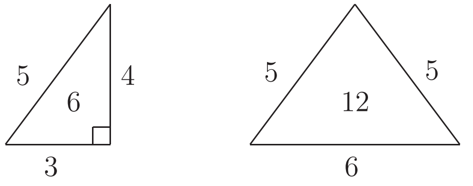


Figure 1: Heronian Triangles

At least one of the two sides of a Pythagorean triangle that sandwiches a right angle is even. Therefore, the area of all Pythagorean triangles is an integer. This property leads to the following proposition:

Proposition 3.2 *A Pythagorean triangle is a Heronian triangle.*

3.2 Formula of parametric version for Heronian triangles

This section is based on Carmichael's description[1].

Proposition 3.3[cf [1]] *Every rational integral triangle has ite sides proportional to numbers of the form*

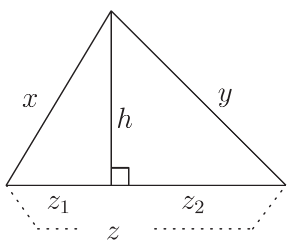
$$n(m^2+h^2), \quad m(n^2+h^2), \quad (m+n)(mn-h^2), \tag{1}$$

where m, n, h are positive intgers and $mn > h^2$.

Proof of Proposition 3.3:

If we denote the sides of the triangle by x, y, z the perpendicular from the opposite angle upon z by h and the segments into which it divides z by z_1 and z_2, z_1 being adjacent to x and z_2 adjacent to y, then we have

$$h^2 = x^2 - z_1^2 = y^2 - z_2^2, \quad z_1 + z_2 = z. \tag{2}$$



These equations must be satisfied if x, y, z are to be the sides of a rational triangle. Moreover, if they are satisfied by positive rational numbers x, y, z, z_1, z_2, h, then x, y, z, h are in order the sides and altitude upon z of a rational triangle. From equations (2), it follows that rational numbers m and n exist such that

$$x + z_1 = m, \ x - z_1 = \frac{h^2}{m},$$

$$y + z_2 = n, \ y - z_2 = \frac{h^2}{n}.$$

Hence x, y, z, where $z = z_1 + z_2$, have the form

$$x = \frac{1}{2}\left(m + \frac{h^2}{m}\right), \quad y = \frac{1}{2}\left(n + \frac{h^2}{n}\right), \quad z = \frac{1}{2}\left(m + n - \frac{h^2}{m} - \frac{h^2}{n}\right),$$

respectively. Where if we multiply both sides by $2mn$, we have

$$\begin{cases} 2mnx = mn\left(m + \frac{h^2}{m}\right) = n(m^2 + h^2), \\ 2mny = mn\left(n + \frac{h^2}{n}\right) = m(n^2 + h^2), \\ 2mnz = mn\left(m + n - \frac{h^2}{m} - \frac{h^2}{n}\right) = (m+n)(mn - h^2), \end{cases} \tag{3}$$

where $2mnx$, $2mny$, $2mnz$ are all rational numbers. By the conditions for becoming a triangle, $x + z_1 > h$, $y + z_2 > h$, therefore we get $mn = (x + z_1)(y + z_2) > h^2$.

In this situation, let d be a least common multiple of denominator of the rational numbers m, n, h. There exist μ, ν, k such that

$$m = \frac{\mu}{d}, \quad n = \frac{\nu}{d}, \quad h = \frac{k}{d}.$$

We get

$$\begin{cases} 2mnx = \frac{\nu}{d}\left(\frac{\mu^2}{d^2} + \frac{k^2}{d^2}\right), \\ 2mny = \frac{\mu}{d}\left(\frac{\nu^2}{d^2} + \frac{k^2}{d^2}\right), \\ 2mnz = \left(\frac{\mu}{d} + \frac{\nu}{d}\right)\left(\frac{\mu}{d}\frac{\nu}{d} - \frac{k^2}{d^2}\right). \end{cases} \tag{4}$$

If we multiply the values $2mnx, 2mny, 2mnz$ in (4) by d^3, and put $\bar{x} = d^3 \times 2mnx$, $\bar{y} = d^3 \times 2mny$, $\bar{z} = d^3 \times 2mnz$ respectively, we get

$$\begin{cases} \bar{x} = d^3\left(\frac{\nu}{d} \times \frac{\mu^2 + k^2}{d^2}\right) = \nu(\mu^2 + k^2), \\ \bar{y} = d^3\left(\frac{\mu}{d} \times \frac{\nu^2 + k^2}{d^2}\right) = \mu(\nu^2 + k^2), \\ \bar{z} = d^3\left(\frac{\mu+\nu}{d} \times \frac{\mu\nu - k^2}{d^2}\right) = (\mu + \nu)(\mu\nu - k^2). \end{cases} \tag{5}$$

With a modified notation, we get the result. ∎

Apply this Proposition 3.3 to calculate a length of less than 100 on all three sides of the triangle.

Proposition 3.4 *If $m \geq 10$ or $n \geq 10$ or $h \geq 10$ then of the values in the three equations in (1), at least one is more than* 100.

Proof. In case $m \geq 10$, n and h are positive integer, therefore we can easily see $n(m^2 + h^2) \geq 100$. Similarly in case $n \geq 10$, we have $m(n^2 + h^2) \geq 100$, and in case $h \geq 10$, we have $n(m^2 + h^2) \geq 100$. ■

By this Proposition 3.4, When we calculating with a computer, it is enough to consider the case of $m, n, h \leq 9$ in (1). So, the results are as follows.

Example 3.5 *The length of the three sides of Heronian triangle calculated by applying the proposition 3.3, with $m, n, h \leq 9$ (See Table 2).*

Table 2: Three sides of Heronian triangle calculated by applying proposition 3.3 ($a \leq b \leq c < 100$)

No.	a	b	c	S
C1	3	4	5	6
C2	6	8	10	24
C3	6	25	29	60
C4	7	65	68	210
C5	8	15	17	60
C6	10	10	12	48
C7	10	24	26	120
C8	12	35	37	210
C9	14	30	40	168
C10	14	48	50	336
C11	15	20	25	150
C12	16	63	65	504
C13	18	80	82	720
C14	20	34	42	336
C15	21	72	75	756

No.	a	b	c	S
C16	24	32	40	384
C17	24	35	53	336
C18	24	78	90	864
C19	25	52	63	630
C20	30	30	48	432
C21	30	39	39	540
C22	30	74	88	1056
C23	36	40	68	576
C24	40	42	58	840
C25	40	51	77	924
C26	45	50	85	900
C27	48	64	80	1536
C28	52	56	60	1344
C29	65	87	88	2640
C30	80	80	96	3072

Remark 3.6 *The numbers shown in Table 2 do not indicate all Heronian triangles with $a, b, c < 100$.*

Example 3.7 *For example, the triangles with side lengths $5, 12, 13$, and with side lengths $7, 24, 25$ are Heronian triangles by Example 2.3, but these triangles do not exist Table 2.*

By describing Carmichael (1) in Proposition 3.3, if $m, n, h \leq 9$, then the following example is shown:

Example 3.8 *The primitive Pythagorean triangle P2 is obtained in Proposition 3.3 by setting $m = 3$, $n = 2$, $h = 2$ and dividing it by 2. Similarly, the triangle P3 is obtained by setting $m = 4$, $n = 3$, $h = 3$ and dividing it by 3 or by setting $m = 7$, $n = 1$, $h = 1$ and dividing by 2. The triangle P5 is*

Table 3: Primitive Pythagorean triples that cannot be displayed in Table 2. ($a \leq b \leq c < 100$)

No.	a	b	c	S
P2	5	12	13	30
P3	7	24	25	84
P5	9	40	41	180
P6	11	60	61	330
P8	13	84	85	546
P10	20	21	29	210

No.	a	b	c	S
P11	28	45	53	630
P12	33	56	65	924
P13	36	77	85	1386
P14	39	80	89	1560
P15	48	55	73	1320
P16	65	72	97	2340

obtained by setting $m = 9$, $n = 1$, $h = 1$ *and dividing it by* 2.

Future research themes We study the relationship between the lengths of the three sides of the Pythagorean triangle and the lengths of the three sides of the Heronian triangle.

Acknowledgment

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References

[1] R.D.Carmichael, *Diophantine Analysis*, John Wiley & Sons (1915), 11-13.

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