A Study of Constraints on Eulerian Circuits

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Abstract

The author calls the maximum of the length of a shortest subcycle of an Eulerian circuit of an Eulerian graph the Eulerian recurrence length and pursues the determination of the Eulerian recurrence length $e(K_n)$ of a complete graph K_n with an odd size of the vertex set. So far, the value of $e(K_n)$ has been found for all $n < 15$, and it has been proved that the inequality $n-4 \leqq e(K_n) \leqq n-3$ holds for all $n \geqq 15$. The author conjectures that $e(K_n) = n-4$ holds for all $n \geqq 15$ and attempts to prove this conjecture by mathematical induction with $e(K_{15}) = 11$ as the basis. However, running a simple search algorithm in the computing environment available to the author, it turns out that the search space is too large to prove $e(K_{15}) = 11$. In this paper, the author proposes to introduce two types of constraints on the edges of the trails to be searched in order to reduce the search space.

Keywords. Eulerian circuit, computer experiment, search space, constraint.

1 Introduction

The shortest subcycle length $s(C)$ for an Eulerian circuit C of an Eulerian graph is defined as

$$s(C) = \min\{(j-i) \bmod m \mid v_i = v_j\}.$$

Some of the terms related to graph theory used in this paper are described in Section 2. For other terms, please refer to the book [2]. For an Eulerian graph G, the maximum value of the shortest subcycle length of an Eulerian circuit C of G is called the Eulerian recurrence length of G and denoted by $e(G)$, that is

$$e(G) = \max\{s(C) \mid C \text{ is an Eulerian circuit of } G\}.$$

The author is working on determining the Eulerian recurrence length $e(K_n)$ of a complete graph K_n consisting of an odd number n of vertices. Currently, for all odd positive integers n less than 15, the value of $e(K_n)$ is determined using computers. On the other hand, although the value of $e(K_n)$ has not been completely determined for odd integers n greater than or equal to 15, the author and others obtain the inequality

$$n-4 \leqq e(K_n) \leqq n-3,$$

and further conjecture that $e(K_n) = n - 4$ holds for all odd integers n greater than or equal to 15[1]. The author and others have a policy of proving the above conjecture by mathematical induction by proving the following two conjectures.

Conjecture 1. Function $f(n) = n - e(K_n)$ defined on odd integers greater than or equal to 3 is weakly increasing.

Conjecture 2. $e(K_{15}) = 11$.

In this paper, we consider the case of performing on a computer the search for an Eulerian circuit in K_{15} whose shortest subcycle length is 12 for the proof of Conjecture 2 above. This paper discusses the introduction of two types of constraints to reduce the size of the search space and the effects of this reduction. In this paper, the search space in the target search is the set consisting of all of the elements of the whole set that were examined during the search. The target search is the search for an Eulerian circuit in K_{15} whose shortest subcycle length is 12, and the whole set is the set consisting of all the trails obtained by extending a particular trail, called the initial configuration.

The next section will discuss the estimated size of the search space in the target search. In Section 3, we will introduce two types of new constraints to reduce the search space of the target search and discuss the effects of their introduction. Section 4 summarizes the contents of this paper and discusses future challenges.

2 The Size of the Search Space

2.1 Preliminaries

Some of the terms related to graph theory used in this paper are described below. For other terms, please refer to the book [2]. In this paper, undirected graphs are simply referred to as graphs. A walk in a graph G is a sequence $(v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$ of alternating vertices and edges that begins with a vertex and ends with a vertex, where each edge e_i is an edge connecting v_{i-1} and v_i. If a graph G is a simple graph, then a walk $W = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$ of G may be denoted by $v_0 \to v_1 \to v_2 \to \cdots \to v_k$. A walk W is said to be a trail if W has no edges that appear more than once. For a walk W in a graph, we write $P_m(W)$ and say that W satisfies condition P_m, if the same vertex does not appear more than twice among any m consecutive vertices on W. Note that if a walk W satisfies condition P_m, then W has no vertex with an occurrence interval less than m.

2.2 Initial Configurations in Trail Extension

Let $\{0, 1, 2, \ldots, 14\}$ be the vertex set of the complete graph K_{15} in what follows. Furthermore, $V(K_{15})$ and $E(K_{15})$ denote the vertex set and the edge set of K_{15}, respectively. The length of an Eulerian circuit of a complete graph K_{15} is $105 = \binom{15}{2}$.

For any Eulerian circuit C satisfying condition P_{12}, we can apply a vertex substitution ρ to C so that C always has the subtrail

$$13 \to 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 0 \to 12.$$

The reasons for this are as follows. The existence of a trail from 0 on the left to 0 on the right is evident from the fact that there is no Eulerian circuit of K_{15} satisfying condition P_{13}. The vertex v adjacent to the left of 0 on the left and the vertex w adjacent to the right of 0 on the right must not both be vertices from 0 to 11, since C is a circuit and satisfies condition P_{12}. Furthermore, if v and w are the same, then the same edge appears twice on C, which is inconsistent with C being a circuit. Therefore, the above trail of length 14 containing 15 vertices may be used as the initial configuration when searching for an Eulerian circuit satisfying condition P_{12} by extending trails. Expression $I(15)$ denotes the initial configuration above. Furthermore, if the initial configuration I is a subtrail of a trail W beginning at the initial vertex of W, then W is called an I-trail.

2.3 Evaluating the Size of the Search Space

The substitution on the vertex set of K_{15},

$$\varphi : V(K_{15}) \to V(K_{15}), \tag{1}$$

is defined by the following equations:

$$\begin{aligned} \varphi(0) = 0, \varphi(1) = 11, \varphi(2) = 10, \varphi(3) = 9, \varphi(4) = 8, \varphi(5) = 7, \\ \varphi(6) = 6, \varphi(7) = 5, \varphi(8) = 4, \varphi(9) = 3, \varphi(10) = 2, \varphi(11) = 1, \\ \varphi(12) = 13, \varphi(13) = 12, \text{ and } \varphi(14) = 14. \end{aligned}$$

By applying the mapping φ to a trail W obtained by extending the initial configuration $I(15)$ forward, i.e., an $I(15)$-trail, we obtain a backward extension of $I(15)$, i.e., a trail whose last subtrail of length 14 is the initial configuration $I(15)$. For a set S whose elements are $I(15)$-trails, $\varphi(S)$ denotes the set consisting of trails obtained by applying φ to each element of S, namely $\{\varphi(s) \mid s \in S\}$. Note that the sizes of S and $\varphi(S)$ are the same.

Expression $X_{I,m}(k)$ denotes the set consisting of all trails satisfying condition P_m obtained by extending a given initial configuration I by length k, and $\mathcal{X}_{I,m}(k)$ is defined as

$$\mathcal{X}_{I,m}(k) = \bigcup_{i=1}^{k} X_{I,m}(i).$$

In particular, $X_{I,12}(k)$ and $\mathcal{X}_{I,12}(k)$ are abbreviated as $X_I(k)$ and $\mathcal{X}_I(k)$, respectively. Furthermore, $X_{I(15)}(k)$ and $\mathcal{X}_{I(15)}(k)$ are abbreviated as $X(k)$ and $\mathcal{X}(k)$, respectively.

If the set of trails $\varphi(X(k))$ is of a size that can be stored in a hash table during the search, then for the trails W obtained by extending the initial configuration $I(15)$ by length $91 - k$, the search for an element of $\varphi(X(k))$ that becomes an Eulerian circuit satisfying P_{12} by connecting with W can be performed in constant time on average by using the hash table. Condition A for a positive integer k is defined as follows, and the maximum value of k satisfying condition A is denoted by k_{H}.

Condition A All of the trails obtained by extending the initial configuration $I(15)$ backward by length k that could possibly yield an Eulerian circuit satisfying P_{12} by continuing the extension can be stored in a hash table.

In what follows, the search for an Eulerian circuit of K_{15} satisfying condition P_{12} is called the target search. Without any modification of the search algorithm to reduce the search space as described in the next section, the size of the search space in the target search can be estimated as $|\mathcal{X}(91 - k_{\mathrm{H}})|$.

The value of the parameter K_{H} could increase as the main memory capacity increases, but also by improving the search algorithm. Although a detailed explanation is omitted, we estimate from computer experiments using random numbers that the value of k_{H} is about 25 when a simple search algorithm is adopted in the computing environment available to the author at present and that the size of the above search space $|\mathcal{X}(91-k_{\mathrm{H}})| = |\mathcal{X}(66)|$ is about 2^{62}. In the computing environment available to the author, the size of this search space is too large to complete the target search.

3 Introducing Constraints for Search Space Reduction

Below, we propose two types of constraints to be introduced to reduce the search space and discuss the effects of these constraints on the reduction of the search space.

3.1 The First Constraint

Let S be a subset of $E(K_{15})$ such that S is invariant to φ, that is S satisfies the equation $S = \varphi(S) = \{\varphi(x)|x \in S\}$, and for a positive integer k less than or equal to 45, define $N(S, k)$, $n(S, k)$, $N'(S, k)$, and $n'(S, k)$ as follows:

$$\begin{aligned} N(S,k) &= \{W \in X(k) \mid |S \cap W| > |S|/2\}, \\ n(S,k) &= |N(S,k)|, \\ N'(S,k) &= \{W \in X(k) \mid |S \cap W| \leqq |S|/2\}, \text{ and} \\ n'(S,k) &= |N'(S,k)| = |X(k)| - n(S,k). \end{aligned}$$

Furthermore, when k is an integer greater than 45, let $N(S, k)$ be the set consisting of all trails obtained by extending trails belonging to $N(S, 45)$ by length $k - 45$, and let $N'(S, k)$ be the set consisting of all trails obtained by extending trails belonging to $N'(S, 45)$ by length $k - 45$. In the expressions above, W is regarded as a set of edges. Note that the composite mapping $\varphi \circ \varphi$ is an identity mapping I.

Let us attempt to take S so that $n(S, k)$ is as large as possible ($n'(S, k)$ is as small as possible) for small k. For example, if

$$n'(S, 45) = 0 \tag{2}$$

holds, then it has been proved that there is no Eulerian circuit of K_{15} satisfying condition P_{12}. If such an Eulerian circuit C exists, then the subtrail of length 45 in its second half contains only less than $|S|/2$ edges belonging to S. However, the Eulerian circuit $\varphi(C)$ obtained by transforming C by the mapping φ in expression (1) satisfies condition P_{12} as well as C, but the subtrail of length 45 in the first half of $\varphi(C)$ contains no more than $|S|/2$ edges belonging to S, a contradiction. However, such an S would either not exist or be very difficult to find.

Even if no such S can be found, if there exists an Eulerian circuit C of K_{15} satisfying condition P_{12}, then, letting W_C be the only element of $X(45)$ that is a subtrail of C, and W'_C the only element of $X(45)$ that is a subtrail of $\varphi(C)$, then $|S \cap W_C| \leqq |S|/2$ or $|S \cap W'_C| \leqq |S|/2$ holds. Hence, if a trail W satisfies the following expression (3), then every trail obtained by extending W may be excluded from the search.

$$|S \cap W| > |S|/2 \tag{3}$$

Thus the only trails obtained by extending the initial configuration $I(15)$ by length k that satisfy condition P_{12} and that actually need to be examined during the search are those belonging to $N'(S,k)$. Define $\mathcal{N}'(S,k)$ by

$$\mathcal{N}'(S,k) = \bigcup_{i=1}^{k} N'(S,i),$$

Under this definition, the search space is $\mathcal{N}'(S, 91 - k_{\mathrm{H}})$ when the first constraint is introduced. However, the first constraint has no effect on increasing the value of k_{H} because it cannot be used to constrain trails obtained by extending $I(15)$ backward.

3.2 The Second Constraint

The initial configuration $I(12)$ is defined to be

$$0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11.$$

Let k be a positive integer less than 94. Then, if k is sufficiently large, we can take a set S consisting of edges connecting vertices belonging to $I(12)$ so that the following holds:

$$W \cap S \neq \emptyset \quad \text{for any } W \in X_{I(12)}(k). \tag{4}$$

Note that $X_{I(12)}(k)$ consists of $I(12)$-trails of length $|I(12)| + k$ satisfying condition P_{12}. In other words, we can take S such that no matter how we extend the initial configuration $I(12)$ by length k to obtain a trail W satisfying condition P_{12}, some element of S will always appear in W. For example, for $k > 6$, expression (4) always holds if we take as S the set consisting of all edges connecting two vertices in $I(12)$. If k is somewhat larger, the size of S could be reduced considerably. A possible way to take S is to choose the elements of S in descending order of the frequency of the edges appearing in $X_{I(12)}(k)$, so that the expression (4) holds.

Let S be a set such that each element is an edge connecting two vertices of $I(12)$. For any $I(15)$-trail W satisfying condition P_{12}, let the subtrail of W of length 11 that ends at the final vertex of W be described as

$$v_0 \to v_1 \to v_2 \to \cdots \to v_{11},$$

where v_{11} is the final vertex of W. Then, the set of edges $\{v_i v_j \mid ij \in S\}$ is called the transformation of S by W and denoted by S^W.

Let $S_1, S_2, \ldots,$ and S_h be sets such that each element is an edge connecting two vertices of $I(12)$, and assume that for each $i \in \{1, 2, \ldots, h\}$, the proposition (4) with S

replaced by S_i is true. Furthermore, let m be a positive integer satisfying $m + k \leqq 91$. Then, if there exists an $i \in \{1, 2, \ldots, h\}$ satisfying

$$S_i^W \subseteq W \tag{5}$$

for element W of $X(m) = X_{I(15),12}(m)$, then we cannot extend W to make an Eulerian circuit C in K_{15} satisfying condition P_{12}. Denote by

$$X[S_1, S_2, \ldots, S_h](m) = X_{I(15)}[S_1, S_2, \ldots, S_h](m)$$

the set consisting of all elements W of $X(m)$ such that, for any subsequence W' of W beginning at the initial vertex of W,

$$\text{if} \quad |W'| + k \leqq |E(K_{15})| = 105, \quad \text{then} \quad S_1^{W'} \nsubseteq W', S_2^{W'} \nsubseteq W', \ldots, S_h^{W'} \nsubseteq W'. \tag{6}$$

If the trail $W \in X(m)$ does not belong to the set $X[S_1, S_2, \ldots, S_h](m)$, we cannot continue to extend W to make an Eulerian circuit in K_{15} satisfying condition P_{12}. A trail that need to be examined before all trails obtained by extending $I(15)$ by length m are explored therefore belongs to the set

$$\mathcal{X}[S_1, S_2, \ldots, S_h](m) = \bigcup_{i=1}^{m} X[S_1, S_2, \ldots, S_h](i). \tag{7}$$

Let $\mathcal{S}$ denote the family of sets $\{S_1, S_2, \ldots, S_h\}$. Then, $X[\mathcal{S}](m)$ and $\mathcal{X}[\mathcal{S}](m)$ are defined as

$$X[\mathcal{S}](m) = X[S_1, S_2, \ldots, S_h](m)$$

and

$$\mathcal{X}[\mathcal{S}](m) = \mathcal{X}_{I(15)}[\mathcal{S}](m) = \mathcal{X}_{I(15)}[S_1, S_2, \ldots, S_h](m),$$

respectively. From this definition, if we denote by R the set of edges that define the first constraint, i.e., S in expression (3), then the search space is expressed as

$$N'(R, 91 - k_{\mathrm{H}}) \cap \mathcal{X}[\mathcal{S}](91 - k_{\mathrm{H}}). \tag{8}$$

Furthermore, k_{H} in expression (8) is the maximum value of k such that $X[\mathcal{S}](k)$ fits into the hash table, which we expect to be larger than the value 25 we estimate in the absence of discussion in this section.

4 Summary

The author aims to complete the search for an Eulerian circuit in K_{15} satisfying condition P_{12}. For any integer k greater than or equal to 7, define proposition $Q(k)$ to be "There exists no Eulerian circuit of K_{2k+1} satisfying condition P_{2k-2}". The author conjectures that the proposition $Q(k)$ holds for any integer k greater than or equal to 7, and considers proving the conjecture by mathematical induction on the basis that $Q(7)$ holds.

In this paper, we have introduced two types of new constraints to reduce the search space of the target search and discussed the effects of these constraints on search space reduction. In particular, the author expects that the effect of reducing the search space by introducing the second constraint depends greatly on the choice of the family of set

$\mathcal{S}$ from which the second constraint is constructed and that finding a good family of sets $\mathcal{S}$ will have a great effect on reducing the search space. Note, however, that the total computation time of the search is affected by the size of the family of set $\mathcal{S}$ in addition to the size of the search space. Here, the size of $\mathcal{S}$ is the sum of the sizes of the sets belonging to $\mathcal{S}$, namely $\sum_{S \in \mathcal{S}} |S|$.

The magnitude of the reduction in total computation time for the search due to the introduction of constraints based on the family of sets $\mathcal{S}$ can be estimated by computer experiments with a sufficient number of random trials. In the future, we aim to discover the best possible family of set $\mathcal{S}$ by incorporating Monte Carlo tree search algorithms, etc., and to complete the search for an Eulerian circuit of K_{15} that satisfies condition P_{12}.

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