On Computational Aspects of Finding Inverse Monoids of Partial Automorphisms

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May 27, 2022

Abstract

A partial automorphism of a combinatorial structure is an isomorphism between its two induced substructures. It is a natural generalization of a total automorphism which is a classical tool to study symmetries. As all total automorphisms of a combinatorial structure form a group, all partial automorphisms form an inverse monoid, called inverse monoid of partial automorphisms of a combinatorial structure. This monoid is a richer and more complex object that contains more information about the structure than its automorphism group used in the classical Group Theory. In our paper, we review the results we obtained for the inverse monoids of partial automorphisms of graphs in the study of questions analogous to those concerning automorphism groups of graphs. We also address some computational aspects of finding these inverse monoids.

1 Introduction

To understand a combinatorial structure $\mathcal{C} = (V, \mathcal{F})$, where V is a (finite) non-empty set of vertices and $\mathcal{F}$ is a family of subsets of V, called blocks, we often look at symmetries of the structure that are represented by automorphisms of $\mathcal{C}$. An automorphism of a combinatorial structure $\mathcal{C}$ is a permutation of its vertices that preserves the structure, i.e. preserves its blocks. All automorphisms form a group, which is called an automorphism group of $\mathcal{C}$ and is denoted by $Aut(\mathcal{C})$, and is a subgroup of the symmetric group $Sym(V)$ on the set of vertices V, $Aut(\mathcal{C}) \leq Sym(V)$. The automorphism group of a combinatorial structure is a powerful classical tool in the study of the structure, allowing one to make various claims about the structure. However, the usefulness of the knowledge of the automorphism group of a combinatorial structure is rather limited if its action on the vertices has a large number of orbits, with the extreme of automorphism group being trivial. This suggests a need for generalization of the classical Group Theory tools. Lately, several works (see for instance [10, 1, 7, 11]) with possible generalizations of automorphisms and automorphism groups appeared. One possible direction in the generalization is relaxing the conditions on permutations of vertices. Instead of using total permutations, so called partial permutations are used. This broadens the study of symmetries of structures from the Group Theory into

*Author supported in part by APVV-19-0308 and VEGA - 1/0423/20.

Inverse Semigroup Theory [10]. The inverse semigroups are very natural setting and natural generalization of the Group Theory tools thanks to Wagner-Preston Theorem 1.1, which is an analogue of Cauley's theorem for groups:

Theorem 1.1 (Wagner-Preston)**.** *Every finite inverse semigroup is isomorphic to an inverse subsemigroup of the symmetric inverse semigroup of all partial bijections of some finite set V.*

We begin with reviewing the concepts of inverse monoid and the full symmetric inverse monoid from Inverse Semigroup Theory in Section 2.

Even though the results that we will present here, apply to more general combinatorial sturctures $\mathcal{C} = (V, \mathcal{F})$, for the purposes of this paper we will restrict our attention to particular case of simple non-oriented graphs, as the most familiar and widely used examples of such structures and also to avoid cumbersome technical details in some arguments. For a more general approach see [7]. We will mention some open problems from Graph Theory to argue that study of *partial* symmetries can be very beneficial in the settings of the graphs, and it make sense to concentrate on them specifically. As another argument for the need to generalize the notion of the classical symmetry in Graph Theory, just recall Erdős's and Rényi's (1963) fact (see [5]), that almost all finite graphs are asymmetric, i.e. with the trivial automorphism group.

We will review structural results and characterization of Partial inverse monoids of graphs together with pointing out computational aspects of finding these monoids. We will end by mentioning our projects where catalogues of Inverse monoids of certain classes of graphs were created.

2 Preliminaries

All the graphs, groups, inverse semigroups and monoids considered in our paper are finite.

2.1 Partial automorphisms of graphs

A *graph* is an ordered pair $\Gamma = (V, E)$, where V is the finite non-empty set of vertices, and E is the set of (undirected) edges, which is a set of 2-element subsets of V.

Classically, automorphism group $Aut(\Gamma) \leq Sym(V)$ is studied using tools of Group Theory to describe graphs and their symmetries. Much attention is given to graphs with rich automorphism groups. We have nice result of Frucht [3]

Theorem 2.1 (Frucht 1938)**.** *For any finite group G there exists a graph Γ such that $Aut(\Gamma) \cong G$.*

However, the result of Erdős and Rényi (see also [5]) shows that the graphs with no non-trivial symmetries are prevalent. A graph Γ is called *asymmetric* if it does not have a non-trivial automorphism, i.e. its automorphism group is trivial.

Theorem 2.2 (Erdös, Rényi, 1963)**.** *Almost all finite graphs are asymmetric.*

In the Figure 1 we show two examples of small asymmetric graphs.

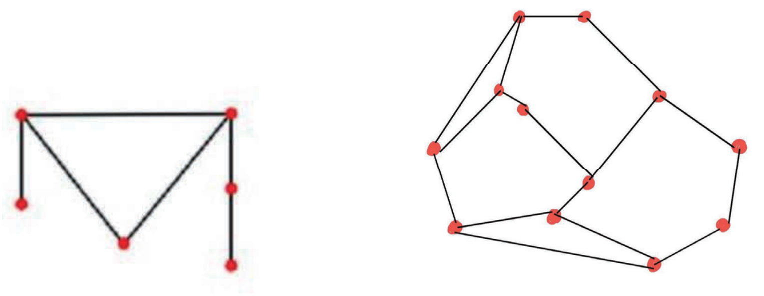

Figure 1: The smallest asymmetric graph and the Frucht graph, one of the five smallest asymmetric cubic graphs.

Theorem 2.2 implies that almost all graphs Γ have the same (trivial) automorphism group $Aut(\Gamma)$, which is not revealing much information about the graphs. To overcome this shortcomings of total automorphisms groups, we turn to a relaxed concept of a partial automorphism.

Definition 2.3. *Let* $\Gamma = (V, E)$ *be a finite graph. A partial automorphism of* $\Gamma = (V, E)$ *is an isomorphism between its two* induced *subgraphs.*

Note that a partial automorphism of a graph Γ can be an isomorphism between two different induces subgraphs Γ_1 and Γ_2 of Γ or it can be a total automorphism of one induce subgraph Γ_1, as is schematically depicted in the Figure 2.

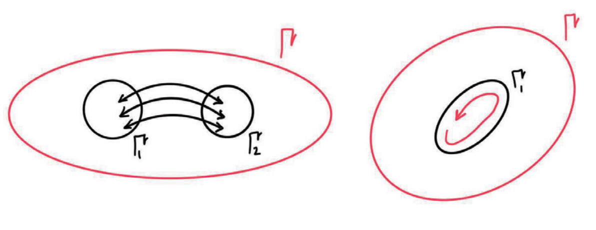

Figure 2: Two situations for partial automorphism of a graph.

As we mentioned in the introduction, the graph theory provides one of the motivations to study partial automorphisms. As an example, let us mention the long-standing open problem called *Graph Reconstruction Conjecture*. (For the details and overview see [8, 9].) Given a finite graph $\Gamma = (\{v_1, \ldots, v_n\}, E)$ on n vertices, consider all induced subgraphs $\Gamma - v_i$ $(1 \leq i \leq n)$. The multiset of these subgraphs is called *deck of* Γ, The Graph Reconstruction Conjecture predicts the unique reconstructability of any graph Γ of order at least 3 from its deck. This problem is closely related to partial automorphisms. Namely, any two induced subgraphs $\Gamma - v_i$ and $\Gamma - v_j$ $(i \neq j)$ admit at least one partial isomorphism φ between $(\Gamma - v_i) - v_j$ and $(\Gamma - v_j) - v_i$. Clearly, if $\Gamma - v_i$ and $\Gamma - v_j$ admit exactly one partial isomorphism φ with domain of size $n-2$, Γ is reconstructable from $\Gamma - v_i$ and $\Gamma - v_j$, by identifying v and $\varphi(v)$, for each v in the domain of φ. Two induced subgraphs $\Gamma - v_i$ and $\Gamma - v_j$ of Γ are said to be *pseudo-similar* if there is a partial automorphism mapping $\Gamma - v_i$ to $\Gamma - v_j$ that cannot be extended to a full automorphism of Γ. The Graph Reconstruction Conjecture holds for graphs containing no pseudo-similar vertices [9].

Similarly as in Group Theory, the set of *all partial automorphisms*, together with the composition and partial inverse of partial maps, forms an inverse monoid. We call this inverse monoid *the partial automorphism monoid of a finite graph* Γ, and denote it by $PAut(\Gamma)$. This object will be the main interest of our study.

2.2 Inverse semigroups

Before we continue, we will briefly review relevant concepts from Inverse Semigroup Theory.

A non-empty set together with an associative operation is called a semigroup, and a semigroup admitting an identity (neutral) element is called a monoid. A monoid $\mathcal{M}$ is an *inverse monoid* if for every $a \in \mathcal{M}$, there exists a unique element $a^{-1} \in \mathcal{M}$, called the inverse of a, such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$ hold. Note that the operation of taking inverse has the properties that $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$ for any $a, b \in \mathcal{M}$. An element e of an inverse monoid $\mathcal{M}$ is called an idempotent, if $e^2 = e$. The set of all idempotents of $\mathcal{M}$ is denoted $E(\mathcal{M})$, and $\forall a \in \mathcal{M}$, $aa^{-1}, a^{-1}a \in E(\mathcal{M})$ and are generally different. In inverse monoids idempotents commute and form a subsemilattice. The partial order induced by this semilattice extends naturally to the whole inverse monoid: $s \leq t \Leftrightarrow \exists$ an idempotent e such that $s = te$. This is called the *natural partial order*.

2.3 PSym(X)

The archetypal inverse monoid is the *symmetric inverse monoid* on a set X, denoted $PSym(X)$, and defined as follows: The underlying set of $PSym(X)$ is the set of all bijections between subsets of X, including the empty set. The elements of $PSym(X)$ are called *partial permutations* of X. If $\varphi \colon Y \to Z \in PSym(X)$ then Y and Z are the *domain* and *range* of φ denoted $dom\varphi$ and $ran\varphi$, respectively. The common size $|dom\varphi| = |ran\varphi|$ of the sets $dom\varphi$ and $ran\varphi$ is called the *rank* of φ. The cycle notation of classical permutations generalizes by the addition of a notion called a path, which (unlike a cycle) ends when it reaches the "undefined" element: $dom(x_1, x_2 ... x_k] = \{x_1, x_2, ..., x_{k-1}\}$ and $ran(x_1, x_2 ... x_k] = \{x_2, x_3, ..., x_k\}$.

The operation on $PSym(X)$ is the usual composition of partial maps defined for a pair of partial permutations $\varphi_1 \colon Y_1 \to Z_1$ and $\varphi_2 \colon Y_2 \to Z_2$ to be the partial permutation $\varphi_2\varphi_1 \colon \varphi_1^{-1}(Z_1 \cap Y_2) \to \varphi_2(Z_1 \cap Y_2)$ where $(\varphi_2\varphi_1)(x) = \varphi_2(\varphi_1(x))$ for any $x \in \varphi_1^{-1}(Z_1 \cap Y_2)$. For every $\varphi \in PSym(X)$, the inverse in $PSym(X)$ is just the usual inverse φ^{-1} of the bijection $\varphi \colon dom\varphi \to ran\varphi$. The identity element of $PSym(X)$ is the identity map id_X on X, and $PSym(X)$ also has a zero element, the empty map $id_\emptyset$. Other noteworthy elements of $PAut(\Gamma)$ are so called *local identities* id_A, for $A \subset X$. Local identities are idempotents of $PAut(\Gamma)$ and the natural partial order is defined by restriction of domains. It is clear that if Γ is a graph then $PAut(\Gamma)$ is an inverse submonoid of $PSym(V(\Gamma))$, $PAut(\Gamma) \leq PSym(V)$.

It is clear that all restrictions of a total automorphism of a graph are partial automorphisms. But not all partial automorphisms extend to a total automorphism. For further details on Inverse Monoids see [10].

3 Inverse monoids of partial graph automorphisms

We have characterization of those inverse monoids that appear as Partial Automorphism Monoids of graphs [7]. In this section we review these results. For related concepts and details of proofs see [7].

In our study, we address the question of structure of such monoids as well as computational aspects of finding them and two closely related classification problems:

1. Classify finite inverse monoids that are *isomorphic* to inverse monoids of partial automorphisms of a graph

2. For a specific class of representations of finite inverse semigroups (e.g., those given by Wagner-Preston theorem) classify finite inverse semigroups that admit a graph for which the inverse semigroup of partial automorphisms is *equal to* the partial bijections from the representation.

The first classification question is an analogue of Frucht's Theorem (Theorem 2.1) for groups. Here the situation is very different, as no finite graph on at least two vertices admits a trivial inverse monoid of partial automorphisms. Indeed, partial identical maps are always partial automorphisms, and these already account for exponentially more elements than the number of vertices of the graph. In addition, as we will see, there are usually many more partial automorphisms. It turns out that the class of finite inverse monoids arising as partial automorphism monoids of graphs is quite restrictive. This is in contrast to the result of Frucht (Theorem 2.1). There were several attempts to establish Frucht type of results in the setting of inverse semigroups, by further restricting partial automorphisms of graphs, see for instance [13, 14].

The second classification question is an analogous to the more specialized problem from the group theory of the classification of the finite groups that admit so called a Graphical Regular Representation (GRR) - see for instance [16, 6]. The *Graphical Regular Representation Problem* (the GRR problem) asks for the classification of finite groups G that admit the existence of an edge set E with the property that the full automorphism group of the graph (G, E) acts *regularly* on G. Such groups are said to *admit a GRR* and include almost all finite groups with the exception of abelian groups of exponent at least 3, generalized dicyclic groups, and thirteen sporadic groups.

To address the structural questions for $PAut(\Gamma)$, for a graph $\Gamma(V, E)$, recall that in $PSym(X)$ idempotents are the partial identical maps on subsets of X and the natural partial order is defined by restriction of domains. Moreover, $PAut(\Gamma)$ of a graph Γ is a *full* (i.e. contains all idempotents) submonoid of $PSym(V)$.

To study structure of inverse monoids, in general, five (the two of which coincide in the finite case we work with) equivalence relations, called *Green's relations* are crucial. For $s, t \in \mathcal{M}$, we define $\mathcal{L}$ and $\mathcal{R}$:

$s\,\mathcal{L}\,t \Leftrightarrow \exists\ x, y \in \mathcal{M}$ s.t. $xs = t\ \&\ yt = s$,
$s\,\mathcal{R}\,t \Leftrightarrow \exists\ x, y \in \mathcal{M}$ s.t. $sx = t\ \&\ ty = s$.

In $PSym(X)$ these translate to pleasant and easy to understand relations on domains and ranges of partial permutations:

$\varphi_1\,\mathcal{L}\,\varphi_2 \Leftrightarrow dom\varphi_1 = dom\varphi_2$,
$\varphi_1\,\mathcal{R}\,\varphi_2 \Leftrightarrow ran\varphi_1 = ran\varphi_2$.

The next Green's relation is $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, which in a symmetric inverse monoid $PSym(X)$ translates to $\varphi_1\ \mathcal{H}\ \varphi_2$ if and only if $dom\varphi_1 = dom\varphi_2$ and $ran\varphi_1 = ran\varphi_2$. Each $\mathcal{R}$-class and each $\mathcal{L}$-class contain precisely one idempotent, and the $\mathcal{H}$-classes containing these idempotents are the maximal subgroups of the inverse monoid.

The last relation is $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, in a symmetric inverse monoid means that $\varphi_1\,\mathcal{D}\,\varphi_2$ if and only φ_1 and φ_2 have the same rank.

However, for a inverse monoid of partial automorphisms of a graph even finer distinction is needed ([7]):

Proposition 3.1. *For any graph Γ, the $\mathcal{D}$-classes of $PAut(\Gamma)$ correspond to the isomorphism classes of induced subgraphs of Γ, that is, two elements are $\mathcal{D}$-related if and only if the subgraphs induced by their respective domains are isomorphic.*

This means, that the partial order for $\mathcal{D}$-classes corresponds to induced subgraph relation. This fact is used havily when finding inverse monoids of partial automorphisms for particular graphs. It also indicates that computationally this problem is hard, as we have to go through all induced subgraphs of a given graph.

We are now ready to answer questions from the beginning of this section. The Theorem 3.2 answer the question (analouge of GRR) when is an inverse monoid of partial permutations the partial automorphism monoid of a graph. For details and proof see[7]

Theorem 3.2. *Given an inverse submonoid $S \leq PSym(X)$, where X is a finite set, there exists a graph with vertex set X whose partial automorphism monoid is S if and only if the following conditions hold:*

1. *S is a full inverse submonoid of $PSym(X)$,*
2. *for any compatible subset $A \subseteq S$ of rank 1 partial permutations, if S contains the join of any two elements of A, then S contains the join of the set A,*
3. *the rank 2 elements of S form at most two D-classes,*
4. *the $\mathcal{H}$-classes of rank 2 elements are nontrivial.*

In Theorem 3.3 we give the classification of those (abstract) inverse monoids that are *isomorphic* to the partial automorphism monoids of a finite graph. The transition between the partial permutation representation case and the abstract case is provided by a slightly altered version of the Munn representation for inverse monoids.

Theorem 3.3. *Given a finite inverse monoid S, there exists a finite graph whose partial automorphism monoid is isomorphic to S if and only if the following conditions hold:*

1. *S is Boolean,*

2. *S is fundamental,*

3. *for any subset $A \subseteq S$ of compatible 0-minimal elements, if all 2-element subsets of A have a join in S, then the set A has a join in S,*

4. *the 0-minimal elements of S are $\mathcal{D}$-equivalent,*

5. *S has at most two $\mathcal{D}$-classes of height 2,*

6. *the $\mathcal{H}$-classes of the height 2 $\mathcal{D}$-classes of S are nontrivial.*

The above presented results are of theoretical nature, and therefore it is interesting to ask how difficult it is to find or compute $PAut(\Gamma)$ for a given graph Γ. Or how difficult it is to decide when is a given abstract monoid isomorphic to the partial automorphism monoids of some graph. We do not have definite answers, but have some preliminary results based on experiments we were running as a part of our project.

To answer the last question for a given abstract monoid, one must check, if all the conditions of the Theorem 3.3 hold. Here it is important how the monoid is presented or given. It is not clear whether there is an effective algorithm to verify these conditions for a fixed presentation or even if all the conditions are decidable.

If the $\mathcal{D}$-structure of a inverse monoid satisfying conditions of Theorem3.3 is known, constructing a graph (up to complements) is easily done from $\mathcal{D}$-classes of height 2, corresponding to edges and non-edges of the graph.

We know that $PAut(\Gamma)$ of a graph Γ is a rich and complex structure, much more complex then classical automorphism group $Aut(\Gamma)$. As we mentioned above just number of local identities in $PAut(\Gamma)$ is exponential to order of the graph. We now have catalogues of explicitly described monoids for certain (easy) classes of graphs, like cycles, trees, etc. and for many small graphs [12]. Note that computing $PAut(\Gamma)$ for a graph Γ entails computing automorphism groups repeatedly in several stages. As a final top $\mathcal{D}$-class the classical $Aut(\Gamma)$ must be found, as well as automorphism groups of all induced subgraphs. We know that, in general, constructing the automorphism group is at least as difficult as solving the graph isomorphism problem. Of course, for many classes of graphs it can be done effectively.

Just comparing the number of elements in $PAut(\Gamma)$, we can use some results about $PSym(V)$, [4]. We consider an upper bound, but we know that some graphs, namely complete or empty graphs on n vertices have $PAut(\Gamma) = PSym(\{1, 2, \dots n\})$, and there are $|PSym(\{1, 2, \dots n\})| = \Sigma_{i=0}^{n} \binom{n}{i}^2 i!$ elements in $PAut(\Gamma)$.

Another very interesting family of graphs for which we do have the catalogue [2] of inverse monoids of partial automorphisms is the family of *minimal asymmetric graphs* (Figure 3). This family of graphs was just quite recently completely characterized by P. Schweitzer and P. Schweitzer in [15]. The graphs are asymmetric, so their $Aut(\Gamma)$ are trivial and classical group theory approach using automorphism groups does not distinguish among them, but their $PAut(\Gamma)$ are very interesting and suprisingly rich. This graphs are minimal with respect to asymmetry, i.e. non of their induced subgraphs on at least two vertices is asymmetric, which gives their $PAut(\Gamma)$ interesting structure.

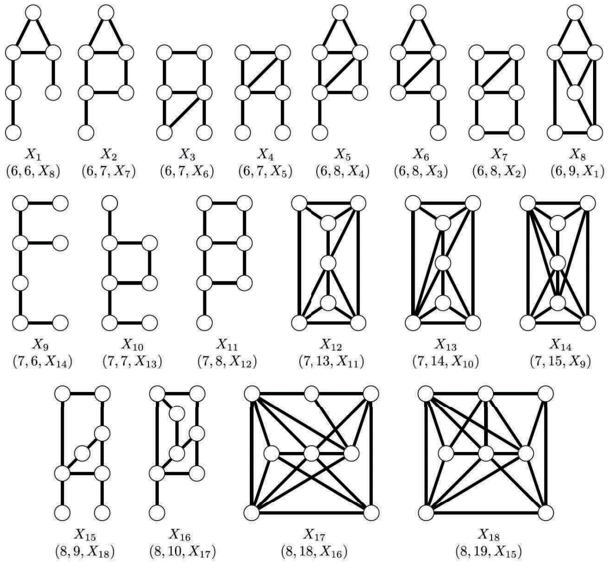

Figure 3: Minimal asymmetric graphs

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