R. Thompson's group F and its group algebras

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We have continued to study on group algebras of R. Thompson's group F. We would like to know whether these algebras satisfy the Ore condition or not. This question is directly connected to a well-known problem on F called the amenability problem. Unfortunately we have not been able to reached the answer. On the other hand, recently some research papers on this problem have been published. In this note, we first see a brief introduction for the amenability problem on Thompson's group F and then we consider Guba's recent results ([4], [6] and [5]) on this problem. Finally, we introduce our approach and the current progress.

1 Introduction

An amenable group is a group whose subsets admit an invariant finitely additive probability measure. That is, a group G is amenable if for the power set $\mathcal{P}(G)$ of G, there exists $\mu : P(G) \longrightarrow [0, 1]$ such that $\mu(G) = 1$, if S and T are disjoint subsets of G then $\mu(S \cup T) = \mu(S) + \mu(T)$, and if $S \in P(G)$ and $g \in G$ then $\mu(gS) = \mu(S)$. Amenable groups was introduced by von Neumann in 1929 in response to the Banach-Tarski paradox. Many equivalent conditions for amenability are now known.

Neumann himself showed that finite groups and abelian groups, more generally, compact groups and soluble groups are amenable. Moreover, he showed also that a group which contains a non-abelian free subgroup (i.e., a subgroup which is a non-abelian free group) is non-amenable. On the other hand, it was very difficult to find a non-amenable group which has no non-abelian free subgroups. In such a situation, a conjecture was stated: a non-amenable group always contains a non-abelian free subgroup. It is called the von Neumann conjecture (it is said that von Neumann's name was apparently attached to it by Day in the 1950s).

First counter example for von Neumann conjecture was given in 1980 by Ol'shanskiĭ [9], and then Adian showed that free Burnside groups of large exponent are also counter examples. In 2003, Ol'shanskiĭ and Sapir [10] gave the first finitely presented example.

Now, it is known that R. Thompson's group F does not contain a free subgroup, but it is a torsion free group unlike groups seen just above. In 1979, Geoghegan conjectured that R. Thompson's group F is not amenable. Since then, it has been the focus of a large amount of subsequent research, but this problem is still open.

There is a well-known manuscript entitled "Thompson's group at 40 years. Preliminary problem list" by Guba [3], in which he announced that if Thompson's group F is amenable then a group algebra KF of F over a field K is an Ore domain, and asked whether the converse is true or not. In 2019, Kielak gave a positive solution for this Guba's question.

Recently, motivated by Kielak's result, Guba has studied on amenability of F again and submitted several papers. In the next section, we will see Guba's recent results and consider what they suggest. After that, we will introduce our approach and the current progress.

2 Recent progress on the problem

Thompson's group F is the group (under composition) of those homeomorphisms of the interval $[0, 1]$, which are piecewise linear and orientation-preserving, in the pieces where the maps are linear, the slope is always a power of 2, and the breakpoints are dyadic. Thompson's groups

$F \subseteq T \subseteq V$ were originally defined by Richard Thompson in 1965 to construct finitely-presented groups with unsolvable word problems [7]. All of F, T and V are finitely generated non-noetherian groups. T and V are simple groups but F is not so. We refer the reader to Cannon, Floyd and Parry [2] for a more detailed discussion of the Thompson's groups F, T and V.

Now, it is known that F is torsion free, finitely presented and orderable. On the other hand, it has no non-abelian free subgroups unlike T and V, which leads to our question of whether F is amenable or not.

In 2019, Kielak [1] proved the next beautiful theorem:

Theorem 2.1. (Tamari [11], 1954, Kielak [1], 2019) *Let G be a group and K a field. Suppose that the group algebra KG is a domain. Then G is amenable if and only if KG is an Ore domain.*

Recall, in the above, that a domain (i.e., it is a ring with no nonzero divisors) R is a (right) Ore domain provided that for each $A, B \in R$ with $B \neq 0$, there exist $X, Y \in R$ with $Y \neq 0$ such that $AY = BX$. As is well known, if R is a (right) Ore domain then R has the (right) classical ring of quotients which is a division ring (a noncommutative field).

In the above theorem, the necessity for amenability has been well known for a long time by V. S. Guba's manuscript [3], in which Guba asked whether the converse was true or not. Theorem 2.1 gave a positive answer for his question. Since Thompson's group F is orderable, the group algebra KF is a domain for any field K, and so Theorem 2.1 translates amenability of F into the Ore condition of KF.

Now, F has the following presentation:

$$F = \langle x_0, x_1, x_2, \cdots x_n, \cdots, \mid x_i^{-1} x_j x_i = x_{j+1}, \text{ for } i < j \rangle.$$

For the above presentation, every non-trivial element of F can be expressed in unique normal form

$$x_0^{\beta_0} x_1^{\beta_0} \cdots x_n^{\beta_n} x_n^{-\alpha_n} \cdots x_1^{-\alpha_1} x_0^{-\alpha_0},$$

where n, $\alpha_0, \ldots, \alpha_n, \beta_0, \cdots, \beta_n$ are non-negative integers such that

1. exactly one of α_n and β_n is non-zero and
2. if $\alpha_k > 0$ and $\beta_k > 0$ for some integer k with $0 \leq k < n$, then $\alpha_{k+1} > 0$ or $\beta_{k+1} > 0$.

Let M be a positive monoid of $F = \langle x_0, x_1, \cdots \mid x_i^{-1} x_j x_i = x_{j+1}, \text{ for } i < j \rangle$ and KM the monoid ring of M over a field K. In [5], Guba first shows that KF satisfies Ore condition if and only if so does KM, and then it is shown that KM satisfies Ore condition, provided for any homogeneous elements $A, B \in KM$ of same degree, there exist $U, V \in KM$ such that $AU = BV$. After that, he gives a partial solution to equations appeared in Ore condition for KM:

Theorem 2.2. (Guba [5], 2021, [6], 2022) *Let $F = \langle x_0, x_1, \cdots \mid x_i^{-1} x_j x_i = x_{j+1}, \text{ for } i < j \rangle$, and let M be the positive monoid M of F, K a field and KM (resp. KF) the monoid algebra of M (resp. the group algebra of F) over K. Then the following* (1), (2) *and* (3) *hold.*

(1) *If $A = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_m x_m$ and $B = \beta_0 x_0 + \beta_1 x_1 + \cdots + \beta_m x_m$ ($\alpha_i, \beta_i \in K$), then*

$$AU = BU \text{ holds for some } U, V \in KM \setminus \{0\}.$$

(2) *If $A = \alpha_{00} x_0^2 + \alpha_{01} x_0 x_1 + \alpha_{02} x_0 x_2 + \alpha_{11} x_1^2 + \alpha_{12} x_1 x_2$ and $B = \beta_{00} x_0^2 + \beta_{01} x_0 x_1 + \beta_{02} x_0 x_2 + \beta_{11} x_1^2 + \beta_{12} x_1 x_2$ ($\alpha_{ij}, \beta_{ij} \in K$), then*

$$AU = BU \text{ holds for some } U, V \in KM \setminus \{0\}.$$

(3) *If $A = (1 - x_0)$ then for any $B \in KF \setminus \{0\}$, $AU = BV$ holds for some U, V in $KF \setminus \{0\}$.*

Guba's results above suggest that if we would like to find elements A, B in KF which fail to satisfy the Ore condition, these elements need to be of somewhat large degree.

3 Find elements not satisfying the Ore condition

Let KF be the group algebra of F over a field K. For elements A, B in $KF^* = KF \setminus \{0\}$, we consider the following condition $(\mathrm{O})_{A,B}$:

$$(\mathrm{O})_{A,B} \quad \text{There exist } U, V \in KF^* \text{such that } AU = BV.$$

By Theorem 2.1, if any two elements A and B in KF^* satisfy the condition $(\mathrm{O})_{A,B}$, then F is amenable. Actually, we would like to show non-amenability of F. Hence, in order to do it, we should just find elements A and B which fail to satisfy the condition $(\mathrm{O})_{A,B}$.

Let $A = a_1 + \cdots + a_m$ and $B = b_1 + \cdots + b_n$ be given nonzero elements in KF^*, where $a_i, b_i \in F$ such that $a_i \neq a_j$ and $b_i \neq b_j$ if $i \neq j$. Suppose, to the contrary, that these two elements satisfy the condition $(\mathrm{O})_{A,B}$. That is,

$$(a_1 + \cdots + a_m)U = b_1 + \cdots + b_n V,$$

for some $U = \gamma_1 u_1 + \cdots + \gamma_s u_s$ and $V = \delta_1 u_1 + \cdots + \delta_t u_t$, where $\gamma, \delta \in K \setminus \{0\}$, $u_i \in Supp(U)$ and $v_i \in Supp(V)$. We have thus

$$\sum_{i,j} \alpha_i \gamma_j a_i u_j - \sum_{i,j} \beta_i \delta_j b_i v_j = 0.$$

The above equation means that $a_i u_j$'s and $b_i v_j$'s are canceled each other because of $\alpha_i \gamma_j \neq 0$ and $\beta_i \delta_j \neq 0$, and so for each i, j, there exist p, q such that

$$(\mathrm{P}) \begin{cases} a_i u_j = a_p u_q \quad (i \neq p, j \neq q) \quad \text{or} \quad a_i u_j = b_p v_q, \\ \\ b_i v_j = b_p v_q \quad (i \neq p, j \neq q) \quad \text{or} \quad b_i v_j = a_p u_q. \end{cases}$$

Let $V = \{(a, i, j), (b, p, q) \mid 1 \leq i \leq m, 1 \leq j \leq s, 1 \leq p \leq n, 1 \leq q \leq t\}$, $\rho(a, i, j) = a_i u_j$ and $\rho(b, p, q) = b_p v_q$. We consider here a two edge coloured graph on V whose two edge sets E and F as follows:

$$\begin{aligned} E = & \ \{vw \mid v, w \in V, v \neq w, \rho v = \rho w\} \\ F = & \ \{(v, w) \mid v = (a, i, j), w = (a, i+1, j) \text{ and } v = (b, p, q), \\ & \qquad\qquad w = (a, p+1, q) \text{ with } m+1 = 1, n+1 = 1\}, \end{aligned}$$

where E is undirected and F is directed.

We call this triple (V, E, F) a DSR-graph (see [8] for details). A cycle in an DSR-graph (V, E, F) is called an DSR-cycle if its edges belong alternatively to E and F; more formally, we call cycle (V', E') an DSR-cycle if there is labeling $V' = \{v_1, v_2, \ldots, v_c\}$ and $E' = \{v_1v_2, v_2v_3, \ldots, v_{2m-1}v_{2m}, v_{2m}v_1\}$ so that $v_{2i-1}v_{2i} \in E$ and $(v_{2i}, v_{2i+1}) \in F$.

By elementary graphical consideration, we can see that the property (P) implies the corresponding DSR-graph has a DSR-cycle, which implies an equation $w_1 \cdots w_k = 1$ in F, where w_i in $\mathcal{W}(a_i, b_j | m, n) = \{a_i a_{i+1}^{-1}, b_j b_{j+1}^{-1} \mid 1 \leq i \leq m, 1 \leq j \leq n, m+1 = n+1 = 1\}$ and they satisfy the following condition:

$$(\mathrm{Q}) \begin{cases} w_i = a_j a_{j+1}^{-1} \Longrightarrow w_{i+1} \neq a_{j+1} a_{j+2}^{-1}, \\ \\ w_i = b_j b_{j+1}^{-1} \Longrightarrow w_{i+1} \neq b_{j+1} b_{j+2}^{-1}. \end{cases}$$

We have thus seen that in order to find elements A and B of KF which fail to satisfy condition $(\mathrm{O})_{A,B}$, we have only to find elements $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ in F^* such that for any finite number of elements $w_1, \ldots . w_k \in \mathcal{W}(a_i, b_j | m, n)$, $w_1 \cdots w_k \neq 1$ under the condition (Q). That is,

Proposition 3.1. *If there exists elements $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ in F^* such that for each finite number of elements $w_1, \ldots . w_k \in \mathcal{W}(a_i, b_j | m, n)$, $w_1 \cdots w_k \neq 1$ under the condition* (Q)*, then F is non-amenable.*

When we try to find elements satisfy the condition in Proposition 3.1, we should search ones of somewhat large degree because of Theorem

2.2. We need also to know properties of chosen elements a_i's and b_i's to confirm that those elements satisfy the property needed in Proposition 3.1.

For example, if we choose elements

$$\begin{array}{lll} a_1 = x_0^4 x_{11}^{12}, & a_2 = x_0^8 x_{11}^{-8}, & a_3 = x_{10}^{24}, \\ b_1 = x_0^{10} x_{11}^{8}, & b_2 = x_0^{14} x_{11}^{-12}, & b_3 = x_0^2 x_{10}^{24}, \end{array}$$

then $\mathcal{W} = \mathcal{W}(a_i, b_j|3,3) = \{a_1a_2^{-1}, a_2a_3^{-1}, a_3a_1^{-1}, b_1b_2^{-1}, b_2b_3^{-1}, b_3b_1^{-1}\}$,

$$\begin{array}{lll} a_1a_2^{-1} = x_7^{20}, & a_2a_3^{-1} = x_0^8 x_{11}^{-8} x_{12}^{-24}, & a_3a_1^{-1} = x_{10}^{24} x_{11}^{-12} x_0^{-4}, \\ b_1b_2^{-1} = x_1^{20} x_0^{-4}, & b_2b_3^{-1} = x_0^{12} x_9^{-12} x_b^{-24}, & b_3b_1^{-1} = x_8^{24} x_9^{-8} x_0^{-8}, \end{array}$$

and elements in $\mathcal{W}$ satisfy the following properties. If $a_1a_2^{-1}$ is annihilated, then there exists $w_1, \ldots, w_k, w'_1, \ldots, w'_l \in \mathcal{W}$ $(k, l \geq 3)$ such that

$$(a_1a_2^{-1})w_1 \cdots w_k(a_2a_3^{-1})w'_1 \cdots w'_l(a_3a_1^{-1}).$$

For any $a_ia_{i+1}^{-1}$ and $b_ib_{i+1}^{-1}$ in $\mathcal{W}$, we can see that all of those elements have similar property.

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