Whittaker functions on $\mathrm{GL}(4,\mathbb{R})$ and archimedean zeta integrals

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1 Introduction

Let $F = \mathbb{R}$ or $\mathbb{C}$. Explicit formulas of Whittaker functions on $\mathrm{GL}(2,F)$ and $\mathrm{GL}(3,F)$ and related archimedean zeta integrals have been studied by several authors (cf. [2]). In this note we give explicit formulas of Whittaker functions on $G = \mathrm{GL}(4,\mathbb{R})$ and its application to archimedean zeta integrals. Since the case of principal series are already done in [4], we consider the remaining cases –$P_{(2,1,1)}$ and $P_{(2,2)}$-principal series. As in our previous work [2], we derive a system of partial differential equations satisfied by Whittaker functions, and give Mellin-Barnes integral representations of moderate growth solutions. By using these explicit formulas, we compute the archimedean parts of Bump-Friedberg zeta integrals ([1]) to give test vectors.

2 Basic notation

We define subgroups N, A and K of $G = \mathrm{GL}(4,\mathbb{R})$ by

$$\begin{aligned} N &= \{x = (x_{i,j}) \in G \mid x_{i,i} = 1 \ (1 \le i \le 4),\ x_{j,k} = 0 \ (1 \le k < j \le 4)\}, \\ A &= \{y = \mathrm{diag}(y_1y_2y_3y_4,\ y_2y_3y_4,\ y_3y_4,\ y_4) \mid y_1, y_2, y_3, y_4 > 0\}, \\ K &= \mathrm{O}(4). \end{aligned}$$

Then we have an Iwasawa decomposition $G = NAK$. Let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ be the associated Lie algebras of G, N, A and K, respectively. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form, that is, $\mathfrak{p} = \{X \in \mathfrak{g} \mid {}^tX = X\}$. For $1 \le i, j \le 4$, the symbol $E_{i,j}$ denotes the matrix in $\mathfrak{g}$ with 1 at the (i,j)-th entry and 0 at other entries. We set $E^{\mathfrak{k}}_{i,j} = E_{i,j} - E_{j,i}$ and $E^{\mathfrak{p}}_{i,j} = E_{i,j} + E_{j,i}$ $(1 \le i, j \le 4)$. Then $\{E_{i,j} \mid 1 \le i, j \le 4\}$, $\{E^{\mathfrak{k}}_{i,j} \mid 1 \le i < j \le 4\}$ and $\{E^{\mathfrak{p}}_{i,j} \mid 1 \le i \le j \le 4\}$ are bases of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$, respectively.

Let $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$, and $U(\mathfrak{g}_\mathbb{C})$ the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. We define a matrix $\mathcal{E} = (\mathcal{E}_{i,j})_{1\le i,j\le 4}$ of size 4 with entries in $U(\mathfrak{g}_\mathbb{C})$ by

$$\mathcal{E}_{i,j} = \begin{cases} E_{i,i} - \frac{5-2i}{2} & \text{if } i=j, \\ E_{i,j} & \text{if } i \neq j. \end{cases}$$

We define the Capelli elements $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ by the identity

$$\mathrm{Det}(t1_4 + \mathcal{E}) = t^4 + \mathcal{C}_1 t^3 + \mathcal{C}_2 t^2 + \mathcal{C}_3 t + \mathcal{C}_4$$

in a variable t. Here Det means the vertical determinant defined by

$$\mathrm{Det}(X) = \sum_{w\in\mathfrak{S}_4} \mathrm{sgn}(w) X_{1,w(1)} X_{2,w(2)} X_{3,w(3)} X_{4,w(4)}, \qquad X = (X_{i,j})_{1\le i,j\le 4}$$

with the symmetric group $\mathfrak{S}_4$ of degree 4. It is known that the Capelli elements $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ generate the center $Z(\mathfrak{g}_\mathbb{C})$ of $U(\mathfrak{g}_\mathbb{C})$ as a $\mathbb{C}$-algebra.

We define a character $\psi_{(c_1,c_2,c_3)}$ of N by

$$\psi_{(c_1,c_2,c_3)}(x) = \exp\{2\pi\sqrt{-1}(c_1x_{1,2} + c_2x_{2,3} + c_3x_{3,4})\} \qquad (x = (x_{i,j}) \in N)$$

for $(c_1,c_2,c_3) \in \mathbb{R}^3$. Then unitary characters of N are exhausted by the characters of this form. We say that $\psi_{(c_1,c_2,c_3)}$ is non-degenerate if $(c_1,c_2,c_3) \in (\mathbb{R}^\times)^3$. For $c \in \mathbb{R}$, we denote $\psi_{(c,c,c)}$ simply by ψ_c.

We regard $C^\infty(G)$ as a G-module via the right translation. For a non-degenerate character ψ of N, let $C^\infty(N\backslash G;\psi)$ be the subspace of $C^\infty(G)$ consisting of all functions f satisfying $f(xg) = \psi(x)f(g)$ $(x\in N,\ g\in G)$. For an admissible representation (Π, H_Π) of G, let

$$\mathcal{I}_{\Pi,\psi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(H_{\Pi,K}, C^\infty(N\backslash G;\psi)_K).$$

Here $H_{\Pi,K}$ and $C^\infty(N\backslash G;\psi)_K$ are the subspaces of H_Π and $C^\infty(N\backslash G;\psi)$ consisting of all K-finite vectors, respectively. We define the subspace $\mathcal{I}_{\Pi,\psi}^{\mathrm{mg}}$ of $\mathcal{I}_{\Pi,\psi}$ consisting of all homomorphisms Φ such that $\Phi(f)$ $(f\in H_{\Pi,K})$ are moderate growth functions. We define the space $\mathrm{Wh}(\Pi,\psi)$ of Whittaker functions for (Π,ψ) by

$$\mathrm{Wh}(\Pi,\psi) = \mathbb{C}\text{-span}\{\Phi(f) \mid f\in H_{\Pi,K},\ \Phi\in\mathcal{I}_{\Pi,\psi}\},$$

and define the subspace $\mathrm{Wh}(\Pi,\psi)^{\mathrm{mg}}$ of $\mathrm{Wh}(\Pi,\psi)$ by

$$\mathrm{Wh}(\Pi,\psi)^{\mathrm{mg}} = \mathbb{C}\text{-span}\{\Phi(f) \mid f\in H_{\Pi,K},\ \Phi\in\mathcal{I}_{\Pi,\psi}^{\mathrm{mg}}\}.$$

Let $\varphi\colon V_\tau \to \mathrm{Wh}(\Pi,\psi)$ be a K-embedding with a K-type (τ, V_τ) of Π. By definition, we have

$$\varphi(v)(xgk) = \psi(x)\varphi(\tau(k)v)(g) \qquad (v\in V_\tau,\ x\in N,\ g\in G,\ k\in K).$$

Because of the Iwasawa decomposition $G = NAK$, φ is characterized by its restriction $v\mapsto \varphi(v)|_A$ to A. We call $v\mapsto\varphi(v)|_A$ the radial part of φ.

Assume that Π is irreducible. Then the multiplicity one theorem tells that the intertwining space $\mathcal{I}^{\mathrm{mg}}_{\Pi,\psi}$ is at most one dimensional. It is known that $\mathcal{I}_{\Pi,\psi} \neq 0$ if and only if Π is large in the sense of Vogan [5].

For $(c_1, c_2, c_3) \in (\mathbb{R}^\times)^3$, there is a G-isomorphism

$$\Xi_{(c_1,c_2,c_3)} \colon C^\infty(N\backslash G; \psi_1) \to C^\infty(N\backslash G; \psi_{(c_1,c_2,c_3)})$$

defined by $\Xi_{(c_1,c_2,c_3)}(f)(g) = f(\mathrm{diag}(c_1c_2c_3,\ c_2c_3,\ c_3,\ 1)g)$ $(g \in G)$. Hence, it suffices to consider the case of ψ_1. In this note, we give explicit formulas of the radial part of a K-embedding $\varphi\colon V_\tau \to \mathrm{Wh}(\Pi,\psi_1)^{\mathrm{mg}}$ for an irreducible admissible large representation Π of G and the minimal K-type (τ, V_τ) of Π.

3 Representation theory of K

Let us briefly explain a way of construction of irreducible representation of K. We define a representation $(\tau_{\mathrm{st}}, V_{\mathrm{st}})$ of K by $V_{\mathrm{st}} = \mathrm{M}_{4,1}(\mathbb{C}) \simeq \mathbb{C}^4$ and $\tau_{\mathrm{st}}(h)v = hv$ $(h \in K,\ v \in V_{\mathrm{st}})$. Here hv is the ordinal product of matrices h and v. For $1 \le i \le 4$, the symbol ξ_i denotes the matrix unit in $V_{\mathrm{st}} = \mathrm{M}_{4,1}(\mathbb{C})$ with 1 at $(i,1)$-th entry and 0 at other entries. For $1 \le i, j \le 4$, we define $\xi_{ij} = \xi_i \wedge \xi_j \in V_{\mathrm{st}} \wedge_{\mathbb{C}} V_{\mathrm{st}}$. We define the graded $\mathbb{C}$-algebra $\mathcal{R} = \bigoplus_{\lambda_1 \ge \lambda_2 \ge 0} \mathcal{R}_{(\lambda_1,\lambda_2)}$ by

$$\mathcal{R} = \mathrm{Sym}(V_{\mathrm{st}}) \otimes_{\mathbb{C}} \mathrm{Sym}(V_{\mathrm{st}} \wedge_{\mathbb{C}} V_{\mathrm{st}}), \qquad \mathcal{R}_\lambda = \mathrm{Sym}^{\lambda_1-\lambda_2}(V_{\mathrm{st}}) \otimes_{\mathbb{C}} \mathrm{Sym}^{\lambda_2}(V_{\mathrm{st}} \wedge_{\mathbb{C}} V_{\mathrm{st}}).$$

Here $\mathrm{Sym}(V) = \bigoplus_{m \ge 0} \mathrm{Sym}^m(V)$ is the symmetric algebra on V with the usual grading for a $\mathbb{C}$-vector space V. We regard $\mathcal{R}$ as a K-module via the action $\mathcal{T}$ which is induced from τ_{st}. Then $\mathcal{R}_{(\lambda_1,\lambda_2)}$ is a K-submodule of $\mathcal{R}$.

For $v \in V_{\mathrm{st}}$ and $v' \in V_{\mathrm{st}} \wedge_{\mathbb{C}} V_{\mathrm{st}}$, the elements $v \otimes 1$ and $1 \otimes v'$ of $\mathcal{R}$ are denoted simply by v and v', respectively. Then we note that $\{\xi_i \mid 1 \le i \le 4\} \cup \{\xi_{jk} \mid 1 \le j < k \le 4\}$ is a system of generators of $\mathcal{R}$ as a $\mathbb{C}$-algebra.

For $1 \le i, j, k \le 4$, we define elements $\widehat{\xi}$, $\widehat{\xi}_i$, $\widehat{\xi}_{ijk}$, $\widehat{\xi}_{ij}$, $\widehat{\xi}_{1234}$ of $\mathcal{R}$ by

$$\begin{aligned}
&\widehat{\xi} = (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 + (\xi_4)^2, \\
&\widehat{\xi}_i = \xi_1\xi_{i1} + \xi_2\xi_{i2} + \xi_3\xi_{i3} + \xi_4\xi_{i4}, && \widehat{\xi}_{ijk} = \xi_i\xi_{jk} - \xi_j\xi_{ik} + \xi_k\xi_{ij}, \\
&\widehat{\xi}_{ij} = \xi_{i1}\xi_{j1} + \xi_{i2}\xi_{j2} + \xi_{i3}\xi_{j3} + \xi_{i4}\xi_{j4}, && \widehat{\xi}_{1234} = \xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23}.
\end{aligned}$$

Let $I_{\mathcal{R}}$ be the ideal of $\mathcal{R}$ generated by

$$\{\widehat{\xi}, \widehat{\xi}_{1234}\} \cup \{\widehat{\xi}_i \mid 1 \le i \le 4\} \cup \{\widehat{\xi}_{ij} \mid 1 \le i \le j \le 4\} \cup \{\widehat{\xi}_{ijk} \mid 1 \le i < j < k \le 4\}.$$

We can show that the ideal $I_{\mathcal{R}}$ is K-invariant. The action of K on $\mathcal{R}/I_{\mathcal{R}}$ induced from $\mathcal{T}$ is denoted by $\widehat{\mathcal{T}}$. Let $\mathrm{q}_{\mathcal{R}} \colon \mathcal{R} \ni r \mapsto r + I_{\mathcal{R}} \in \mathcal{R}/I_{\mathcal{R}}$ be the natural surjection. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^2 \times \{0,1\}$ with $\lambda_1 \ge \lambda_2 \ge 0$. We define a representation $(\tau_\lambda, V_\lambda)$ of K by

$$\tau_\lambda(k) = (\det k)^{\lambda_3}\widehat{\mathcal{T}}(k) \quad (k \in K), \qquad V_\lambda = \mathrm{q}_{\mathcal{R}}(\mathcal{R}_{(\lambda_1,\lambda_2)}).$$

Then we can show the following:

Proposition 1. *Let*

$$\Lambda_K = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^2 \times \{0,1\} \mid \lambda_1 \geq \lambda_2 \geq 0,\ \ \lambda_2\lambda_3 = 0\}.$$

The correspondence $\lambda \leftrightarrow \tau_\lambda$ gives a bijection between Λ_K and the set of equivalence classes of irreducible representations of K.

Let S_λ be the set of $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in (\mathbb{Z}_{\geq 0})^{10}$ satisfying

$$l_1 + l_2 + l_3 + l_4 = \lambda_1 - \lambda_2, \qquad l_{12} + l_{13} + l_{14} + l_{23} + l_{24} + l_{34} = \lambda_2.$$

For $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in S_\lambda$, we set

$$u_l = \mathrm{q}_{\mathcal{R}}\left(\prod_{1 \leq i \leq 4} (\xi_i)^{l_i} \prod_{1 \leq j < k \leq 4} (\xi_{jk})^{l_{jk}}\right).$$

We note that $\{u_l\}_{l \in S_\lambda}$ forms a system of generators of V_λ as a $\mathbb{C}$-vector space. It is convenient to set $u_l = 0$ if $l \notin (\mathbb{Z}_{\geq 0})^{10}$. We set $\mathbf{0} = (0,0,0,0,0,0,0,0,0,0)$ and

$$\begin{aligned}
&e_1 = (1,0,0,0,0,0,0,0,0,0), && e_2 = (0,1,0,0,0,0,0,0,0,0),\\
&e_3 = (0,0,1,0,0,0,0,0,0,0), && e_4 = (0,0,0,1,0,0,0,0,0,0),\\
&e_{12} = e_{21} = (0,0,0,0,1,0,0,0,0,0), && e_{13} = e_{31} = (0,0,0,0,0,1,0,0,0,0),\\
&e_{14} = e_{41} = (0,0,0,0,0,0,1,0,0,0), && e_{23} = e_{32} = (0,0,0,0,0,0,0,1,0,0),\\
&e_{24} = e_{42} = (0,0,0,0,0,0,0,0,1,0), && e_{34} = e_{43} = (0,0,0,0,0,0,0,0,0,1).
\end{aligned}$$

For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda_K$, we have the following relations.

- When $\lambda_1 - \lambda_2 \geq 2$, for $l \in S_{\lambda - (2,0,0)}$, we have

$$u_{l+2e_1} + u_{l+2e_2} + u_{l+2e_3} + u_{l+2e_4} = 0.$$

- When $\lambda_1 > \lambda_2 > 0$, for $l \in S_{\lambda-(2,1,0)}$, we have

$$\begin{aligned}
&\textstyle\sum_{1 \leq j \leq 4,\, j \neq i} \mathrm{sgn}(j-i) u_{l+e_j+e_{ij}} = 0 \quad (1 \leq i \leq 4),\\
&u_{l+e_i+e_{jk}} - u_{l+e_j+e_{ik}} + u_{l+e_k+e_{ij}} = 0 \quad (1 \leq i < j < k \leq 4).
\end{aligned}$$

- When $\lambda_2 \geq 2$, for $l \in S_{\lambda-(2,2,0)}$, we have

$$\begin{aligned}
&\textstyle\sum_{1 \leq k \leq 4,\, k \notin \{i,j\}} \mathrm{sgn}((k-i)(k-j)) u_{l+e_{ik}+e_{jk}} = 0 \quad (1 \leq i, j \leq 4),\\
&u_{l+e_{12}+e_{34}} - u_{l+e_{13}+e_{24}} + u_{l+e_{14}+e_{23}} = 0.
\end{aligned}$$

4 Generalized principal series representations of G

We recall the definition of generalized principal series representations of G and the associated L- and ε-factors. We specify certain representations of $G_1 = \mathrm{GL}(1,\mathbb{R})$ and $G_2 = \mathrm{GL}(2,\mathbb{R})$ as follows:

- For $\nu \in \mathbb{C}$ and $\delta \in \{0,1\}$, we define a character $\chi_{(\nu,\delta)}$ of G_1 by $\chi_{(\nu,\delta)}(t) = \mathrm{sgn}(t)^\delta |t|^\nu$ $(t \in G_1)$.
- For $\nu \in \mathbb{C}$ and $\kappa \in \mathbb{Z}_{\geq 2}$, let $D_{(\nu,\kappa)}$ be an irreducible Hilbert representation of G_2 such that $D_{(\nu,\kappa)}(t1_2) = t^{2\nu}$ $(t \in \mathbb{R}_+)$ and $D_{(\nu,\kappa)} \simeq D_\kappa^+ \oplus D_\kappa^-$ as $(\mathfrak{sl}(2,\mathbb{R}), \mathrm{SO}(2))$-modules, where $D_\kappa^\pm$ is the discrete series representations of $\mathrm{SL}(2,\mathbb{R})$ with the minimal $\mathrm{SO}(2)$-type: $\mathrm{SO}(2) \ni \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{\pm\sqrt{-1}\kappa\theta} \in \mathbb{C}^\times$.

For $\mathbf{n} \in \{(1,1,1,1),(2,1,1),(2,2)\}$, we associate the block upper triangular parabolic subgroup $P_\mathbf{n} = N_\mathbf{n} M_\mathbf{n}$ of G as usual manner. Here $N_\mathbf{n}$ and $M_\mathbf{n}$ are the unipotent radical and the Levi part of $P_\mathbf{n}$, respectively. Because of Vogan's characterization, any irreducible admissible large representation Π of G is infinitesimally equivalent to some $\Pi_\sigma := \mathrm{Ind}_{P_\mathbf{n}}^G(\sigma)$, which is induced from one of the following representations:

- Case 1 ($\mathbf{n} = (1,1,1,1)$):
  $\sigma = \chi_{(\nu_1,\delta_1)} \boxtimes \chi_{(\nu_2,\delta_2)} \boxtimes \chi_{(\nu_3,\delta_3)} \boxtimes \chi_{(\nu_4,\delta_4)}$ where $\nu_1,\nu_2,\nu_3,\nu_4 \in \mathbb{C}$ and $\delta_1,\delta_2,\delta_3,\delta_4 \in \{0,1\}$ with $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4$.
- Case 2 ($\mathbf{n} = (2,1,1)$):
  $\sigma = D_{(\nu_1,\kappa_1)} \boxtimes \chi_{(\nu_2,\delta_2)} \boxtimes \chi_{(\nu_3,\delta_3)}$ where $\nu_1,\nu_2,\nu_3 \in \mathbb{C}$, $\kappa_1 \in \mathbb{Z}_{\geq 2}$ and $\delta_2,\delta_3 \in \{0,1\}$ with $\delta_2 \geq \delta_3$.
- Case 3 ($\mathbf{n} = (2,2)$):
  $\sigma = D_{(\nu_1,\kappa_1)} \boxtimes D_{(\nu_2,\kappa_2)}$ where $\nu_1,\nu_2 \in \mathbb{C}$ and $\kappa_1,\kappa_2 \in \mathbb{Z}_{\geq 2}$ with $\kappa_1 \geq \kappa_2$.

We call Π_σ a generalized principal series representation of G and the representation space of Π_σ is denoted by $H(\sigma)$. To discuss three kinds of the generalized principal series representations simultaneously, we set $\kappa_1,\kappa_2 \in \mathbb{Z}$, $\delta_1,\delta_2,\delta_3 \in \{0,1\}$ and $\nu_1' \in \mathbb{C}$ as follows:

- Case 1: $\kappa_1 := \delta_1 - \delta_4$, $\kappa_2 := \delta_2 - \delta_3$, $\nu_1' := \kappa_1\nu_1 + \kappa_2\nu_4$.
- Case 2: $\delta_1 \equiv \kappa_1 \pmod 2$, $\kappa_2 := \delta_2 - \delta_3$, $\nu_1' := \nu_1$.
- Case 3: $\delta_1 \equiv \kappa_1 \pmod 2$, $\delta_2 \equiv \kappa_2 \pmod 2$, $\delta_3 := 0$, $\nu_1' := \nu_1$.

Then we know $\tau_{(\kappa_1,\kappa_2,\delta_3)}$ is the minimal K-type of Π_σ.

Let us recall the L- and ε-factors of Π_σ. See [2, §5.1, §5.2] for the precise. The equivalence classes of irreducible representations of the Weil group $W_\mathbb{R}$ is exhausted by the characters ϕ_ν^δ $(\nu \in \mathbb{C}, \delta \in \{0,1\})$ and the two dimensional representations $\phi_{\nu,\kappa}$ $(\nu \in \mathbb{C}, \kappa \in \mathbb{Z}_{\geq 1})$. The associated L- and ε- factors are

$$L(s,\phi_\nu^\delta) = \Gamma_\mathbb{R}(s+\nu+\delta), \qquad \epsilon(s,\phi_\nu^\delta,\psi_1) = (\sqrt{-1})^\delta,$$

$$L(s,\phi_{\nu,\kappa}) = \Gamma_{\mathbb{C}}(s+\nu+\tfrac{\kappa}{2}), \qquad \epsilon(s,\phi_{\nu,\kappa},\psi_1) = (\sqrt{-1})^{\kappa+1}.$$

Here we set $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. For a finite dimensional semisimple representation ϕ of $W_{\mathbb{R}}$, we define the corresponding L- and ε-factors by

$$L(s,\phi) = \prod_{i=1}^{m} L(s,\phi_i), \qquad \varepsilon(s,\phi,\psi_1) = \prod_{i=1}^{m} \varepsilon(s,\phi_i,\psi_1),$$

where $\phi \simeq \bigoplus_{i=1}^{m} \phi_i$ is the irreducible decomposition of ϕ. The local Langlands correspondence is a bijection between the set of infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}(n,\mathbb{R})$ and the set of equivalence classes of n-dimensional semisimple representations of $W_{\mathbb{R}}$. For an irreducible admissible representation Π of $\mathrm{GL}(n,\mathbb{R})$, the corresponding representation $\phi[\Pi]$ of $W_{\mathbb{R}}$ is called Langlands parameter of Π. We define the local L-factors $L(s,\Pi)$, $L(s,\Pi,\wedge^2)$ and ε-factors $\varepsilon(s,\Pi,\psi_1)$, $\varepsilon(s,\Pi,\wedge^2,\psi_1)$ by

$$L(s,\Pi) = L(s,\phi[\Pi]), \qquad L(s,\Pi,\wedge^2) = L(s,\wedge^2(\phi[\Pi])),$$
$$\varepsilon(s,\Pi,\psi_1) = \varepsilon(s,\phi[\Pi],\psi_1), \qquad \varepsilon(s,\Pi,\wedge^2,\psi_1) = \varepsilon(s,\wedge^2(\phi[\Pi]),\psi_1).$$

Here $\wedge^2 : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(\frac{n(n-1)}{2},\mathbb{C})$ is the exterior square representation.

Since the Langlands parameter $\phi[\Pi_\sigma]$ of irreducible generalized principal series Π_σ is given by

$$\phi[\Pi_\sigma] = \begin{cases} \phi_{\nu_1}^{\delta_1} \oplus \phi_{\nu_2}^{\delta_2} \oplus \phi_{\nu_3}^{\delta_3} \oplus \phi_{\nu_4}^{\delta_4} & \text{case 1,} \\ \phi_{\nu_1,\kappa_1-1} \oplus \phi_{\nu_2}^{\delta_2} \oplus \phi_{\nu_3}^{\delta_3} & \text{case 2,} \\ \phi_{\nu_1,\kappa_1-1} \oplus \phi_{\nu_2,\kappa_2-1} & \text{case 3,} \end{cases}$$

we know that

$$L(s,\Pi_\sigma) = \begin{cases} \prod_{1\le i\le 4} \Gamma_{\mathbb{R}}(s+\nu_i+\delta_i) & \text{case 1,} \\ \Gamma_{\mathbb{C}}(s+\nu_1+\frac{\kappa_1-1}{2})\Gamma_{\mathbb{R}}(s+\nu_2+\delta_2)\Gamma_{\mathbb{R}}(s+\nu_3+\delta_3) & \text{case 2,} \\ \Gamma_{\mathbb{C}}(s+\nu_1+\frac{\kappa_1-1}{2})\Gamma_{\mathbb{C}}(s+\nu_2+\frac{\kappa_2-1}{2}) & \text{case 3,} \end{cases}$$

$$L(s,\Pi_\sigma,\wedge^2) = \begin{cases} \prod_{1\le i<j\le 4} \Gamma_{\mathbb{R}}(s+\nu_i+\nu_j+|\delta_i-\delta_j|) & \text{case 1,} \\ \Gamma_{\mathbb{C}}(s+\nu_1+\nu_2+\frac{\kappa_1-1}{2})\Gamma_{\mathbb{C}}(s+\nu_1+\nu_3+\frac{\kappa_1-1}{2}) & \\ \quad \times\Gamma_{\mathbb{R}}(s+2\nu_1+\delta_1)\Gamma_{\mathbb{R}}(s+\nu_2+\nu_3+|\delta_2-\delta_3|) & \text{case 2,} \\ \Gamma_{\mathbb{C}}(s+\nu_1+\nu_2+\frac{\kappa_1-\kappa_2}{2})\Gamma_{\mathbb{C}}(s+\nu_1+\nu_2+\frac{\kappa_1+\kappa_2-2}{2}) & \\ \quad \times\Gamma_{\mathbb{R}}(s+2\nu_1+\delta_1)\Gamma_{\mathbb{R}}(s+2\nu_2+\delta_2) & \text{case 3,} \end{cases}$$

$$\varepsilon(s,\Pi_\sigma,\psi) = \begin{cases} (\sqrt{-1})^{\delta_1+\delta_2+\delta_3+\delta_4} & \text{case 1,} \\ (\sqrt{-1})^{\kappa_1+\delta_2+\delta_3} & \text{case 2,} \\ (\sqrt{-1})^{\kappa_1+\kappa_2} & \text{case 3,} \end{cases}$$

$$\varepsilon(s,\Pi_\sigma,\wedge^2,\psi) = \begin{cases} (\sqrt{-1})^{\delta_{1,2}+\delta_{1,3}+\delta_{1,4}+\delta_{2,3}+\delta_{2,4}+\delta_{3,4}} & \text{case 1,} \\ (\sqrt{-1})^{\delta_1+\delta_{2,3}+2\kappa_1} & \text{case 2,} \\ (\sqrt{-1})^{\delta_1+\delta_2+2\kappa_1} & \text{case 3.} \end{cases}$$

Here we define $\delta_{i,j} \in \{0,1\}$ by $\delta_{i,j} \equiv \delta_i + \delta_j \pmod 2$.

5 Differential equations

We give a system of partial differential equations satisfied by Whittaker functions belonging to minimal K-type $\tau_{(\kappa_1,\kappa_2,\delta_3)}$ of Π_σ. For $1 \le i \le 4$, let $S_i(a_1,a_2,a_3,a_4)$ be the i-th elementary symmetric function:

$$S_1(a_1,a_2,a_3,a_4) = \textstyle\sum_{1\le i\le 4} a_i, \qquad S_2(a_1,a_2,a_3,a_4) = \textstyle\sum_{1\le i<j\le 4} a_i a_j,$$
$$S_3(a_1,a_2,a_3,a_4) = \textstyle\sum_{1\le i<j<k\le 4} a_i a_j a_k, \qquad S_4(a_1,a_2,a_3,a_4) = a_1a_2a_3a_4.$$

For $1 \le i \le 4$, we set

$$\gamma_i = \begin{cases} S_i(\nu_1,\nu_2,\nu_3,\nu_4) & \text{case 1,} \\ S_i(\nu_1 + \frac{\kappa_1-1}{2}, \nu_1 - \frac{\kappa_1-1}{2}, \nu_2, \nu_3) & \text{case 2,} \\ S_i(\nu_1 + \frac{\kappa_1-1}{2}, \nu_1 - \frac{\kappa_1-1}{2}, \nu_2 + \frac{\kappa_2-1}{2}, \nu_2 - \frac{\kappa_1-1}{2}) & \text{case 3.} \end{cases}$$

Then we have

$$\Pi_\sigma(\mathcal{C}_i) f = \gamma_i f \quad \text{for } f \in H(\sigma)_K. \tag{5.1}$$

As in [2], we can find that $\mathrm{Hom}_K(V_{(\kappa_1,\kappa_2,\delta_3)}, H(\sigma)_K) = \mathbb{C}\hat{\eta}_\sigma$ and show the following:

- Assume that $\kappa_1 > \kappa_2$. For $l \in S_{(\kappa_1-1,\kappa_2,\delta_3)}$ and $1 \le i \le 4$, we have

$$2\nu_1' \hat{\eta}_\sigma(u_{l+e_i}) = \sum_{k=1}^{4} \Pi_\sigma(E^{\mathfrak{p}}_{i,k}) \hat{\eta}_\sigma(u_{l+e_k}), \tag{5.2}$$

  where $\nu_1' = \begin{cases} (\delta_1-\delta_2)\nu_1 + (\delta_3-\delta_4)\nu_4 & \text{case 1,} \\ \nu_1 & \text{cases 2,3.} \end{cases}$

- Assume that $\kappa_2 \ge 1$. For $l \in S_{(\kappa_1-1,\kappa_2-1,0)}$ and $1 \le i < j \le 4$, we have

$$\begin{aligned} 2(\nu_1+\nu_2)\, \hat{\eta}_\sigma(u_{l+e_{ij}}) &= \Pi_\sigma(E^{\mathfrak{p}}_{i,i} + E^{\mathfrak{p}}_{j,j}) \hat{\eta}_\sigma(u_{l+e_{ij}}) \\ &+ \sum_{1\le k\le 4,\, k\notin\{i,j\}} \{\mathrm{sgn}(j-k)\Pi_\sigma(E^{\mathfrak{p}}_{i,k})\hat{\eta}_\sigma(u_{l+e_{kj}}) + \mathrm{sgn}(k-i)\Pi_\sigma(E^{\mathfrak{p}}_{j,k})\hat{\eta}_\sigma(u_{l+e_{ik}})\}. \end{aligned} \tag{5.3}$$

Let $\varphi : V_{(\kappa_1,\kappa_2,\delta_3)} \to \mathrm{Wh}(\Pi_\sigma, \psi_1)$ be a K-homomorphism. For $l \in S_{(\kappa_1,\kappa_2,\delta_3)}$, we give a system of partial differential equations for $\varphi(u_l)(y)$ ($y = \mathrm{diag}(y_1y_2y_3y_4, y_2y_3y_4, y_3y_4, y_4) \in A$). Let $\partial_i = y_i \frac{\partial}{\partial y_i}$. Since (5.1) with $i=1$ implies that

$$(\partial_4 - \gamma_1)\varphi(u_l)(y) = 0,$$

we can define a function $\hat{\varphi}_l$ on $(\mathbb{R}_+)^3$ by

$$\varphi(u_l)(y) = (\sqrt{-1})^{-l_1+l_3-l_{13}+l_{24}}(-1)^{l_2+l_{14}+l_{23}} y_1^{3/2} y_2^2 y_3^{3/2-\kappa_2} y_4^{\gamma_1} \hat{\varphi}_l(y_1,y_2,y_3)$$

for $l \in S_{(\kappa_1,\kappa_2,\delta_3)}$. From (5.1), (5.2) and (5.3) we get the following system of partial differential equations for $\hat{\varphi}_l$.

Proposition 2. *(i) For $l \in S_{(\kappa_1,\kappa_2,\delta_3)}$, we have*

$$\{\Delta_2 - (2\pi y_1)\mathfrak{K}_{12} - (2\pi y_2)\mathfrak{K}_{23} - (2\pi y_3)\mathfrak{K}_{34}\}\hat{\varphi}_l = 0,$$

$$\begin{aligned}\{\Delta_3 &+ (2\pi y_1)(\partial_2 - \gamma_1)\mathfrak{K}_{12} + (2\pi y_2)(-\partial_1 + \partial_3 - \gamma_1 - \kappa_2)\mathfrak{K}_{23} \\ &- (2\pi y_3)\partial_2\mathfrak{K}_{34} + (2\pi y_1)(2\pi y_2)\mathfrak{K}_{13} + (2\pi y_2)(2\pi y_3)\mathfrak{K}_{24}\}\hat{\varphi}_l = 0\end{aligned}$$

and

$$\begin{aligned}[\Delta_4 &- (2\pi y_1)\{(-\partial_2 + \partial_3 - \kappa_2)(-\partial_3 + \gamma_1 + \kappa_2) + (2\pi y_3)^2\}\mathfrak{K}_{12} \\ &- (2\pi y_2)\partial_1(-\partial_3 + \gamma_1 + \kappa_2)\mathfrak{K}_{23} - (2\pi y_3)\{\partial_1(-\partial_1 + \partial_2) + (2\pi y_1)^2\}\mathfrak{K}_{34} \\ &+ (2\pi y_1)(2\pi y_2)(-\partial_3 + \gamma_1 + \kappa_2)\mathfrak{K}_{13} + (2\pi y_2)(2\pi y_3)\partial_1\mathfrak{K}_{24} \\ &+ (2\pi y_1)(2\pi y_2)(2\pi y_3)\mathfrak{K}_{14} + (2\pi y_1)(2\pi y_3)\mathfrak{K}_{12,34}]\hat{\varphi}_l = 0,\end{aligned}$$

where Δ_2, Δ_3 and Δ_4 are the differential operators defined by

$$\begin{aligned}\Delta_2 = &-\partial_1^2 - \partial_2^2 - (\partial_3 - \kappa_2)^2 + \partial_1\partial_2 + \partial_2(\partial_3 - \kappa_2) + \gamma_1(\partial_3 - \kappa_2) \\ &+ (2\pi y_1)^2 + (2\pi y_2)^2 + (2\pi y_3)^2 - \gamma_2,\end{aligned}$$

$$\begin{aligned}\Delta_3 = &\,\partial_2(\partial_1 - \partial_3 + \kappa_2)(\partial_1 - \partial_2 + \partial_3 - \kappa_2) - \gamma_1(\partial_1^2 + \partial_2^2 - \partial_1\partial_2 - \partial_2(\partial_3 - \kappa_2)) \\ &+ (2\pi y_1)^2(-\partial_2 + \gamma_1) + (2\pi y_2)^2(\partial_1 - \partial_3 + \gamma_1 + \kappa_2) + (2\pi y_3)^2\partial_2 - \gamma_3\end{aligned}$$

and

$$\begin{aligned}\Delta_4 = &\,\partial_1(\partial_2 - \partial_1)(\partial_3 - \partial_2 - \kappa_2)(-\partial_3 + \gamma_1 + \kappa_2) \\ &+ (2\pi y_1)^2(\partial_3 - \partial_2 - \kappa_2)(-\partial_3 + \gamma_1 + \kappa_2) + (2\pi y_2)^2\partial_1(-\partial_3 + \gamma_1 + \kappa_2) \\ &+ (2\pi y_3)^2\partial_1(\partial_2 - \partial_1) + (2\pi y_1)^2(2\pi y_3)^2 - \gamma_4,\end{aligned}$$

respectively. Here

$$\begin{aligned}\mathfrak{K}_{12}\hat{\varphi}_l &= l_1\hat{\varphi}_{l-e_1+e_2} + l_2\hat{\varphi}_{l-e_2+e_1} \\ &\quad + l_{13}\hat{\varphi}_{l-e_{13}+e_{23}} + l_{14}\hat{\varphi}_{l-e_{14}+e_{24}} + l_{23}\hat{\varphi}_{l-e_{23}+e_{13}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{14}}, \\ \mathfrak{K}_{23}\hat{\varphi}_l &= l_2\hat{\varphi}_{l-e_2+e_3} + l_3\hat{\varphi}_{l-e_3+e_2} \\ &\quad + l_{12}\hat{\varphi}_{l-e_{12}+e_{13}} + l_{13}\hat{\varphi}_{l-e_{13}+e_{12}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{34}} + l_{34}\hat{\varphi}_{l-e_{34}+e_{24}}, \\ \mathfrak{K}_{34}\hat{\varphi}_l &= l_3\hat{\varphi}_{l-e_3+e_4} + l_4\hat{\varphi}_{l-e_4+e_3} \\ &\quad + l_{13}\hat{\varphi}_{l-e_{13}+e_{14}} + l_{14}\hat{\varphi}_{l-e_{14}+e_{13}} + l_{23}\hat{\varphi}_{l-e_{23}+e_{24}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{23}}, \\ \mathfrak{K}_{13}\hat{\varphi}_l &= l_1\hat{\varphi}_{l-e_1+e_3} + l_3\hat{\varphi}_{l-e_3+e_1} \\ &\quad - l_{12}\hat{\varphi}_{l-e_{12}+e_{23}} + l_{14}\hat{\varphi}_{l-e_{14}+e_{34}} + l_{23}\hat{\varphi}_{l-e_{23}+e_{12}} - l_{34}\hat{\varphi}_{l-e_{34}+e_{14}}, \\ \mathfrak{K}_{24}\hat{\varphi}_l &= l_2\hat{\varphi}_{l-e_2+e_4} - l_4\hat{\varphi}_{l-e_4+e_2} \\ &\quad + l_{12}\hat{\varphi}_{l-e_{12}+e_{14}} - l_{14}\hat{\varphi}_{l-e_{14}+e_{12}} - l_{23}\hat{\varphi}_{l-e_{23}+e_{34}} + l_{34}\hat{\varphi}_{l-e_{34}+e_{23}}, \\ \mathfrak{K}_{14}\hat{\varphi}_l &= -l_1\hat{\varphi}_{l-e_1+e_4} - l_4\hat{\varphi}_{l-e_4+e_1} \\ &\quad + l_{12}\hat{\varphi}_{l-e_{12}+e_{24}} + l_{13}\hat{\varphi}_{l-e_{13}+e_{34}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{12}} + l_{34}\hat{\varphi}_{l-e_{34}+e_{13}}\end{aligned}$$

and

$$
\begin{aligned}
&\mathfrak{K}_{12,34}\hat{\varphi}_l \\
&= l_1 l_3 \hat{\varphi}_{l-e_1-e_3+e_2+e_4} + l_1 l_4 \hat{\varphi}_{l-e_1-e_4+e_2+e_3} + l_2 l_3 \hat{\varphi}_{l-e_2-e_3+e_1+e_4} + l_2 l_4 \hat{\varphi}_{l-e_2-e_4+e_1+e_3} \\
&+ l_1 (l_{13}\hat{\varphi}_{l-e_1+e_2-e_{13}+e_{14}} + l_{14}\hat{\varphi}_{l-e_1+e_2-e_{14}+e_{13}} + l_{23}\hat{\varphi}_{l-e_1+e_2-e_{23}+e_{24}} + l_{24}\hat{\varphi}_{l-e_1+e_2-e_{24}+e_{23}}) \\
&+ l_2 (l_{13}\hat{\varphi}_{l-e_2+e_1-e_{13}+e_{14}} + l_{14}\hat{\varphi}_{l-e_2+e_{23}-e_{14}+e_{13}} + l_{23}\hat{\varphi}_{l-e_2+e_1-e_{23}+e_{24}} + l_{24}\hat{\varphi}_{l-e_2+e_1-e_{24}+e_{23}}) \\
&+ l_3 (l_{13}\hat{\varphi}_{l-e_3+e_4-e_{13}+e_{23}} + l_{14}\hat{\varphi}_{l-e_3+e_4-e_{14}+e_{24}} + l_{23}\hat{\varphi}_{l-e_3+e_4-e_{23}+e_{13}} + l_{24}\hat{\varphi}_{l-e_3+e_4-e_{24}+e_{14}}) \\
&+ l_4 (l_{13}\hat{\varphi}_{l-e_4+e_3-e_{13}+e_{23}} + l_{14}\hat{\varphi}_{l-e_4+e_3-e_{14}+e_{24}} + l_{23}\hat{\varphi}_{l-e_4+e_3-e_{23}+e_{13}} + l_{24}\hat{\varphi}_{l-e_4+e_3-e_{24}+e_{14}}) \\
&+ l_{13}(l_{14}+l_{23}+1)\hat{\varphi}_{l-e_{13}+e_{24}} + l_{24}(l_{14}+l_{23}+1)\hat{\varphi}_{l-e_{24}+e_{13}} \\
&+ l_{14}(l_{13}+l_{24}+1)\hat{\varphi}_{l-e_{14}+e_{23}} + l_{23}(l_{13}+l_{24}+1)\hat{\varphi}_{l-e_{23}+e_{14}} \\
&+ l_{13}l_{24}(\hat{\varphi}_{l-e_{13}-e_{24}+2e_{14}} + \hat{\varphi}_{l-e_{13}-e_{24}+2e_{23}}) + l_{14}l_{23}(\hat{\varphi}_{l-e_{14}-e_{23}+2e_{13}} + \hat{\varphi}_{l-e_{14}-e_{23}+2e_{24}}) \\
&+ l_{13}(l_{13}-1)\hat{\varphi}_{l-2e_{13}+e_{14}+e_{23}} + l_{24}(l_{24}-1)\hat{\varphi}_{l-2e_{24}+e_{14}+e_{23}} \\
&+ l_{14}(l_{14}-1)\hat{\varphi}_{l-2e_{14}+e_{13}+e_{24}} + l_{23}(l_{23}-1)\hat{\varphi}_{l-2e_{23}+e_{13}+e_{24}}.
\end{aligned}
$$

(ii) Assume that $\kappa_1 > \kappa_2$. For $l \in S_{(\kappa_1-1,\kappa_2,\delta_3)}$, we have

$$(\partial_1 - \nu_1' - \tfrac{\kappa_1-1}{2})\hat{\varphi}_{l+e_1} + 2\pi y_1 \hat{\varphi}_{l+e_2} = 0,$$

$$
\begin{aligned}
&(-\partial_1 + \partial_2 - \nu_1' - \tfrac{\kappa_1-1}{2} + l_1)\hat{\varphi}_{l+e_2} - 2\pi y_1 \hat{\varphi}_{l+e_1} + 2\pi y_2 \hat{\varphi}_{l+e_3} \\
&+ l_2 \hat{\varphi}_{l-e_2+2e_1} + l_{13}\hat{\varphi}_{l+e_1-e_{13}+e_{23}} + l_{14}\hat{\varphi}_{l+e_1-e_{14}+e_{24}} \\
&+ l_{23}\hat{\varphi}_{l+e_1-e_{23}+e_{13}} + l_{24}\hat{\varphi}_{l+e_1-e_{24}+e_{14}} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&(-\partial_2 + \partial_3 - \nu_1' + \tfrac{\kappa_1-1}{2} - \kappa_2 - l_4)\hat{\varphi}_{l+e_3} - 2\pi y_2 \hat{\varphi}_{l+e_2} + 2\pi y_3 \hat{\varphi}_{l+e_4} \\
&- l_3 \hat{\varphi}_{l-e_3+2e_4} - l_{13}\hat{\varphi}_{l+e_4-e_{13}+e_{14}} - l_{14}\hat{\varphi}_{l+e_4-e_{14}+e_{13}} \\
&- l_{23}\hat{\varphi}_{l+e_4-e_{23}+e_{24}} - l_{24}\hat{\varphi}_{l+e_4-e_{24}+e_{23}} = 0,
\end{aligned}
$$

$$(-\partial_3 + \gamma_1 - \nu_1' + \tfrac{\kappa_1-1}{2} + \kappa_2)\hat{\varphi}_{l+e_4} - 2\pi y_3 \hat{\varphi}_{l+e_3} = 0.$$

(iii) Assume that $\kappa_2 \geq 1$. For $l \in S_{(\kappa_1-1,\kappa_2-1,0)}$, we have

$$(\partial_2 - \nu_1 - \nu_2 - \tfrac{\kappa_1+\kappa_2}{2} + 1)\hat{\varphi}_{l+e_{12}} + 2\pi y_2 \hat{\varphi}_{l+e_{13}} = 0,$$

$$
\begin{aligned}
&(\partial_1 - \partial_2 + \partial_3 - \nu_1 - \nu_2 - \tfrac{\kappa_1+\kappa_2}{2} - l_{13} - l_{14} - l_{23} - l_{24})\hat{\varphi}_{l+e_{13}} \\
&+ 2\pi y_1 \hat{\varphi}_{l+e_{23}} - 2\pi y_2 \hat{\varphi}_{l+e_{12}} + 2\pi y_3 \hat{\varphi}_{l+e_{14}} \\
&+ l_2 \hat{\varphi}_{l-e_2+e_3+e_{12}} + l_3 \hat{\varphi}_{l-e_3+e_2+e_{12}} + l_{13}\hat{\varphi}_{l-e_{13}+2e_{12}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{12}+e_{34}} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&(\partial_1 - \partial_3 + \gamma_1 - \nu_1 - \nu_2 - \tfrac{\kappa_1-3\kappa_2}{2} + l_4)\hat{\varphi}_{l+e_{14}} \\
&+ 2\pi y_1 \hat{\varphi}_{l+e_{24}} - 2\pi y_3 \hat{\varphi}_{l+e_{13}} + l_2 \hat{\varphi}_{l-e_2+e_4+e_{12}} + l_3 \hat{\varphi}_{l-e_3+e_4+e_{13}} = 0,
\end{aligned}
$$

$$(-\partial_1+\partial_3-\nu_1-\nu_2+\tfrac{\kappa_1-3\kappa_2}{2}-l_4)\hat{\varphi}_{l+e_{23}}$$
$$-2\pi y_1\hat{\varphi}_{l+e_{13}}+2\pi y_3\hat{\varphi}_{l+e_{24}}-l_2\hat{\varphi}_{l-e_2+e_4+e_{34}}-l_3\hat{\varphi}_{l-e_3+e_4+e_{24}}=0,$$

$$(-\partial_1+\partial_2-\partial_3+\gamma_1-\nu_1-\nu_2+\tfrac{\kappa_1+\kappa_2}{2}+l_{13}+l_{14}+l_{23}+l_{24})\hat{\varphi}_{l+e_{24}}$$
$$-2\pi y_1\hat{\varphi}_{l+e_{14}}+2\pi y_2\hat{\varphi}_{l+e_{34}}-2\pi y_3\hat{\varphi}_{l+e_{23}}$$
$$-l_2\hat{\varphi}_{l-e_2+e_3+e_{34}}-l_3\hat{\varphi}_{l-e_3+e_2+e_{34}}-l_{13}\hat{\varphi}_{l-e_{13}+e_{12}+e_{34}}-l_{24}\hat{\varphi}_{l-e_{24}+2e_{34}}=0,$$

$$(-\partial_2+\gamma_1-\nu_1-\nu_2+\tfrac{\kappa_1+\kappa_2}{2}-1)\hat{\varphi}_{l+e_{34}}-2\pi y_2\hat{\varphi}_{l+e_{24}}=0.$$

6 Explicit formulas of Whittaker functions

We shall give moderate growth solutions of the system above. Since the case 1 is already done in [4], we treat the cases 2 and 3. Our key observation is that the function $\hat{\varphi}_{(0,0,0,l_4,l_{12},0,0,0,0,l_{34})}(y)$ satisfies similar system of partial differential equations as that for the class one Whittaker functions (case 1 with $\delta_1=\delta_2=\delta_3=\delta_4=0$). Then we know Mellin-Barnes integral representation of $\hat{\varphi}_{(0,0,0,l_4,l_{12},0,0,0,0,l_{34})}(y)$ and the other $\hat{\varphi}_l(y)$ can be determined by the equations (ii) and (iii) in Proposition 2.

Here are our main results. In the following, the path of integration is a vertical line in the complex plane, of sufficiently large real part to keep the poles of integrand on its left.

Theorem 3. *(Case 2) Let $\sigma=D_{(\nu_1,\kappa_1)}\boxtimes\chi_{(\nu_2,\delta_2)}\boxtimes\chi_{(\nu_3,\delta_3)}$ with $\nu_1,\nu_2,\nu_3\in\mathbb{C}$, $\kappa_1\in\mathbb{Z}_{\geq 2}$, $\delta_2,\delta_3\in\{0,1\}$ and $\delta_2\geq\delta_3$ such that Π_σ is irreducible. There exists a K-homomorphism $\varphi_\sigma:V_{(\kappa_1,\delta_2-\delta_3,\delta_3)}\to \mathrm{Wh}(\Pi_\sigma,\psi_1)^{\mathrm{mg}}$ whose radial part is given by*

$$\varphi_\sigma(u_l)(y)=y_1^{3/2}y_2^2y_3^{3/2}y_4^{2\nu_1+\nu_2+\nu_3}\cdot(\sqrt{-1})^{-l_1+l_3-l_{13}+l_{24}}(-1)^{l_2+l_{14}+l_{23}}$$
$$\times\frac{1}{(4\pi\sqrt{-1})^3}\int_{s_3}\int_{s_2}\int_{s_1}V_l(s_1,s_2,s_3)\,y_1^{-s_1}y_2^{-s_2}y_3^{-s_3}\,ds_1ds_2ds_3$$

with $l=(l_1,l_2,l_3,l_4,l_{12},l_{13},l_{14},l_{23},l_{24},l_{34})\in S_{(\kappa_1,\delta_2-\delta_3,\delta_3)}$. Here

$$\begin{aligned}
&V_l(s_1,s_2,s_3)\\
&=\Gamma_{\mathbb{C}}(s_1+\nu_1+\tfrac{\kappa_1-1}{2})\Gamma_{\mathbb{R}}(s_2+2\nu_1+l_3+l_4+l_{12}+l_{34})\\
&\times\Gamma_{\mathbb{R}}(s_2+\nu_2+\nu_3+l_1+l_2+l_{12}+l_{34})\Gamma_{\mathbb{C}}(s_3+\nu_1+\nu_2+\nu_3+\tfrac{\kappa_1-1}{2})\\
&\times\sum_{i_{14}=0}^{l_{14}}\sum_{i_{23}=0}^{l_{23}}\sum_{i_{13}=0}^{l_{13}}\sum_{i_{24}=0}^{l_{24}}\\
&\times\frac{1}{4\pi\sqrt{-1}}\int_q\frac{\Gamma_{\mathbb{R}}(s_1-q+l_1+i_{14}+i_{23}+i_{13}+i_{24})\Gamma_{\mathbb{C}}(s_2-q+\nu_1+\frac{\kappa_1-1}{2})}{\Gamma_{\mathbb{R}}(s_1+s_2-q+2\nu_1+l_1+l_3+l_4+l_{12}+l_{34}+i_{14}+i_{23}+i_{13}+i_{24})}\\
&\times\frac{\Gamma_{\mathbb{R}}(s_3-q+2\nu_1+l_4+l_{14}+l_{23}-i_{14}-i_{23}+i_{13}+i_{24})}{\Gamma_{\mathbb{R}}(s_2+s_3-q+2\nu_1+\nu_2+\nu_3+l_1+l_2+l_4+l_{12}+l_{34}+l_{14}+l_{23}-i_{14}-i_{23}+i_{13}+i_{24})}\\
&\times\Gamma_{\mathbb{R}}(q+\nu_2+l_{34}+l_{24}+l_{23}-i_{23}+i_{14}+i_{13}-i_{24})\\
&\times\Gamma_{\mathbb{R}}(q+\nu_3+l_{12}+l_{13}+l_{14}-i_{14}+i_{23}-i_{13}+i_{24})\,dq.
\end{aligned}$$

Theorem 4. *(Case 3) Let $\sigma = D_{(\nu_1,\kappa_1)} \boxtimes D_{(\nu_2,\kappa_2)}$ with $\nu_1, \nu_2 \in \mathbb{C}$, $\kappa_1, \kappa_2 \in \mathbb{Z}_{\geq 2}$ and $\kappa_1 \geq \kappa_2$ such that Π_σ is irreducible. There exists a K-homomorphism $\varphi_\sigma : V_{(\kappa_1,\kappa_2,0)} \to \mathrm{Wh}(\Pi_\sigma, \psi_1)^{\mathrm{mg}}$ whose radial part is given by*

$$\begin{aligned}\varphi_\sigma(u_l)(y) &= y_1^{3/2} y_2^2 y_3^{3/2} y_4^{2\nu_1+2\nu_2} \cdot (\sqrt{-1})^{-l_1+l_3-l_{13}+l_{24}} (-1)^{l_2+l_{14}+l_{23}} \\ &\quad \times \frac{1}{(4\pi\sqrt{-1})^3} \int_{s_3}\int_{s_2}\int_{s_1} V_l(s_1,s_2,s_3)\, y_1^{-s_1} y_2^{-s_2} y_3^{-s_3}\, ds_1 ds_2 ds_3\end{aligned}$$

with $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in S_{(\kappa_1,\kappa_2,0)}$. Here

$$\begin{aligned}&V_l(s_1,s_2,s_3) \\ &= \Gamma_{\mathbb{C}}(s_1+\nu_1+\tfrac{\kappa_1-1}{2})\Gamma_{\mathbb{R}}(s_2+2\nu_1+l_3+l_4+l_{12}+l_{34}) \\ &\times \Gamma_{\mathbb{R}}(s_2+2\nu_2+l_1+l_2+l_{12}+l_{34})\Gamma_{\mathbb{C}}(s_3+\nu_1+2\nu_2+\tfrac{\kappa_1-1}{2}) \\ &\times \sum_{i=0}^{l_{14}+l_{23}} \sum_{j=0}^{l_{13}+l_{24}} \binom{l_{14}+l_{23}}{i}\binom{l_{13}+l_{24}}{j} \\ &\times \frac{1}{4\pi\sqrt{-1}} \int_q \frac{\Gamma_{\mathbb{R}}(s_1-q+l_1+i+j)\Gamma_{\mathbb{C}}(s_2-q+\nu_1+\frac{\kappa_1-1}{2})}{\Gamma_{\mathbb{R}}(s_1+s_2-q+2\nu_1+l_1+l_3+l_4+l_{12}+l_{34}+i+j)} \\ &\times \frac{\Gamma_{\mathbb{R}}(s_3-q+2\nu_1+l_4+l_{14}+l_{23}-i+j)\Gamma_{\mathbb{C}}(q+\nu_2+\frac{\kappa_2-1}{2})}{\Gamma_{\mathbb{R}}(s_2+s_3-q+2\nu_1+2\nu_2+l_1+l_2+l_4+l_{12}+l_{14}+l_{23}+l_{34}-i+j)}\, dq.\end{aligned}$$

7 Archimedean Bump-Friedberg zeta integrals

As an application of our explicit formulas, we evaluate the archimedean part of Bump-Friedberg zeta integral ([1]) which represents the product of the standard and the exterior square L-functions on $\mathrm{GL}(4)$ at the unramified non-archimedean places. Let $G_2 = \mathrm{GL}(2,\mathbb{R})$, $N_2 = \{\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) \mid x \in \mathbb{R}\}$, $K_2 = \mathrm{O}(2)$. We define an embedding $\tilde{\iota} : G_2 \times G_2 \to G$ by

$$\left(g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \mapsto \tilde{\iota}(g_1,g_2) := \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}.$$

Let $\mathcal{S}(\mathbb{R}^2)$ be the space of Schwartz functions on $\mathbb{R}^2$. For $s_1, s_2 \in \mathbb{C}$, $\Phi \in \mathcal{S}(\mathbb{R}^2)$ and $W \in \mathrm{Wh}(\Pi_\sigma, \psi_1)^{\mathrm{mg}}$, we consider the following archimedean zeta integral:

$$\begin{aligned}Z(s_1,s_2,W,\Phi) &= \int_{N_2\backslash G_2}\int_{N_2\backslash G_2} W(\tilde{\iota}(g_1,g_2))\Phi((0,1)g_2) \\ &\qquad \times |\det g_1|^{s_1-\frac{1}{2}} |\det g_2|^{-s_1+s_2+\frac{1}{2}}\, d\dot{g}_1 d\dot{g}_2.\end{aligned}$$

Here $d\dot{g}$ is the right G_2-invariant measure on $N_2\backslash G_2$ normalized so that

$$\int_{N_2\backslash G_2} f(g)\, d\dot{g} = \int_0^\infty\int_0^\infty\int_{K_2} f\big(\mathrm{diag}(y_1y_2, y_2)k\big)\, dk \frac{2dy_1}{y_1^2}\frac{2dy_2}{y_2}$$

for any compactly supported continuous function f on $N_2\backslash G_2$. Here dk is the normalized Haar measure on K_2 such that $\int_{K_2} dk = 1$. By using our explicit formulas we can compute the integral $Z(s_1, s_2, W, \Phi)$ to get the following:

Theorem 5. *For each Π_σ, there exists $(W, \Phi) \in \mathrm{Wh}(\Pi_\sigma, \psi_1)^{\mathrm{mg}} \times \mathcal{S}(\mathbb{R}^2)$ such that*

$$\begin{aligned} Z(s_1, s_2, W, \Phi) &= L(s_1, \Pi_\sigma) L(s_2, \Pi_\sigma, \wedge^2), \\ Z(1-s_1, 1-s_2, \widetilde{W}, \widehat{\Phi}) &= \varepsilon(s_1, \Pi_\sigma, \psi_1)\varepsilon(s_2, \Pi_\sigma, \psi_1, \wedge^2) L(1-s_1, \widetilde{\Pi}_\sigma) L(1-s_2, \widetilde{\Pi}_\sigma, \wedge^2). \end{aligned}$$

Here $\widetilde{W}$ is the contragredient Whittaker function of W defined by $\widetilde{W}(g) = W(\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} {}^t g^{-1})$ and $\widehat{\Phi}$ is the Fourier transform of Φ.

Here are test vectors, that is, pairs of (W, Φ) such that the above identities hold. The symbol R means the right differential. Since the case 1 is done in [3], we treat the cases 2 and 3. We define $\delta_1, \delta_2, \delta_3$ as before, and define $\delta \in \{0, 1\}$ by $\delta \equiv \delta_1 + \delta_2 + \delta_3 \pmod 2$.

- Case 2: $\sigma = D_{(\nu_1,\kappa_1)} \boxtimes \chi_{(\nu_2,\delta_2)} \boxtimes \chi_{(\nu_3,\delta_3)}$ with $\delta_2 \geq \delta_3$. We have

$$\Phi(x_1, x_2) = (\sqrt{-1}x_1 + x_2)^\delta \exp\{-\pi(x_1^2 + x_2^2)\}$$

and

$$W = (-\sqrt{-1})^{\kappa_1} \times \begin{cases} \varphi_\sigma(w_0) & (\delta_1,\delta_2,\delta_3) = (0,0,0), \\ \varphi_\sigma(w_2 + \sqrt{-1}w_4) & (\delta_1,\delta_2,\delta_3) = (1,0,0), \\ (4\pi)^{-1}\{R(E^{\mathfrak{p}}_{1,2})\varphi_\sigma(w_{3,4}) - R(E^{\mathfrak{p}}_{2,3})\varphi_\sigma(w_{1,4}) & \\ \qquad + R(E^{\mathfrak{p}}_{3,4})\varphi_\sigma(w_{1,2}) - R(E^{\mathfrak{p}}_{1,4})\varphi_\sigma(w_{2,3})\} & (\delta_1,\delta_2,\delta_3) = (0,1,1), \\ (4\pi)^{-1}\{R(-E^{\mathfrak{p}}_{3,4} + \sqrt{-1}E^{\mathfrak{p}}_{2,3})\varphi_\sigma(w_1) & \\ \qquad + R(E^{\mathfrak{p}}_{1,4} - \sqrt{-1}E^{\mathfrak{p}}_{1,2})\varphi_\sigma(w_3)\} & (\delta_1,\delta_2,\delta_3) = (1,1,1), \\ \varphi_\sigma(-\sqrt{-1}w_{2,24} + w_{4,24}) & (\delta_1,\delta_2,\delta_3) = (0,1,0), \\ (4\pi)^{-1}\{R(E^{\mathfrak{p}}_{1,2})\varphi_\sigma(w_{12}) - R(E^{\mathfrak{p}}_{2,3})\varphi_\sigma(w_{23}) & \\ \qquad + R(E^{\mathfrak{p}}_{3,4})\varphi_\sigma(w_{34}) + R(E^{\mathfrak{p}}_{1,4})\varphi_\sigma(w_{14})\} & (\delta_1,\delta_2,\delta_3) = (1,1,0). \end{cases}$$

Here we define $w_0, w_p, w_{p,q}, w_{pq}, w_{p,qr} \in V_{(\kappa_1,\delta_2,\delta_3)}$ $(1 \leq p, q, r \leq 4)$ by

$$\begin{gathered} w_0 = \mathrm{q}_{\mathcal{R}}\big((\xi_1^2+\xi_3^2)^{\kappa_1/2}\big), \qquad w_p = \mathrm{q}_{\mathcal{R}}\big((\xi_1^2+\xi_3^2)^{(\kappa_1-1)/2}\xi_p\big), \\ w_{p,q} = \mathrm{q}_{\mathcal{R}}\big((\xi_1^2+\xi_3^2)^{(\kappa_1-2)/2}\xi_p\xi_q\big), \qquad w_{pq} = \mathrm{q}_{\mathcal{R}}\big((\xi_1^2+\xi_3^2)^{(\kappa_1-1)/2}\xi_{pq}\big), \\ w_{p,qr} = \mathrm{q}_{\mathcal{R}}\big((\xi_1^2+\xi_3^2)^{(\kappa_1-2)/2}\xi_p\xi_{qr}\big). \end{gathered}$$

- Case 3: $\sigma = D_{(\nu_1,\kappa_1)} \boxtimes D_{(\nu_2,\kappa_2)}$ with $\kappa_1 \geq \kappa_2$. We have

$$\Phi(x_1, x_2) = (\sqrt{-1}x_1 + x_2)^\delta \exp\{-\pi(x_1^2 + x_2^2)\}$$

and

$$W=\begin{cases}\varphi_\sigma(w_0) & (\delta_1,\delta_2)=(0,0),\\ R(E^{\mathfrak{p}}_{1,2})\varphi_\sigma(w_{12})-R(E^{\mathfrak{p}}_{2,3})\varphi_\sigma(w_{23}) & \\ \quad +R(E^{\mathfrak{p}}_{3,4})\varphi_\sigma(w_{34})+R(E^{\mathfrak{p}}_{1,4})\varphi_\sigma(w_{14}) & (\delta_1,\delta_2)=(1,1),\\ \varphi_\sigma(w_2+\sqrt{-1}w_4) & \delta_1\neq\delta_2.\end{cases}$$

Here we define $w_0, w_p, w_{pq} \in V_{(\kappa_1,\kappa_2,0)}$ $(1 \le p, q \le 4)$ by

$$w_0=\mathrm{q}_{\mathcal{R}}\big((\xi_1^2+\xi_3^2)^{(\kappa_1-\kappa_2)/2}\xi_{24}^{\kappa_2}\big),\qquad w_p=\mathrm{q}_{\mathcal{R}}\big((\xi_1^2+\xi_3^2)^{(\kappa_1-\kappa_2-1)/2}\xi_p\xi_{24}^{\kappa_2}\big),$$
$$w_{pq}=\mathrm{q}_{\mathcal{R}}\big((\xi_1^2+\xi_3^2)^{(\kappa_1-\kappa_2)/2}\xi_{pq}\xi_{24}^{\kappa_2-1}\big).$$

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