An explicit lifting construction of CAP forms on $\mathrm{O}(1,5)$

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Abstract

This article is the write-up of what the fist named author presented on January 25th in 2022 during the RIMS workshop. We explicitly construct non-tempered cusp forms on the orthogonal group O(1,5) of signature $(1+,5-)$. Given a definite quaternion algebra B over $\mathbb{Q}$, the orthogonal group is attached to the indefinite quadratic space of rank 6 with the anisotropic part defined by the reduced norm of B. As well as the explicit construction we study the cuspidal representations generated by our cusp forms in detail. We determine all local components of the cuspidal representations and show that our cusp forms are CAP forms. Our construction can be viewed as a generalization of [8] to the case of any definite quaternion algebras, for which we note that [8] takes up the case where the discriminant of B is two. Unlike [8] the method of the construction is to consider the theta lifting from Maass cusp forms to $O(1,5)$, following the formulation by Borcherds.

1 Preliminaries

Let $A_0 \in M_4(\mathbb{Q})$ be a positive definite symmetric matrix, and put $A = \begin{bmatrix} & & 1 \\ & -A_0 & \\ 1 & & \end{bmatrix}$. By $\mathcal{G}$ and $\mathcal{H}$ we denote the $\mathbb{Q}$-algebraic groups defined by

$$\mathcal{G}(\mathbb{Q}) = \{g \in \mathrm{GL}_6(\mathbb{Q}) \mid {}^t g A g = A\}, \quad \mathcal{H}(\mathbb{Q}) = \{h \in \mathrm{GL}_4(\mathbb{Q}) \mid {}^t h A_0 h = A_0\}$$

respectively. Both $\mathcal{G}$ and $\mathcal{H}$ are referred to as orthogonal groups. We introduce the standard proper $\mathbb{Q}$-connected parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ defined by the Levi decomposition $\mathcal{P} = \mathcal{N}\mathcal{L}$ with

$$\mathcal{N}(\mathbb{Q}) = \left\{ n(x) = \begin{pmatrix} 1 & {}^t x A_0 & \frac{1}{2}{}^t x A_0 x \\ & 1_4 & x \\ & & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{Q}^4 \right\},$$

$$\mathcal{L}(\mathbb{Q}) = \left\{ a_\alpha = \begin{pmatrix} \alpha & & \\ & h & \\ & & \alpha^{-1} \end{pmatrix} \;\middle|\; \alpha \in \mathbb{Q}^\times,\ h \in \mathcal{H}(\mathbb{Q}) \right\}.$$

Assume that L_0 is a maximal even integral lattice in $\mathbb{Q}^4$ with respect to A_0. We put

$$L := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x, z \in \mathbb{Z},\ y \in L_0 \right\} = L_0 \oplus \mathbb{Z}^2.$$

This is a maximal lattice with respect to A. We let $\Gamma := \{\gamma \in \mathcal{G}(\mathbb{Q}) \mid \gamma L = L\}$.

Now let B be any definite quaternion algebra over $\mathbb{Q}$ with the reduced trace tr and reduced norm Nrd and $\mathcal{O}$ be any maximal order of B. We regard $(\mathcal{O}, \mathrm{Nrd})$ as a quadratic $\mathbb{Z}$ module of rank 4. We are interested in the case where $(\mathbb{Z}^4, A_0) \simeq (\mathcal{O}, \mathrm{Nrd})$. In what follows, we identify these two quadratic modules.

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\mathbb{A}_f$ be the set of finite adeles in $\mathbb{A}$. We consider the adelizations of the $\mathbb{Q}$-algebraic groups above, denoted by $\mathcal{G}(\mathbb{A})$, $\mathcal{H}(\mathbb{A})$, $\mathcal{P}(\mathbb{A})$, $\mathcal{N}(\mathbb{A})$ and so on. Let $L_p := L \otimes \mathbb{Z}_p$ and $L_{0,p} := L_0 \otimes \mathbb{Z}_p$ and we put $K_f := \prod_{p<\infty} K_p$ and $U_f := \prod_{p<\infty} U_p$ with

$$K_p := \{k \in \mathcal{G}(\mathbb{Q}_p) \mid kL_p = L_p\}, \quad U_p := \{u \in \mathcal{H}(\mathbb{Q}_p) \mid uL_{0,p} = L_{0,p}\}$$

for each finite prime $p < \infty$. Let K_∞ be the maximal compact subgroup of $\mathcal{G}(\mathbb{R})$ given by

$$\left\{ g \in \mathcal{G}(\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} 1 & & \\ & A_0 & \\ & & 1 \end{pmatrix} g = \begin{pmatrix} 1 & & \\ & A_0 & \\ & & 1 \end{pmatrix} \right\}.$$

With $A_\infty := \left\{ a_y = \begin{pmatrix} y & & \\ & 1_4 & \\ & & y^{-1} \end{pmatrix} \;\middle|\; y \in \mathbb{R}^+ \right\}$ the Iwasawa decomposition $\mathcal{G}(\mathbb{R}) = \mathcal{N}(\mathbb{R}) A_\infty K_\infty$ gives us the 5-dimensional hyperbolic space $\mathbb{H}_5$ as follows.

$$\mathbb{R}^4 \times \mathbb{R}^+ \ni (x, y) \mapsto n(x) a_y \in \mathcal{G}(\mathbb{R})/K_\infty.$$

Definition. 1.1. *For $r \in \mathbb{C}$ we denote by $\mathcal{M}(\Gamma, r)$ the space of smooth functions F on $\mathcal{G}(\mathbb{R})$ satisfying the following conditions:*

i) $\Omega \cdot F = \frac{1}{8}\left(r^2 - 4\right) F$, where Ω is the Casimir operator defined in [7, (2.3)],

ii) for any $(\gamma, g, k) \in \Gamma \times \mathcal{G}(\mathbb{R}) \times K_\infty$, we have $F(\gamma g k) = F(g)$,

iii) F is of moderate growth.

As usual we say that $F \in \mathcal{M}(\Gamma, r)$ is a cusp form if it vanishes at all the cusps of Γ.

From Proposition 2.3 of [7], we see that a cusp form F in $\mathcal{M}(\Gamma, r)$ has the Fourier expansion

$$F(n(x)a_y) = \sum_{\beta \in L_0' \setminus \{0\}} A(\beta) y^2 K_r(4\pi \sqrt{Q_{A_0}(\beta)} y) e({}^t\beta A_0 x), \tag{1}$$

with the dual lattice L_0' of L_0. Here, Q_{A_0} is the quadratic form corresponding to A_0.

2 Vector valued modular forms and theta lifts

2.1 Vector valued modular forms

Let $d_B = N$ be the discriminant of a definite quaternion algebra B over $\mathbb{Q}$. By definition this is a square-free integer. Let $\mathcal{O}$ be any maximal order of B with $\mathcal{O} \simeq (\mathbb{Z}^4, A_0)$. Let Q_{A_0}, L and A be as in Section 1. Let $\mathcal{O}'$ and L' be the dual of $\mathcal{O}$ and L respectively with respect to bilinear forms B_{A_0} and B_A defined by A_0 and A. We have described the dual $\mathcal{O}'$ in the previous section. We have

$$L' = \{ \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} : a, b \in \mathbb{Z}, \alpha \in \mathcal{O}' \}.$$

Define the discriminant form D by $D = L'/L$. From the description of L' above, we have $D = L'/L = \mathcal{O}'/\mathcal{O}$. D inherits the quadratic form Q_D and bilinear form B_D (with values in $\mathbb{Q}/\mathbb{Z}$) from those of $\mathcal{O}'$

considered modulo 1. The level of D is the smallest positive integer n such that $nQ_D(\mu) \equiv 0 \pmod 1$ for all $\mu \in D$. Since $\mathrm{Nrd}(\mathcal{O}') = \frac{1}{N}\mathbb{Z}$, we see that the level of D is N.

The group algebra $\mathbb{C}[D]$ is a $\mathbb{C}$-vector space generated by the formal basis vectors $\{e_\mu : \mu \in D\}$ with product defined by $e_\mu e_{\mu'} = e_{\mu+\mu'}$. The inner product on $\mathbb{C}[D]$ (anti-linear in the second argument) is defined by $\langle e_\mu, e_{\mu'} \rangle = \delta_{\mu,\mu'}$. Hereafter we will often use the notation

$$e(x) := \exp(2\pi\sqrt{-1}x)$$

for $x \in \mathbb{R}$. We will now define a representation ρ_D of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ by specifying it on the generators of $\mathrm{SL}_2(\mathbb{Z})$ given by $T = \left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$.

$$\begin{aligned} \rho_D(T)e_\mu &= e(Q_D(\mu))e_\mu, \\ \rho_D(S)e_\mu &= \frac{e(-\mathrm{sgn}(D)/8)}{\sqrt{|D|}} \sum_{\mu' \in D} e(-B_D(\mu,\mu'))e_{\mu'} = -\frac{1}{N} \sum_{\mu' \in D} e(-B_D(\mu,\mu'))e_{\mu'}. \end{aligned}$$

This action extends to a unitary representation ρ_D of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ called the Weil representation of D.

To construct a vector valued modular form for $\mathrm{SL}_2(\mathbb{Z})$ with values in $\mathbb{C}[D]$, one has to start with a scalar valued modular form of level N. We let $S(\Gamma_0(N), r)$ be the space of Maass cusp form of weight 0 with respect to $\Gamma_0(N)$ with Laplace eigenvalue $(r^2+1)/4$. According to the Selberg conjecture on the minimal Laplace eigenvalue for Maass cusp forms, r should be real (cf. [4, Section 11.3 Conjecture]). The Fourier expansion of $f \in S(\Gamma_0(N), r)$ is given by

$$f(u+iv) = \sum_{n \neq 0} c(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|v)e(nu).$$

for $\mathfrak{h} := \{u+iv \in \mathbb{C} : v > 0\}$. Define $\mathcal{L}_D(f) : \mathfrak{h} \to \mathbb{C}[D]$ by

$$\mathcal{L}_D(f) = \sum_{M \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} f|M \rho_D(M)^{-1} e_0, \tag{2}$$

where $(f|M)(\tau) = f(M\cdot\tau) := f((a\tau+b)/(c\tau+d))$ for $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R})$.

Proposition. 2.1. *Let $f \in S(\Gamma_0(N), r)$. The function $\mathcal{L}_D(f)$ is well-defined and satisfies*

$$\mathcal{L}_D(f)|\gamma = \rho_D(\gamma)\mathcal{L}_D(f),$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

2.2 Theta lifts

We construct the theta lift of $f \in S(\Gamma_0(N), r)$, N square-free, to an automorphic form on 5-dimensional hyperbolic space as in [1]. Also see [7]. More precisely our theta lifts are from vector valued modular forms given above. We will follow the construction of the theta lift in Section 3 of [7]. We recall from Section 1 that if $g \in \mathcal{G}(\mathbb{R})$, then we can write

$$g = n(x)a_y k, \text{ where } n(x) = \begin{bmatrix} 1 & {}^t x A_0 & \frac{1}{2}{}^t x A_0 x \\ & 1_4 & x \\ & & 1 \end{bmatrix}, x \in \mathbb{R}^4, a_y = \begin{bmatrix} y & & \\ & 1_4 & \\ & & y^{-1} \end{bmatrix}, y \in \mathbb{R}^+, k \in K_\infty$$

where K_∞ is the maximal compact subgroup of $\mathcal{G}(\mathbb{R})$ and that

$$\mathbb{R}^4 \times \mathbb{R}^+ \ni (x,y) \mapsto n(x)a_y \in \mathcal{G}(\mathbb{R})/K_\infty$$

gives the 5-dimensional hyperbolic space $\mathbb{H}_5$. Let $V_5 := (\mathbb{R}^6, Q_A)$ and let $\mathcal{D}$ be the Grassmanian of positive oriented lines in the quadratic space V_5. Note that $V_5 = L \otimes \mathbb{R}$, where L was the lattice defined in Section 1. We will identify $\mathbb{H}_5$ with a connected component of $\mathcal{D}$ as follows.

$$\mathbb{H}_5 \ni (x, y) \mapsto \nu(x, y) := \frac{1}{\sqrt{2}}{}^t(y + y^{-1}Q_{A_0}(x), -y^{-1}x, y^{-1}) \in V_5$$

satisfying $B_A(\nu(x, y), \nu(x, y)) = 1$. It generates the positive, oriented line $\mathbb{R} \cdot \nu(x, y)$, which is an element in $\mathcal{D}$. In fact, we see that $\mathcal{D}^+ := \{\mathbb{R} \cdot \nu(x, y) \mid (x, y) \in \mathbb{H}_5\}$ is one of the two connected components of $\mathcal{D}$. We now note that the quadratic space V_5 is isometric to $\mathbb{R}^{1,5}$, where $\mathbb{R}^{1,5}$ denotes the real vector space $\mathbb{R}^6$ with the quadratic form

$$Q_{1,5}(x_1, x_2, \cdots, x_6) := \frac{1}{2}\left(x_1^2 - \sum_{j=2}^{6} x_j^2\right).$$

We slightly abuse the notation by using ν to represent the line generated by $\nu(x, y)$. Every line $\nu \in \mathcal{D}^+$ induces an isometry

$$\begin{aligned} \iota_\nu : V_5 &\to \mathbb{R} \cdot \nu \oplus (\nu^\perp, Q_{A_0}|_{\nu^\perp}) \simeq \mathbb{R}^{1,5} \\ \lambda &\mapsto (\iota_\nu^+(\lambda), \iota_\nu^-(\lambda)), \end{aligned}$$

where

$$\iota_\nu^+(\lambda) := B_A(\lambda, \nu)\nu, \ \iota_\nu^-(\lambda) := \lambda - \iota_\nu^+(\lambda) \in \nu^\perp$$

are the components of λ. Let us remark here that, if we fix $(x, y) \in \mathbb{H}_5$, then we get a corresponding isometry of V_5 into $\mathbb{R}^{1,5}$ where the one dimensional positive definite subspace is the line generated by $\nu(x, y)$.

Let w^+ (respectively w^-) be the orthogonal complement of the line generated by z_{ν^+} (respectively z_{ν^-}) in $\iota_\nu^+(V_5)$ (respectively $\iota_\nu^-(V_5)$). For $\lambda \in V_5$, let λ_{w^+} and λ_{w^-} be the projection of λ to w^+ and w^- respectively. We define the linear map $w : V_5 \to \mathbb{R}^{1,5}$ by $w(\lambda) = (\lambda_{w^+}, \lambda_{w^-})$, so that w is an isomorphism from w^+ and w^- to their images and w vanishes on z_{ν^+} and z_{ν^-}. For our special case, w^+ is trivial, the image of w is 4-dimensional, and the first coordinate of $w(\lambda)$ is 0.

If p is a polynomial on $\mathbb{R}^{1,5}$, we say that p has homogeneous degree (m^+, m^-) if it is homogeneous of degree m^+ in the first variable and homogeneous of degree m^- in the last 5 variables. For h^+, h^- integers satisfying $0 \le h^+ \le m^+$ and $0 \le h^- \le m^-$ define polynomials p_{w,h^+,h^-} on $w(V_5)$ of homogeneous degree $(m^+ - h^+, m^- - h^-)$ by

$$p(\iota_\nu(\lambda)) = \sum_{h^+,h^-} B_A(\lambda, z_{\nu^+})^{h^+} B_A(\lambda, z_{\nu^-})^{h^-} p_{w,h^+,h^-}(w(\lambda)). \tag{3}$$

Let $p : \mathbb{R}^6 \to \mathbb{R}$ be the polynomial given by $p(x_1, \cdots, x_6) = -2^{-2}x_1^2$. We get a polynomial on V_5 defined by $p \circ \iota_\nu$ given by the formula

$$p(\iota_\nu(\lambda)) = -2^{-2}B_A(\lambda, \nu)^2 = -2^{-1}y^2 B_A(\lambda, z_{\nu^+})^2.$$

By (3), we have

$$p_{w,h^+,h^-} = \begin{cases} -2^{-1}y^2 & \text{if } (h^+, h^-) = (2, 0); \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Note that the polynomial p_{w,h^+,h^-} is a constant in this case.

Let Δ be the Laplacian on $\mathbb{R}^{1,5}$. For $\tau \in \mathfrak{h}$, $(x, y) \in \mathbb{H}_5$ and $\mu \in D = L'/L$, define

$$\theta_\mu^L(\tau, \nu(x, y), p) := \sum_{\lambda \in L+\mu} \left(exp(\frac{-\Delta}{8\pi v})(p)\right)(\iota_\nu(\lambda)) exp(2\pi\sqrt{-1}\left(Q_A(\iota_\nu^+(\lambda))\tau + Q_A(\iota_\nu^-(\lambda))\bar{\tau}\right)),$$

$$\Theta_L(\tau,\nu(x,y),p) := \sum_{\mu\in D} e_\mu \theta_\mu^L(\tau,\nu(x,y),p).$$

Proposition. 2.2. *For* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, *we have*

$$\Theta_L(\frac{a\tau+b}{c\tau+d},\nu(x,y),p) = |c\tau+d|^5 \rho_D(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right])\Theta_L(\tau,\nu(x,y),p).$$

Let $f \in S(\Gamma_0(N),r)$, N square-free, be an Atkin-Lehner eigenform with eigenvalues ε_c for all $c|N$. Let $\mathcal{L}_D(f)$ be the $\mathbb{C}[D]$ valued modular form as defined in (2). Let $\Theta_L(\tau,\nu(x,y),p)$ be the theta function defined in the previous section. Define

$$\Phi_L(\nu(x,y),p,f) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}} (\mathcal{L}_D(f))(\tau)\overline{\Theta_L(\tau,\nu(x,y),p)} v^{\frac{5}{2}} \frac{dudv}{v^2}.$$

Here, complex conjugation on $\mathbb{C}[D]$ is given by $\overline{e_\mu} := e_{-\mu}$. In the product of Θ_L and $\mathcal{L}_D(f)$, we are taking the inner product in $\mathbb{C}[D]$ to get a $\mathbb{C}$-valued function. By Propositions 2.1 and 2.2, we see that the integrand is indeed invariant under $\mathrm{SL}_2(\mathbb{Z})$.

Lemma. 2.3. *Let* $\gamma \in \Gamma = \{\gamma \in \mathcal{G}(\mathbb{Q}) : \gamma L = L\}$. *Then*

$$\Phi_L(\gamma\nu(x,y),p,f) = \Phi_L(\nu(x,y),p,f).$$

We give a formula for the Fourier coefficients of $\Phi_L(\nu(x,y),f)$ in terms of the Fourier coefficients of f. To be precise, we provide a formula for $A(\beta)$ in terms of the Fourier coefficients $c(n)$ of f. Let us define the primitive elements of $\mathcal{O}'$ by

$$\mathcal{O}'_{\text{prim}} := \{\beta \in \mathcal{O}' : \frac{1}{n}\beta \notin \mathcal{O}' \text{ for all positive integers } n > 1\}.$$

Proposition. 2.4. *Write* $\beta \in \mathcal{O}'$ *as*

$$\beta = \prod_{p|N} p^{u_p} n\beta_0, \qquad u_p \geq 0, n > 0, \gcd(n,N) = 1 \textit{ and } \beta_0 \in \mathcal{O}'_{\text{prim}}.$$

Let $q_{\beta_0} = q_{\mu_{\beta_0}}$. *For* $p|N$, *set*

$$\delta_p = \begin{cases} 0 & \textit{if } p|q_{\beta_0}; \\ 1 & \textit{if } p \nmid q_{\beta_0}. \end{cases}$$

Then

$$A(\beta) = \sqrt{Q_{A_0}(\beta)} \sum_{p|N} \sum_{t_p=0}^{2u_p+\delta_p} \sum_{d|n} c(\frac{-Q_{A_0}(\beta)}{\prod_{p|N} p^{t_p-1} d^2}) \prod_{p|N} (-\varepsilon_p)^{t_p-1}. \tag{5}$$

We can also verify the following:

Proposition. 2.5. *For each representative c of the Γ-cusps, $\Phi_L(c\nu(x,y),p,f)$ has no constant term. Namely, our lifts $\Phi_L(\nu(x,y),p,f)$ are cuspidal.*

As a result of this we have an enough knowledge of the Foureir expansion of our theta lifts. From Lemma 2.3 and the above Fourier expansion (compare to (1)), we get

Theorem. 2.6. $\Phi_L(\nu(x,y),f)$ *is a cusp form belonging to* $\mathcal{M}(\Gamma,\sqrt{-1}r)$.

3 Hecke Theory

3.1 Adelization of automorphic forms

To study the action of the Hecke operators on our cusp forms constructed by the lift, we need the adelic as well as non-adelic treatment of automorphic forms.

For $h \in \mathcal{H}(\mathbb{A})$, we have the decomposition $h = au^{-1}$ with $(a, u) \in \mathrm{GL}_4(\mathbb{Q}) \times (\Pi_{p<\infty}\mathrm{SL}_4(\mathbb{Z}_p) \times \mathrm{SL}_4(\mathbb{R}))$. Let $\mathcal{O}_h := (\Pi_{p<\infty} h_p \mathbb{Z}_p^4 \times \mathbb{R}^4) \cap \mathbb{Q}^4$ for $h = (h_v)_{v\leq\infty} \in \mathcal{H}(\mathbb{A})$. Then, we have $\mathcal{O}_h = a\mathcal{O}$ (c.f. [7, Section 3.3]). The dual lattice $\mathcal{O}'_h$ is then equal to $a^{-1}\mathcal{O}'$.

To obtain an adelic Fourier expansion, let $f \in S(\Gamma_0(N), r)$ be a Maass cusp form with the Fourier expansion $f(z) = \sum_{n\neq 0} c(n) W_{0,\frac{\sqrt{-1}r}{2}}(4\pi|n|y)e(x)$. Let Λ be the standard additive character of $\mathbb{A}/\mathbb{Q}$. We introduce the following Fourier series

$$F_f(n(x)a_y kg) := \sum_{\lambda\in\mathbb{Q}^4\setminus\{0\}} F_{f,\lambda}(n(x)a_y kg) \quad \forall (x, y, k, g) \in \mathbb{A}^4 \times \mathbb{R}_+^\times \times K_\infty \times \mathcal{G}(\mathbb{A}_f) \tag{6}$$

with

$$F_{f,\lambda}(n(x)a_y kg) := A_\lambda(g) y^2 K_{\sqrt{-1}r}(4\pi|\lambda|_A y)\Lambda({}^t\lambda Ax),$$

where $A_\lambda(g)$ is defined by the following conditions:

$$A_\lambda\left(\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right) := \begin{cases} \sqrt{Q_{A_0}(\lambda)} \sum\limits_{p|N} \sum\limits_{t_p=0}^{2u_p+\delta_p} \sum\limits_{d|n} c\Big(\frac{-Q_{A_0}(\lambda)}{\prod_{p|N} p^{t_p-1}d^2}\Big) \prod\limits_{p|N} (-\varepsilon_p)^{t_p-1} & (\lambda \in \mathcal{O}'_h) \\ 0 & (\lambda \in \mathbb{Q}^4 \setminus \mathcal{O}'_h) \end{cases}$$

$$A_\lambda\left(\begin{pmatrix}s & & \\ & h & \\ & & s^{-1}\end{pmatrix}\right) := ||s||_{\mathbb{A}}^2 A_{||s||_{\mathbb{A}}^{-1}\lambda}\left(\begin{pmatrix}1 & & \\ & h & \\ & & 1\end{pmatrix}\right)$$

$$A_\lambda(n(x)gk) := \Lambda({}^t\lambda Ax)A_\lambda(g) \qquad \forall (x, g, k) \in \mathbb{A}_f^4 \times \mathcal{G}(\mathbb{A}_f) \times K_f.$$

Here

1. u_p, δ_p and n are as defined in Proposition 2.4 for $\beta = h^{-1}\lambda$.
2. $(s, h) \in \mathbb{A}_f^\times \times \mathcal{H}(\mathbb{A}_f)$ and $||s||_{\mathbb{A}}$ denotes the idele norm of s.

For $r \in \mathbb{C}$, let $\mathcal{M}(\mathcal{G}(\mathbb{A}), r)$ denote the space of smooth functions F on $\mathcal{G}(\mathbb{A})$ satisfying the following conditions:

1. $\Omega \cdot F = \frac{1}{8}(r^2 - 4)F$, where Ω is the Casimir operator defined in [7].
2. For any $(\gamma, g, k) = \mathcal{G}(\mathbb{Q}) \times \mathcal{G}(\mathbb{A}) \times K$, we have $F(\gamma gk) = F(g)$.
3. F is of moderate growth.

Note that $F \in \mathcal{M}(\mathcal{G}(\mathbb{A}), r)$ has the Fourier expansion

$$F(g) = \sum_{\lambda\in Q^4} F_\lambda(q), \quad F_\lambda(g) := \int_{\mathbb{A}^4/\mathbb{Q}^4} F(n(x)g)\Lambda({}^t\lambda Ax)dx,$$

where dx is the invariant measure normalized so that the volume of $\mathbb{A}^4/\mathbb{Q}^4$ is one. The adelic function F is called a cusp form if $F_0 \equiv 0$ in the Fourier expansion. By the argument similar to [7, Theorem 3.3] we deduce the following proposition from the Fourier expansion discussed in Section 2.2.

Proposition. 3.1. *The adelic function F_f is a cusp form belonging to $\mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$*

3.2 Sugano Theory

We will show that if f is a Hecke eigenform then F_f is an Hecke eigenform by using the non-archimedean local theory of Sugano [16, Section 7]. For a prime p, let $F = \mathbb{Q}_p$ with the ring of integers $\mathbb{Z}_p$. Let $n_0 \le 4$ and let $S_0 \in M_{n_0}(F)$ be an anisotropic even symmetric matrix of degree n_0. For the $m \times m$ matrix $J_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$, let G_m denote the group of F-valued points of the orthogonal group of degree $2m + n_0$, defined by the matrix $Q = \begin{pmatrix} & & J_m \\ & S_0 & \\ J_m & & \end{pmatrix}$. Denote by $L_m := \mathbb{Z}_p^{2m+n_0}$ the maximal lattice with respect to Q_m and let K_m be the maximal compact open subgroup of G_m defined by the lattice

$$K_m := \{g \in G_m \mid gL_m = L_m\}. \tag{7}$$

Let $\mathcal{H}_m$ be Hecke algebra for (G_m, K_m) and define $C_m^{(r)} \in \mathcal{H}_m$ to be the double cosets $K_m c_m^{(r)} K_m$, where

$$c_m^{(r)} := \mathrm{diag}(p, \ldots, p, 1, \ldots 1, p^{-1}, \ldots, p^{-1}) \in G_m$$

which is a diagonal matrix whose first r and last r entries are p and p^{-1} respectively. By [16, Section 7], $\{C_m^{(r)} \mid 1 \le r \le m\}$ forms generators of the Hecke algebra $\mathcal{H}_m$.

We embed G_i for $i \le m$ in G_m as a subgroup by the middle $(2i + n_0) \times (2i + n_0)$ block. We regard K_i as subgroup of K_m similarly. The invariant measure of G_m is normalized so that the volume of K_i is one for each $i \le m$.

For a prime $p \nmid N$, we have $n_0 = 0$ and $m = 3$. In this case, the lattice L_3 is self-dual. For a non-negative integer k, let

$$f_{k,j} := \frac{p^{j-1}(p^{k-j+1} - 1)(p^{k-j} + 1)}{p^j - 1} \quad (\forall j \in \mathbb{Z} \setminus \{0\}), \tag{8}$$

a special case of [16, 7.11] for $n_0 = \delta = 0$. For positive integers k, r, set $R_k^{(r)} := K_k/(K_k \cap c_k^{(r)} K_k (c_k^{(r)})^{-1})$, and let $|R_k^{(r)}|$ denote the cardinality of $R_k^{(r)}$. We have

$$|R_k^{(r)}| = \begin{cases} \Pi_{j=1}^r f_{k,j} & (1 \le r \le k); \\ 1 & (r = 0). \end{cases} \tag{9}$$

Following the methods in Section 4 of [7], we get the following theorem (essentially Theorem 4.11 of [7] for $n = 1/2$).

Theorem. 3.2. *Suppose that f is a Hecke eigenform and let λ_p be the Hecke eigenvalue of f at $p < \infty$ with $p \nmid N$. Then the following holds.*

i) F_f is a Hecke eigenform.

ii) Let μ_i be the Hecke eigenvalue with respect to the Hecke operator $C_3^{(i)}$ for $1 \le i \le 3$. We have

$$\mu_1 = p^2(\lambda_p^2 - 2) + pf_{2,1} = p^2(\lambda_p^2 + p + p^{-1});$$

$$\mu_i = |R_2^{(i-1)}| \left(\mu_1 - \frac{p^{i-1} - 1}{p^i - 1} f_{3,1}\right), (i = 2, 3).$$

3.3 The case $p \mid N$

When $p \mid N$, we have $m = 1$ and $n_0 = 4$. Hence, the Hecke algebra $\mathcal{H}_1$ is generated by $C_1^{(1)}$ which is the double coset $K_1 c_1^{(1)} K_1$ as defined in (3.2). Let $n(x) \in G_1$ be as defined in Section 1 and let $(t, g) := \mathrm{diag}(t, g, t^{-1}) \in G_1$ for $t \in \mathbb{Q}_p^\times$ and $g \in G_0$.

Lemma. 3.3.

$$C_1^{(1)} = \bigsqcup_{x \in \mathfrak{X}_1} (p, 1_4) n(x) K_1 \sqcup \bigsqcup_{x \in \mathfrak{X}_3} (1, 1_4) n(x) K_1 \sqcup (p^{-1}, 1_4) K_1$$

where

$$\mathfrak{X}_1 = \left\{x \in p^{-1}\mathcal{O}/\mathcal{O}\right\}, \mathfrak{X}_3 = \{x \in (\mathcal{O}' - \mathcal{O})/\mathcal{O}\}.$$

We can now describe the action of $C_1^{(1)}$ with the invariant measure dx of G_1 normalized so that the volume $\int_{K_1} dx = 1$. Define

$$(C_1^{(1)} \cdot \Phi)(g) := \int_{G_1} \mathrm{char}_{K_1 c_1^{(1)} K_1}(x) \Phi(gx) dx$$

for $\Phi \in \mathcal{M}(\mathcal{G}(\mathbb{A}), r)$.

The following proposition derives the action of $C_1^{(1)}$ on Fourier coefficients of Φ.

Proposition. 3.4. *Let $\Phi \in \mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$ be a lift. Then*

$$(C_1^{(1)} \cdot \Phi)(n(x)a_y) = \sum_{\lambda \in \mathcal{O}' \setminus \{0\}} A'_\lambda(1) y^2 K_{\sqrt{-1}r}(4\pi \sqrt{Q_{A_0}(\lambda)} y) \Lambda({}^t\lambda A_0(x)),$$

where

$$A'_\lambda(1) = \begin{cases} p^2 A_{p\lambda}(1) - A_\lambda(1) + p^2 A_\lambda(1) + p^2 A_{p^{-1}\lambda}(1) & \text{if } \lambda \in p\mathcal{O}' \setminus \{0\}; \\ p^2 A_{p\lambda}(1) - A_\lambda(1) + p^2 A_\lambda(1) & \text{if } \lambda \in \mathcal{O} \setminus p\mathcal{O}'; \\ p^2 A_{p\lambda}(1) - A_\lambda(1) & \text{if } \lambda \in \mathcal{O}' \setminus \mathcal{O}. \end{cases}$$

To write the action of the Hecke operator in terms of Fourier coefficients given in Proposition 2.4, we write $A_\lambda(1) = A(\beta)$ where $\beta = \prod_{p|N} p^{u_p} n \beta_0$ as in the proposition. Note, for $\lambda \in \mathcal{O}'$ and $\beta \in \mathcal{O}'$ the conditions for $A'_\lambda(1)$ on λ from Proposition 3.4 above translate to conditions on β as follows:

$$\begin{aligned} \lambda \in p\mathcal{O}' \setminus \{0\} &\iff u_p \geq 1; \\ \lambda \in \mathcal{O} \setminus p\mathcal{O}' &\iff u_p = 0, \delta_p = 1; \\ \lambda \in \mathcal{O}' \setminus \mathcal{O} &\iff u_p = 0, \delta_p = 0. \end{aligned}$$

Then, as

$$A_{p\lambda}(1) = A(p\beta); \quad A_{p^{-1}\lambda}(1) = A(p^{-1}\beta)$$

we can rewrite the $A'_\lambda(1)$ in terms of β as

$$A'_\lambda(1) = \begin{cases} p^2 A(p\beta) + (p^2 - 1)A(\beta) + p^2 A(p^{-1}\beta) & \text{if } u_p \geq 1; \\ p^2 A(p\beta) + (p^2 - 1)A(\beta) & \text{if } u_p = 0, \delta_p = 1; \\ p^2 A(p\beta) - A(\beta) & \text{if } u_p = 0, \delta_p = 0. \end{cases} \tag{10}$$

Let $f \in S(\Gamma_0(N), r)$ be a new form with Hecke eigenvalue λ_p for the operator defined by the action of the double coset $\Gamma_0(N)\left[\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right]\Gamma_0(N)$ at prime p. Assuming it is an Atkin Lehner eigenform with eigenvalue ϵ_p, it can be shown that

$$\lambda_p = -\epsilon_p. \tag{11}$$

Using the single coset decomposition ([6, Lemma 9.14])

$$\Gamma_0(N)\begin{bmatrix}1 & \\ & p\end{bmatrix}\Gamma_0(N) = \bigsqcup_{b=0}^{p-1}\Gamma_0(N)\begin{bmatrix}1 & b\\ & p\end{bmatrix}$$

we have

$$\sum_{b=0}^{p-1} f(\frac{z+b}{p}) = \lambda_p f(z).$$

In terms of Fourier coefficients, using (11), we get

$$c(pm) = \frac{\lambda_p}{p}c(m) = \frac{-\epsilon_p}{p}c(m) \quad \forall m \in \mathbb{Z}.$$

The discussion above and the explicit formula for Fourier coefficients of F_f provide us with enough ingredients to show the following:

Theorem. 3.5. *Let $f \in S(\Gamma_0(N), r)$ be a new form and eigenfunction of the Atkin Lehner involution with eigenvalue ϵ_p at each $p|N$. Let F_f be the lift of f defined in (6). Then F_f is a Hecke eigenform with*

$$C_1^{(1)} \cdot F_f = (p^3 + p^2 + p - 1)F_f.$$

4 Non-vanishing of the lift

In this section, we will obtain the non-vanishing of the map $f \to F_f$ constructed in Section 2.2. Let us start by observing that the proof of Lemma 4.5 of [8] can be used to conclude that there exists $M > 0$ such that the Fourier coefficient $c(-M)$ of f is non-zero. If f is a Hecke eigenform, then this implies that $c(-1) \neq 0$. Using the explicit formula (5) for the Fourier coefficients for F_f, we can see that in this case we get $A(1) \neq 0$. Hence, the map $f \to F_f$ is injective when restricted to Hecke eigenforms f. We will now prove the injectivity for all f.

Consider a basis of Hecke eigenforms $\{f_1, \cdots, f_k\}$ of $S(\Gamma_0(N), r)$. Since this is a finite set, we can find a prime $p \nmid N$ such that the Hecke eigenvalues $\lambda_p^{(i)}$ of f_i for $i = 1, \cdots k$ satisfy $|\lambda_p^{(i)}| \neq |\lambda_p^{(j)}|$ for all $i \neq j$. This follows from Corollary 4.1.3 of [12]. Let $F_1, \cdots, F_k$ be the lifts of $f_1, \cdots, f_k$. By Theorem 3.2, we know that F_i are Hecke eigenforms with eigenvalues $\mu_{p,1,i} = p^2\Big((\lambda_p^{(i)})^2 + p + p^{-1}\Big)$. Because of the choice of p, we again see that $\mu_{p,1,i} \neq \mu_{p,1,j}$ for all $i \neq j$. We then verify the non-vanishing of our theta lifts by an elementary argument of the linear algebra though there is the well known approach of the inner product formula initiated by Rallis [11].

Theorem. 4.1. *The map $f \to F_f$ is an injective linear map on $S(\Gamma_0(N), r)$.*

5 CAP representation associated to the lift

Assume that $f \in S(\Gamma_0(N), r)$ is a newform, and let $F_f \in \mathcal{M}(\mathcal{G}(\mathbb{A}), \sqrt{-1}r)$ be the corresponding lift defined in (6). Let π_F be the representation of $\mathcal{G}(\mathbb{A})$ generated by F_f.

5.1 Local components of the representation

5.1.1 The archimedean component

Let

$$N_\infty := \{n(x) \mid x \in \mathbb{R}^4\}, \qquad A_\infty := \{a_y \mid y \in \mathbb{R}^+\}$$

for $n(x)$ and a_y as defined in Section 1. Let $\delta_s : A_\infty \to \mathbb{C}^\times$ be a quasi-character given by $\delta_s(y) = y^s$ for a parameter $s \in \mathbb{C}$. We can trivially extend δ_s to the parabolic subgroup P_∞ with Langlands decomposition $P_\infty = N_\infty A_\infty M_\infty$ for $M_\infty := \left\{ \begin{pmatrix} 1 & & \\ & m & \\ & & 1 \end{pmatrix} \middle| m \in \mathcal{H}(\mathbb{R}) \right\}$. We define the normalized parabolic induction induced from δ_s by $I_{P_\infty}^{G_\infty}(\delta_s)$. Proposition 5.5 of [7] for $N = 4$ gives us the following:

Proposition. 5.1. *The archimedean component of π_F is isomorphic to $I_{P_\infty}^{G_\infty}(\delta_{\sqrt{-1}r})$ as admissible G_∞ module, and irreducible. If r is real, namely, f satisfies the Selberg conjecture on the minimal eigenvalue of the hyperbolic Laplacian, π_F is tempered at the archimedean place.*

Using Theorem 3.1 of [9] and Proposition 3.1, we see that π_F is irreducible. Since F_f is a cusp form, we can conclude that π_F is an irreducible, cuspidal representation of $\mathcal{G}(\mathbb{A})$. Hence, we can decompose $\pi_F = \otimes'_v \pi_v$, where π_v is an irreducible, admissible representation of $\mathcal{G}(\mathbb{Q}_v)$. We have obtained the description of π_∞ above. Next we will describe π_p for finite primes p.

5.1.2 Non-archimedean component: $p \nmid N$ case

Let p be a prime with $p \nmid N$. Let χ_1, χ_2, χ_3 be unramified characters of $\mathbb{Q}_p^\times$. We get a character χ of the split torus of $\mathcal{G}(\mathbb{Q}_p)$ via

$$\mathrm{diag}(a_1, a_2, a_3, a_3^{-1}, a_2^{-1}, a_1^{-1}) \to \chi_1(a_1)\chi_2(a_2)\chi_2(a_3).$$

Extend this to a character of the minimal parabolic subgroup of $\mathcal{G}(\mathbb{Q}_p)$ by setting it to be trivial on the unipotent radical. By unramified principal series representation of $\mathcal{G}(\mathbb{Q}_p)$ we mean the normalized parabolic induction $I(\chi)$ of $\mathcal{G}(\mathbb{Q}_p)$ induced from χ, the character of the minimal parabolic group.

The argument of the proof of [7, Theorem 5.6] works also for our setting. From Theorem 3.2 we thus deduce the following:

Proposition. 5.2. *For primes $p \nmid N$, the local component π_p of π_F is the spherical constituent of the unramified principal series representation $I(\chi)$ of $\mathcal{G}(\mathbb{Q}_p)$ where the character χ corresponds to the three unramified characters χ_1, χ_2, χ_3 given by*

$$\chi_1(\varpi_p) = \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2} \right)^2, \chi_2(\varpi_p) = p, \chi_3(\varpi_p) = 1.$$

Here, ϖ_p is an uniformizer in $\mathbb{Q}_p$. Hence, π_p is non-tempered for every $p \nmid N$.

5.1.3 Non-archimedean component: $p|N$ case

Let p be a prime with $p|N$. For an unramified character χ of $\mathbb{Q}_p^\times$, we get a character of the torus of $\mathcal{G}(\mathbb{Q}_p)$ via

$$\mathrm{diag}(y, 1, 1, 1, 1, y^{-1}) \to \chi(y).$$

We can extend this to a character of the maximal parabolic subgroup P by setting it to be trivial on the unipotent radical. The modulus character is given by

$$\delta_P(a_y n(x)) = |y|^4.$$

Define the normalized unramified principal series $I(\chi)$ consisting of all smooth functions $f : \mathcal{G}(\mathbb{Q}_p) \to \mathbb{C}$ satisfying

$$f(a_y n(x) g) = |y|^2 \chi(y) f(g) \quad \text{for all } y \in \mathbb{Q}_p^\times, x \in \mathbb{Q}_p^4, g \in \mathcal{G}(\mathbb{Q}_p).$$

If f_1 is an unramified vector in $I(\chi)$, then the Hecke operator $C_1^{(1)}$ acts on f_1 by a constant. To obtain the constant, using Lemma 3.3, we see that

$$\begin{aligned}\left(C_1^{(1)} f_1\right)(1) &= \int_{\mathcal{G}(\mathbb{Q}_p)} \mathrm{char}_{K_1 c_1^{(1)} K_1}(x) f_1(x) dx \\ &= \sum_{x \in \mathfrak{X}_1} f_1(a_p n(x)) + \sum_{x \in \mathfrak{X}_1} f_1(n(x)) + f_1(a_{p^{-1}}) \\ &= p^4 |p|^2 \chi(p) f_1(1) + (p^2 - 1) f_1(1) + |p^{-1}|^2 \chi(p^{-1}) f_1(1) \\ &= \left(p^2 \chi(p) + p^2 - 1 + p^2 \chi(p^{-1})\right) f_1(1). \end{aligned} \tag{12}$$

From this we can deduce the following:

Proposition. 5.3. *Let $p|N$. The local representation π_p is the spherical constituent of the unramified principal series $I(\chi)$ with $\chi(\varpi_p) = p^{\pm 1}$. The representation π_p is non-tempered.*

5.2 Cuspidal representation generated by F_f and its CAP property

Following the description of the local components, we can now state the result for the explicit determination of the cuspidal representation generated by F_f.

Theorem. 5.4. *Let f be a new form in $S(\Gamma_0(N), r)$ and let π_F be the cuspidal representation generated by F_f. Then,*

i) π_F is irreducible and decomposes into the restricted tensor product $\pi_F = \otimes'_{v \leq \infty} \pi_v$ of irreducible admissible representations π_v of $\mathcal{G}(\mathbb{Q}_v)$.

ii) For $v = p < \infty$, if $p \nmid N$ then π_p is the spherical constituent of the unramified principal series representation of $\mathcal{G}_p$ with the Satake parameters

$$\mathrm{diag}\left(\left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^2, p, 1, 1, p^{-1}, \left(\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}\right)^{-2}\right).$$

iii) For $v = p < \infty$, if $p \mid N$ then π_p is the spherical constituent of the parabolic induction $I(\chi)$ of $\mathcal{G}(\mathbb{Q}_p)$ defined by

$$\chi(p) = p.$$

iv) For every finite prime p, π_p is non-tempered. Suppose that the Selberg conjecture holds for f, namely r is a real number for the Laplace eigenvalue for f. Then π_∞ is tempered.

Proof. This follows from Proposition 5.1, Proposition 5.2, Proposition 5.3 and Theorem 3.1 of [9]. □

We now review the definition of a CAP representation from [8, Definition 6.6].

Definition. 5.5. *Let G_1 and G_2 be two reductive algebraic groups over a number field F such that $G_{1,v} \simeq G_{2,v}$ for almost all places v, where $G_{i,v} = G_i(F_v)$ $(i = 1, 2)$ is the group of F_v-points of G_i for the local field F_v at v. Let P_2 be a parabolic subgroup of G_2 with Levi decomposition $P_2 = M_2 N_2$. An irreducible cuspidal automorphic representation $\pi = \otimes'_v \pi_v$ of $G_1(\mathbb{A})$ is called cuspidal associated to parabolic (CAP) P_2, if there exists an irreducible cuspidal automorphic representation σ of M_2 such that $\pi_v \simeq \pi'_v$ for almost all places v, where $\pi' = \otimes'_v \pi'_v$ is an irreducible constituent of $\mathrm{Ind}_{P_2(\mathbb{A})}^{G_2(\mathbb{A})}(\sigma)$.*

For our case $G_1 = \mathcal{G} = \mathrm{O}(1,5)$ and $G_2 = \mathrm{O}(3,3)$. We have $G_{1,p} = G_{2,p}$ for all $p \nmid N$. Let σ be a cuspidal representation of GL_2 generated by a Maass cusp form f with the trivial central character. Assume that f is a new form. We want to regard the representation $|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma$ of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ (cf.[8, Section 6.2]) as the representation of $\mathbb{A}^\times \times \mathrm{O}(2,2)(\mathbb{A})$, which is isomorphic to a Levi subgroup of a maximal parabolic subgroup $P(\mathbb{A})$ of $\mathrm{O}(3,3)(\mathbb{A})$. Recall that our previous work [8] introduced the parabolic induction from the representation $|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma$ of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ to discuss the CAP property of our lifting for the case of $d_B = 2$ in the setting of GL_2 over B. In the present setting we consider the parabolic induction from the aforementioned representation of $\mathbb{A}^\times \times \mathrm{O}(2,2)(\mathbb{A})$ instead and can show that π_F is a CAP representation attached to this parabolic induction.

To see this we start with recalling the following two isomorphisms

$$\mathrm{GL}_2 \times \mathrm{GL}_2/\{(z,z) \mid z \in \mathrm{GL}_1\} \simeq \mathrm{GSO}(2,2), \quad \mathrm{GO}(2,2) = \mathrm{GSO}(2,2) \rtimes \langle t \rangle.$$

We now note that the representation $|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma$ of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$ can be regarded as the representation of $\mathrm{GSO}(2,2)(\mathbb{A})$ since σ has the trivial central character. We construct a representation of $\mathrm{GO}(2,2)(\mathbb{A})$ by considering its induced representation from $\mathrm{GSO}(2,2)(\mathbb{A})$ to $\mathrm{GO}(2,2)(\mathbb{A})$. Furthermore consider the pull-back of the representation of $\mathrm{GO}(2,2)(\mathbb{A})$ to $\mathbb{A}^\times \times \mathrm{O}(2,2)(\mathbb{A})$ via the surjection $\mathbb{A}^\times \times \mathcal{O}(2,2)(\mathbb{A}) \to \mathrm{GO}(2,2)(\mathbb{A})$. We denote the resulting representation simply by σ and introduce the normalized parabolic induction $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{O}(3,3)(\mathbb{A})}\sigma$, where P is the maximal parabolic subgroup with Levi subgroup isomorphic to $\mathrm{GL}(1) \times \mathrm{O}(2,2)$ and the abelian unipotent radical. Then we have the following:

Proposition. 5.6. *Let π_F be as above and recall that we have assumed that the Maass cusp form f is a new form. The cuspidal representation π_F is CAP to the parabolic induction $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{O}(3,3)(\mathbb{A})}\sigma$.*

5.3 Global standard L-function for F_f

We define the standard L-function of the orthogonal group $\mathcal{G}$, following Sugano [16, Section 7, (7,6)]. The local factors for places $p \nmid d_B$ are well known. We find them in [16, Section 7, (7,6)]. For places $p|d_B$, the case of $(n_0, \partial) = (4,2)$ in [16, Section 7 (7.6)] is valid. We define the standard L-function by the Euler product over all finite primes. Putting the local datum of Theorem 5.4 (ii) and (iii) together, we have the following result with the help of Y. Jo [5, Theorem 5.7] and Gelbert-Jacquet [3]:

Proposition. 5.7. *Suppose that a Maass cusp form f is a new form in $S(\Gamma_0(N), r)$ and recall that σ denotes the cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f. Let $\Pi = \mathrm{Ind}_{P_{2,2}(\mathbb{A})}^{\mathrm{GL}_4(\mathbb{A})}(|\det|_{\mathbb{A}}^{-1/2}\sigma \times |\det|_{\mathbb{A}}^{1/2}\sigma)$, with the parabolic subgroup $P_{2,2}$ of GL_4 with Levi part $\mathrm{GL}_2 \times \mathrm{GL}_2$. By $L(F_f, \mathrm{std}, s)$ (respectively $L(\Pi, \wedge, s)$) we denote the standard L-function for the lift F_f (respectively exterior square L-function of Π). We have*

$$L(F_f, \mathrm{std}, s) = L(\Pi, \wedge, s) = L(\mathrm{sym}^2(f), s)\zeta(s-1)\zeta(s)\zeta(s+1),$$

where the Riemann zeta function $\zeta(s)$ is defined by the Euler product over all finite primes.

Remark. 5.8. *The above coincidence of the two L-functions is expected in the framework of the Langlands L-functions (for instance see [2, Section 4]). We remark that our example is given for non-generic representations while the case of generic representations is known to be proved by Shahidi's theory [15, Theorem 3.5] (see [2, Lemma 3.5]).*

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