ON FINITE LENGTH SMOOTH REPRESENTATIONS OF p-ADIC GENERAL LINEAR GROUPS

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Abstract. We survey qualitative aspects of the study of the decomposition of finite-length smooth representations of the groups $GL_n(F)$, for a p-adic field F, with emphasis on techniques that have been developing in the recent years. We state general goals and questions on decomposition of parabolic induction of irreducible representations, and review applications for branching laws such as the local Gan-Gross-Prasad program.

We give a flavor of the categorical links, and their possible applications, between the p-adic setting and the representation theory of quiver Hecke algebras of type A. Finally, we review, as a case study, the recent RSK classification of irreducible representations, introduced by the author with Lapid.

1. Introduction

This short note is based on an (online) talk given by the author at the January 2022 "Automorphic forms, automorphic L-functions and related topics" RIMS workshop. The author thanks Kazuki Marimoto for his kind invitation to deliver the talk at this workshop.

Our goal is to survey some background material and recent developments in the study of the category of smooth complex representations of the locally compact groups $GL_n(F)$, where F is a non-Archimedean local (p-adic) field.

The family of general linear groups is often considered a prototypical case for the representation theory of p-adic groups. Aside from standing as a natural counterpart to the classical study of continuous symmetry through Lie theory, the rise in interest in the spectral properties of p-adic groups came largely due to their role in arithmetics. Roughly, smooth irreducible representations of p-adic groups appear as local components of the adelic point of view on automorphic forms. As such, their classification and understanding became enmeshed with the development of the Langlands program, that aims to describe number-theoretic phenomena in Lie-theoretic means.

In due course, those themes of thinking gave rise to the rather concrete idea of the local Langlands reciprocity, an arithmetic (in the sense of Galois representations related to the p-adic field) description of the collection of smooth irreducible representations of p-adic groups. Indeed, for the case of general linear groups, such conjectural reciprocity was transformed into celebrated theorems, that paved the path for many ongoing developments in various surrounding fields of research. First, it were the works of Bernstein-Zelevinsky [BZ77, Zel80] which reduced the classification of irreducible $GL_n(F)$-representations to

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the supercuspidal case. The later classification of supercuspidal representations was the arithmetics gist of the matter.

In light of this state of affairs, the case of GL_n from the local Langlands perspective may, somewhat misleadingly, look as a fully charted part of the theory. Yet, in this note we deal with problems arising in modern aspects of the mature theory of $GL_n(F)$-representations, for which the traditional picture of the reciprocity for irreducible representations may not be sufficiently revealing. We discuss some attempts to venture systematically beyond the case of irreducible representations and to obtain meaningful descriptions of (the obviously non-semi-simple) categories of representations.

We will modestly focus on some combinatorial issues around finite-length representations that appear naturally in applications.

In Section 2 we will discuss the basic operation of parabolic induction, which for long is considered as the standard inductive mechanism for producing representations of reductive groups. We will discuss reducibility issues and the main standing problems related to it, sketch the elegant links with the geometric Kazhdan-Lusztig theory and survey some recent results.

In Section 3 we will survey how the study of finite-length representations is, perhaps unexpectedly, crucial for treatment of branching problems, such as the celebrated Gan-Gross-Prasad framework.

Section 4 will attempt to treat some categorical aspects of the theory. In particular, we will exhibit how certain problems on p-adic groups representations are equivalent to problems in the quantum group domain, where further tools, such as an explicit graded structure, are available.

Finally, the last section reviews the newly-developed alternative construction of irreducible representations, developed by the author with Erez Lapid and based on the combinatorial Robinson-Schensted-Knuth transform.

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2. Parabolic induction

Let F be a fixed p-adic field. The well-established Harish-Chandra philosophy of cusp forms, when specialized to the p-adic case, points on the parabolic induction functor as a preferred method for constructing and analyzing smooth representations of groups of the forms $G(F)$, for a connected reductive group G.

In more detail, a parabolic subgroup $P < G$ that projects onto a corresponding reductive Levi subgroup $P \to M < G$, produces a smooth complex $G(F)$-representation $\Pi = i_P^G(\pi)$ out of a smooth complex $M(F)$-representation π, by inflating it to $P(F)$ and further inducing it, with a suitable normalization, to the full group G.

While i_P^G is an exact functor between abelian categories of smooth representations, an irreducible π may give rise to a reducible, albeit still of finite-length, representation Π.

Supposing still that π is an irreducible representation, the following basic questions appear, in order of potentially increasing difficulty:

(1) When is the $G(F)$-representation Π irreducible?
(2) What are the irreducible representations in the Jordan-Hölder series of Π?
(3) What are the possible irreducible sub-representations, or quotient representations, of Π?
(4) What is the description of the full sub-representation lattice of Π?

In this generality the questions may prove hard to approach. For one, a satisfactory dictionary of classes of irreducible representations, a highly non-trivial issue on its own for a general group G, may look as a prerequisite.

One specific case for the above problems is the context of the Langlands Quotient Theorem [BW00, XI.2].

Briefly, for each irreducible smooth $G(F)$-representation κ, the LQT associates a triple (M, τ, ϕ), where $M < G$ is a standard (containing a fixed maximal torus of G) Levi subgroup, τ is an irreducible tempered $M(F)$-representation, and ϕ is an unramified positive (in a suitable fixed sense, relative to the root data of G) character of $M(F)$. Then, σ becomes the unique quotient representation of an induced representation $\Sigma(\kappa) = i_P^G(\tau \cdot \phi)$, known as the standard module of κ.

While the LQT, for many purposes, reduces the classification of irreducible $G(F)$-representation to that of irreducible tempered representations of its Levi subgroups, the fine structure of the resulting standard modules $\Sigma(\kappa)$ remains largely unexplored.

For example, in the active field of relative representation theory, one often studies the space $(\kappa^*)^{H(F)}$ of $H(F)$-invariant linear functionals on the space of the representation κ, where $H < G$ is an algebraic subgroup. Due to the favorable nature of tempered representations and convenience of the parabolic induction functor, it is often feasible to construct invariant functionals on the standard module $\Sigma(\kappa)$ rather than on the actual representation of interest κ. Subsequently, the fine information on which constructed functionals factor through the quotient $\Sigma(\kappa) \to \kappa$ requires an improved understanding of the above mentioned questions when applied to standard modules. See [FLO12, Gur15, Off17, Mat21, Mit19, Suz21] for examples of such analysis.

Let us now focus on the case of $G = GL_n$. We will shortcut $G_n = GL_n(F)$ for the rest of this note.

One advantage of this case in the current context is its immediate links with the pivotal Kazhdan-Lusztig theory. Namely, for irreducible G_n-representations κ_1, κ_2, the multiplicity of κ_1 in the Jordan-Hölder series of $\Sigma(\kappa_2)$ equals the value at 1 of a specified Kazhdan-Lusztig polynomial coming from a finite group of permutations.

We recall that those are the polynomials whose computable coefficients describe certain homological dimensions of singularities of Schubert varieties. They are also long-known to describe the decomposition of Verma modules in the classical theory of semisimple Lie algebras.

The relation of the Lie algebra setting to that of p-adic groups is elegantly revealed through Cherednik's construction, which was formalized in the form of Arakawa-Suzuki functors [AS98, Suz00]. The construction, another special feature of the type A setting, produces (degenerate) Hecke algebra modules, which can then be sent to group representations through standard categorical equivalences (see Section 4), while preserving decomposition properties.

More directly, one can see the involvement of the geometry of Kazhdan-Lusztig polynomials in our setting through the Chriss-Ginzburg description of affine Hecke algebras. This approach expresses multiplicities in standard modules through homological dimensions of singularities of varieties associated to L-parameters [CG97, 8.6.23]. In type A, those varieties may be viewed as orbits on moduli spaces of quiver representations, and subsequently compared with Schubert varieties [Zel81].

A convenient feature of this case is that all Levi subgroups of G_n are themselves isomorphic to groups of the form $G_{n_1} \times \cdots \times G_{n_k}$, with $n_1 + \ldots + n_k = n$. Without loss of generality, we may consider only Levi subgroups $M(F)$ of block-diagonal matrices in G_n, with $P(F)$ being the parabolic subgroup generated by $M(F)$ and the subgroup of upper-triangular matrices.

In this sense, we may take the irreducible $\pi = \pi_1 \otimes \cdots \otimes \pi_k$ to consist of a tuple of irreducible G_{n_i}-representations π_i, $i = 1, \ldots, k$, and write the induced representation in a customary (Bernstein-Zelvinsky) product notation

$$\Pi = \pi_1 \times \cdots \times \pi_k := i_P^{G_n}(\pi) \; .$$

The resulting product is associative, and, up to semisimplification, commutative. Thus, some of the problems arising in decomposition of parabolic induction may be stated in the GL_n case as a quest for the structure constants of an algebra relative to a given basis. The algebra here would come as a sum, over all $n \geq 1$, of the Grothendieck groups of representation categories of G_n, with a product defined in terms of parabolic induction. The given basis is that of the irreducible representations.

In other words, given irreducible representations (of groups G_{n_i} of corresponding ranks) π_1, π_2, π_3, we ask about the multiplicity of π_3 in the product $\pi_1 \times \pi_2$. This picture can now be viewed as an affinization of the classical setting of Littlewood-Richardson coefficients that arise as structure constants for the representation theory of permutation groups.

In certain cases, such as when π_1, π_2 belong to a convenient, yet widespread in applications, class known as ladder representations, full combinatorial formulas for the decomposition of $\pi_1 \times \pi_2$ in the spirit of Littlewood-Richardson rules were indeed devised [Gur21b].

In fact, a (computationally non-trivial) knowledge of the values of the relevant Kazhdan-Lusztig polynomials, gives a precise formula for those multiplicities. Namely, one may decompose $\pi_1 \times \pi_2$ into irreducible representations by transferring the problem to that of standard modules using the matrix given by the Kazhdan-Lusztig theory, and then applying the fact that $\Sigma(\pi_1) \times \Sigma(\pi_2)$ is itself, up to simplification, a standard module (See, for example, [Gur19, Section 3.3]).

We note though, that the Kazhdan-Lusztig theory tools are essentially limited to the semisimplified decomposition problems. We are not aware of a known algorithm to determine the sub-representations lattice of a given general product $\pi_1 \times \pi_2$ of irreducible representations.

We mention here two closely related issues on which recent progress was made.

One is the phenomenon of square-irreducible representations, that is, irreducible G_n-representations π, for which $\pi \times \pi$ remains irreducible.

The first example of a non-square-irreducible irreducible representation tracks back to Leclerc's work [Lec03] in the quantum group setting (and is related to the Kashiwara-Saito goemetric phenomenon [KS97]), which is translated into a construction of a G_8-representation.

Lapid-Mínguez [LM18] have since constructed large families of such representations, and characterized the square-irreducibility property within a class known as regular representations. The general characterization is still an open problem and seems to relate to some deep geometric phenomena [LM20] and properties of canonical bases [Kam22].

The square-irreducibility notion gained prominence in Kang-Kashiwara-Kim-Oh [KKKO18] work on categorification of cluster algebras. This has then influenced the discoveries of Lapid-Míngez on the key role played by same notion in the p-adic setting. For one, it was shown [KKKO15, LM16] that for a square-irreducible π and any irreducible G_n-representation π', the product $\pi \times \pi'$ has a unique irreducible sub-representation.

A second issue is that of "long" products: Given a tuple of irreducible representations $\pi_1, \dots, \pi_k$, the sub-representation lattice of $\pi_1 \times \cdots \times \pi_k$ may quickly become too wild to control.

A typical example of unfavorable behavior would be the product $\Pi = \nu^{1/2} \times \nu^{-1/2} \times \nu^{1/2}$, where $\nu^s(a) = |a|_F^s$, $s \in \mathbb{C}$, is the unramified character of $G_1 = F^\times$. While the Steinberg representation of G_2, St, is known to appear as a sub-representation of $\nu^{1/2} \times \nu^{-1/2}$, the trivial G_2-representation, trv, is known to appear as a sub-representation of $\nu^{-1/2} \times \nu^{1/2}$. Thus, both (irreducible) $St \times \nu^{1/2}$ and $\nu^{1/2} \times trv$ are sub-representations of Π. In fact, Π is isomorphic to their direct sum.

Using the wide collection of square-irreducible representations, we were able to isolate conditions of classes of induced representations, for which a unique irreducible sub-representation (or, dually, a quotient representation) exists. This favorable property is highly sought-after, for example, in applications of the theory to branching laws (Section 3) or to meaningful realizations of irreducible representations (Section 5).

Since a standard module $\Sigma(\kappa)$ is known to have a unique irreducible quotient, any quotient representation of a standard module will still possess this property. Motivated by terminology from quantum affine algebras, we call such representations cyclic.

In a work with Mínguez [GM21], we proved that, in the square-irreducible case, when each of the "short" products $\pi_i \times \pi_j$, $1 \le i < j \le k$ is cyclic, the combined product $\pi_1 \times \cdots \times \pi_k$ will be cyclic as well.

Another criterion for well-behaved products comes from normal sequences, which were introduced by Kashiwara-Kim [KK19] in the quiver Hecke algebras setting.

A work of the author [Gur21c] on the RSK construction (see Section 5) imported the notion into the p-adic setting. These are tuples $\pi_1, \dots, \pi_k$ of square-irreducible representation, satisfying certain computable compatibility conditions. Verifying those conditions, which is an attainable task in many cases, again produces induced representations $\pi_1 \times \cdots \times \pi_k$ with a unique irreducible quotient.

3. Branching laws

One application of the study of the fine structure of finite-length G_n-representations is for the description of p-adic branching laws, or, in other terminology, symmetry-breaking operators.

A copy of the group G_n is naturally found as a subgroup in G_{n+1}, realized as the upper-corner matrices. Given a smooth irreducible G_{n+1}-representation π, its restriction to a subgroup $\pi|_{G_n}$ ceases to be irreducible. In fact, as one can intuitively expect from the drastic change in group dimension, the restricted representation is far from being of finite-length.

In this rather wild setting, a celebrated result [AGRS10] (whose proof relies solely on the geometry of the groups and the invariant distributions it may support) claims that each isomorphism class of an irreducible G_n-representation may appear at most once as a quotient of the restricted representation $\pi|_{G_n}$.

Thus, a question arises: For which pairs of irreducible representations (π, σ) (respectively, of the groups G_{n+1}, G_n), a non-zero G_n-intertwining operator $\pi \to \sigma$ exists?

Special cases of this question that deal with representation classes that are closer to harmonic analysis and number theory (through automorphic representations) received much attention under the framework of the (local) Gan-Gross-Prasad conjectures [GGP12].

More precisely, while the existence of an intertwiner as above for *any* pair of tempered representations (π, σ) is a long-known phenomenon [JPSS83], analogous questions for representations of groups of classical type were an important goal set by the GGP program with clear applications to the study of automorphic forms and their L-functions.

A later development [GGP20] of this program was the formulation of similar precise conjectures (that it, a precise list of pairs (π, σ) with an existing intertwiner) to the case of the so-called Arthur-type representations.

On this frontier, the GL_n case of the conjectures has become a non-trivial step, and was recently resolved in the works of the author [Gur22] (providing a proof of one direction) and ultimately of Chan [Cha22] (which supplied a full stand-alone proof).

A crucial step of both proofs is a reduction of the problem to that of finite-length representations, using what is known as the Bernstein-Zelevinsky filtration of the restricted representation.

Essentially going back to tools from the classical papers [BZ77, Zel80] that developed the basics of the theory of smooth representations of p-adic groups, we can produce a filtration

$$\{0\} = V_{n-1} \subseteq V_{n-2} \subseteq \ldots \subseteq V_0 = V$$

of G_n-representations on the space V of the G_{n+1}-representation π. Thus, the study of irreducible quotients of $\pi|_{G_n}$ is decomposed into the study of quotient representations of V_i/V_{i+1}, for $i = 0, \ldots, n-2$, and of the splittings in between the graded parts of the filtration.

This study is further reduced to morphism spaces between Bernstein-Zelevinsky derivatives of irreducible representations.

A convenient description of the BZ-derivatives is the following functorial diagram. Let $\mathrm{Rep}(G_n)$ denote the abelian category of finite-length G_n-representations. The classical operation of taking Whittaker co-invariants of a representations may be thought of as an exact functor $\mathrm{Wh} : \mathrm{Rep}(G_n) \to Vec$ to the category of finite-dimensional vector spaces.

We also recall the Jacquet functor

$$\mathbf{r}_{n_1,n_2} : \mathrm{Rep}(G_{n_1+n_2}) \to \mathrm{Rep}(G_{n_1} \times G_{n_2}) \cong \mathrm{Rep}(G_{n_1}) \times \mathrm{Rep}(G_{n_2}) ,$$

which is left-adjoint to the previously discussed parabolic induction functor relative to the maximal Levi subgroup $G_{n_1} \times G_{n_2} < G_{n_1+n_2}$.

In these terms, for given $0 < k \leq n$, the left and right Bernstein-Zelevinsky derivatives may be defined as the composed exact functors

$$(Id \times \mathrm{Wh}) \circ \mathbf{r}_{n-k,k} : \begin{array}{ccc} \mathrm{Rep}(G_n) & \to & \mathrm{Rep}(G_{n-k}) \\ \pi & \mapsto & \pi^{(k)} \end{array} ,$$

$$(\mathrm{Wh} \times Id) \circ \mathbf{r}_{k,n-k} : \begin{array}{ccc} \mathrm{Rep}(G_n) & \to & \mathrm{Rep}(G_{n-k}) \\ \pi & \mapsto & {}^{(k)}\pi \end{array} .$$

A given irreducible G_n-representation π produces derivative G_{n-k}-representations $\pi^{(k)}$, ${}^{(k)}\pi$ that are of finite-length.

The following basic property [Gur22, Proposition 5.4] of the Bernstein-Zelevinsky filtration makes the connection between branching laws (i.e. a restriction functor) and the domain of finite-length representations (i.e. constructions coming from *parabolic* restriction):

For irreducible representations (π, σ) of the groups G_n, G_{n+1}, and the filtration of $\pi|_{G_n}$ as above, we have an identification of intertwiner spaces

$$\mathrm{Hom}_{G_n}(V_k/V_{k+1}, \sigma) \cong \mathrm{Hom}_{G_{n-k}}(|\det|_F^{1/2} \pi^{(k+1)}, {}^{(k)}\sigma) ,$$

for all $0 < k \leq n$.

In particular, much of the study of branching laws such as the Gan-Gross-Prasad conjectures is reduced to the characterization of quotient representations of BZ-derivatives of irreducible representations.

Indeed, the key step in the proof of [Gur22] for the GGP branching law was an application of the cyclicity criterion for parabolic induction that was mentioned in Section 2. While the original method suggests applying categorical equivalences that translate the problem into one about modules over quantum affine algebras, a later development allowed this part of the argument to be replaced by the main result of [GM21], which dealt with the cyclicity criterion in 'native' p-adic terms.

4. Categorical equivalences

The structure of the category finite-length smooth G_n-representations and of the parabolic induction functors between them (or their adjoint Jacquet functors) enjoys certain universal properties. By that we mean that near equivalent categorical structures appear in various other type A Lie-theoretic settings. We will now focus on the relation of the questions discussed in previous sections with the representation theory of quiver Hecke algebras.

We first recall the theory of Bernstein decomposition, when specialized to the category $\mathrm{Rep}(G_n)$. It presents the category as a product of smaller, concretely defined, abelian categories known as Bernstein blocks.

The easiest block to define would be the principal, or Iwahori-invariant, block. It is the full sub-category of $\mathrm{Rep}(G_n)$ containing all representations whose irreducible sub-quotients are isomorphic to sub-quotients of parabolic induction of an unramified character of a maximal torus in G_n.

Other Bernstein block are defined by the subtle arithmetic data encoded in supercuspidal representations of Levi subgroups of G_n.

A classical theorem [Bor76] claims that the principal block is naturally equivalent to the category of finite-dimensional modules over $H_n(q)$, the affine Hecke algebra associated with GL_n, a concrete finitely generated complex algebra[1].

This line of reasoning for p-adic groups was later developed into satisfactory descriptions for more general Bernstein blocks. Arguing either through the type theory of Bushnell-Kutzko [BK93], or through Heiermann's description [Hei11] of endomorphism algebras of Bernstein projective generators, the resulting picture for our case of GL_n is especially appeasing.

Any Bernstein block of $\mathrm{Rep}(G_n)$ is equivalent as an abelian category to the category of finite-dimensional modules over an algebra of the form $H_{n_1}(q_1) \otimes \cdots \otimes H_{n_t}(q_t)$.

A corollary of this avenue of results is that all categorical aspects of the finite-length representation theory of the groups $\{G_n\}_n$ remain essentially unchanged when moved to the finite-dimensional representation theory of the algebras $\{H_n(q)\}_n$. In particular, parabolic induction and Jacquet functors become equivalent to natural induction and restriction functors under these identifications [Roc02].

One advantage of this equivalence, as mentioned in the previous section, is the geometric Kazhdan-Lusztig or Chriss-Ginzburg [CG97, Chapter 8] realization of affine Hecke algebras. It allows for an encoding of G_n-representations and their decompositions in categories of perverse sheaves on certain nilpotent cone varieties.

While this deep underlying geometry may be difficult to access directly with combinatorial tools, a valuable aid comes from the algebraic theory of quiver Hecke algebras.

These are a family of finitely-generated ($\mathbb{Z}$-)graded algebras that were introduced by Khovanov-Lauda [KL09] and Rouquier [Rou08] (also commonly known as KLR-algebras) with the aim of categorifying quantum groups. In particular, their approach "algebrizes" the celebrated Lusztig categorification [Lus90, Lus91] of quantum groups by perverse sheaves on moduli spaces of quiver representations.

We will deal with the type A case, in which quiver Hecke algebras may be presented as a sequence of graded algebras $\{R_n\}_n$. In this case, Rouquier [Rou12, Theorem 3.11]

[1]Here q is the cardinality of the finite residue field of F and is used as a parameter in the presentation of the algebra. Yet, its value, beyond being a non-root of unity, is largely irrelevant for the categorical discussion. Thus, $q \in \mathbb{C}$ may be treated as a formal variable.

and Brundan-Kleshchev [BK09] have shown that R_n is closely related to the affine Hecke algebra $H_n(q)$.

Let us write $\mathcal{C}_n$ for the category of graded finite-dimensional modules of R_n, and $\widetilde{\mathcal{C}}_n$ for the same category when forgetting the graded structure of the modules.

Rouquier's equivalence identifies $\widetilde{\mathcal{C}}_n$ with a full sub-category of $H_n(q)$-modules, which essentially captures all of the finite-dimensional representation theory of $H_n(q)$.

Moreover, combining the equivalences for $n \geq 1$, we obtain monoidal functors going from $\oplus_n \mathcal{C}_n$ into $\oplus_n \operatorname{Rep}(G_n)$, where the product structure on the quiver Hecke algebra side is what is known as the convolution product (which plays the role of categorification of the quantum group product) and the product structure on the p-adic side is the Bernstein-Zelevinsky parabolic induction product.

The reader may consult [Gur21c, Section 3] for more details.

Thus, for many purposes we may view $\mathcal{C}_n$ as the category $\operatorname{Rep}(G_n)$ enriched with a (in hindsight, inherent) graded structure. We remark that the mentioned geometric approach already points at a natural graded structure coming from the homological degree of complexes of sheaves. This is the structure algebraically visible in quiver Hecke algebras, while typically being hidden in the p-adic approach.

Indeed, the additional graded structure was recently exploited to tackle specific problems on decomposition of finite-length G_n-representations.

Given a graded module $M = \oplus_{i\in\mathbb{Z}} M_i$ in $\mathcal{C}_n$, we may shift its grading $M\langle r\rangle = \oplus_{i\in\mathbb{Z}} M_{i-r}$ by an integer r to produce a non-isomorphic module. A duality functor $M \mapsto M^*$ on $\mathcal{C}_n$ exists, for which $M\langle r\rangle^* \cong M^*\langle -r\rangle$ holds. When M is a simple module, a unique shift $r_M \in \mathbb{Z}$ exists, for which $M\langle r_M\rangle$ becomes self-dual.

Now, recall again the product decomposition problem of Section 2, where π_1, π_2 were taken as irreducible representations of G_{n_1}, G_{n_2}, and the multiplicities $m^{\pi_3}_{\pi_1,\pi_2}$ of an irreducible π_3 in the Jordan-Hölder series of $\pi_1 \times \pi_2$ were an object of study.

In the graded setting, we may pose an equivalent problem, where $M(\pi_1), M(\pi_2), M(\pi_3)$ are taken as the corresponding self-dual simple quiver Hecke algebra modules. Yet, now we may ask for the multiplicity $m^{\pi_3,r}_{\pi_1,\pi_2}$ of $M(\pi_3)\langle r\rangle$ in the convolution product $M(\pi_1) \circ M(\pi_2)$, separately for any $r \in \mathbb{Z}$.

We see a Laurent polynomial (a quantized multiplicity) $m^{\pi_3}_{\pi_1,\pi_2}(q) = \sum_{r\in\mathbb{Z}} m^{\pi_3,r}_{\pi_1,\pi_2} q^r$, for which the original multiplicity $m^{\pi_3}_{\pi_1,\pi_2}$ constitutes its value at $q = 1$.

Analysis of the quantized multiplicities in a given finite-length module is a computationally convenient middle ground between the semisimplified decompositions in the Grothendieck group and the categorical decomposition with its added subtleties of the sub-module lattice.

Extension properties of perverse sheaves can in these terms have purely algebraic manifestations (see [KKKO18, Theorem 4.2.1] and its proof): A given finite-length M in $\mathcal{C}_n$ admits a graded filtration $\ldots \subseteq M^{r+1} \subseteq M^r \subseteq \ldots \subseteq M$ of R_n-modules, such that each $r \in \mathbb{Z}$, M^r/M^{r+1} is a semisimple module consisting of the irreducible sub-quotients L in M, for which $L\langle -r\rangle$ is self-dual.

In [KKKO18], numerical invariants were produced which shed light on certain decomposition problems. As discussed above, given a square-irreducible module M in $\mathcal{C}_n$ (i.e. $M \circ M$ is simple) and any simple module N, the product module $M \circ N$ has a unique simple quotient L. Let $\Lambda = \widetilde{\Lambda}(M, N) \in \mathbb{Z}$ stand for the integer for which $L\langle -\Lambda \rangle$ becomes self-dual.

Compatibility of the Λ-invraira nts may be used to define and verify the normal sequence condition from [KK19]. Indeed, this is the main technique applied in [Gur21c], where a passage to the graded setting allowed for a computation of such invariants and a construction of families of normal sequences. As a result we are able to produce families of better understood induced modules, such as the RSK-standard modules of the next section.

5. RSK-standard modules

The cornerstone Zelevinsky classification of irreducible G_n-representations is a combinatorial refinement of the non-supercuspidal part of the local Langlands reciprocity.

For simplicity, let us take π to be an irreducible representation in the principal Bernstein block of $\mathrm{Rep}(G_n)$. A combinatorial gadget $\mathfrak{m}$, known as a Zelevinsky multisegment [Zel80], is attached to π. This is a multiset of pairs of numbers $a, b \in \mathbb{C}$, for which $b - a$ is a non-negative integer. It is convenient to write multisets in additive notation, that is,

$$\mathfrak{m} = [a_1, b_1] + \ldots + [a_t, b_t] \; .$$

Without loss of essential information, we may focus on the case when all a_i, b_i are integer numbers. Thus, we think of the collection of multisegments $\mathfrak{M} = \mathbb{Z}_{\geq 0}[\mathrm{Seg}]$ as a monoid with basis $\mathrm{Seg} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \; : \; a \leq b\}$.

The Zelevinsky classification then takes the form of a bijection $Z : \mathfrak{M} \to \mathrm{Irr}_0 = \sqcup_{n \geq 0} \mathrm{Irr}_0(G_n)$, where the sub-collection of irreducible G_n-representations $\mathrm{Irr}_0(G_n)$ captures, up to supercuspidal data, the categorical role of all irreducible objects in $\mathrm{Rep}(G_n)$.

For given $\mathfrak{m} \in \mathfrak{M}$, Zelevinsky locates the irreducible $Z(\mathfrak{m})$ as a sub-representation of a (possibly reducible) Zelevinsky-standard representation $\zeta(\mathfrak{m})$ constructed by parabolic induction. The similarity with the Langlands Quotient Theorem construction is not coincidental, and both approaches may be viewed as dual to each other through the categorical symmetry known as the Zelevinsky involution.

As discussed in Section 2, the fine structure of representations such as $\zeta(\mathfrak{m})$ may be challenging to access. A quest for further models for construction of irreducible representation, perhaps tailored for specific needs, remains a desirable goal.

One such alternative construction was devised in the work of the author with Lapid [GL21]. Our starting point in this approach are the well-behaved properties of the class of irreducible *ladder* representations. These are representations of the form $Z([a_1, b_1] + \ldots + [a_t, b_t])$, with $a_1 < \ldots < a_t$ and $b_1 < \ldots < b_t$.

While the specialty of this subset of Irr_0 was noted in various Lie-theoretic incarnations, their explicit appearance in the p-adic theory begins in [LM14]. Yet, perhaps the most convincing argument for the basic role of ladder representations is visible when passing through the equivalence with quiver Hecke algebra representations, as in Section 4. Ladder representations are those elements $\pi \in \mathrm{Irr}_0$, for which the grading of the self-dual simple module $M = M(\pi)$ is concentrated in a single degree, i.e. $M_i = \{0\}$, for all $i \neq 0$, when writing $M = \oplus_{i\in\mathbb{Z}} M_i$ as a graded vector space. This perspective was the focal point of the Kleshchev-Ram study [KR10] of homogenous modules.

Now, to each $\pi \in \mathrm{Irr}_0$ we associate a tuple of ladder representations $\sigma_1, \ldots, \sigma_\omega$, and locate π as a sub-representation [GL21, Theorem 1.1] of the parabolic induction representation

$$\Lambda(\pi) = \sigma_1 \times \cdots \times \sigma_\omega \ .$$

We call $\Lambda(\pi)$ the RSK-standard module of π, due to the Robinson-Schensted-Knuth transform which is used to produce its inducing data.

The number $\omega = \omega(\pi)$, which we call the width of π, can be read directly from the Zelevinsky data of π. It is the minimal length of a product of ladder representations, in which π may be found as a sub-quotient representation [Gur19]. Thus, our model $\Lambda(\pi)$ is in a suitable sense minimal, when taking the view of ladder representations as basic construction blocks.

The RSK construction becomes somewhat more transparent when moved into the quiver Hecke algebra setting, as was conducted in [Gur21c]. The categorical equivalences of Section 4 were a crucial tool to show that π is indeed the *unique* sub-representation of $\Lambda(\pi)$. In fact, the tuple of ladder representations $\sigma_\omega, \ldots, \sigma_1$ was shown to be an example of a normal sequence.

Moreover, one can apply the mechanism of BZ-derivatives on RSK-standard to produce a plethora of additional "derived" models for irreducible representations. In [Gur21a] it was observed that by picking suitable derivatives on the RSK construction, one can reconstruct the LQT standard module, the Zelevinsky-standard module and the various Specht modules inflated from cyclotomic Hecke algebra quotients. Thus, the new approach seems to be of a universal nature, whose underlying geometric pinning may yet to be discovered.

Some open intriguing problems regarding the RSK model are still standing. Specifically, the nature of the Jordan-Hölder series of the representations $\Lambda(\pi)$, though should conceptually be shorter than that of the standard $\Sigma(\pi)$, remains shrouded with mystery, on which some ideas were conjectured in [GL21, Section 6].

We remark that the derivative procedures of [Gur21a] preserve the decomposition multiplicities. Thus, the multiplicities hidden in RSK-standard modules should in principle contain the information encoded both in Kazhdan-Lusztig polynomials and in Specht decomposition numbers (which are usually a common source of interest in the modular setting).

References

[AGRS10] Avraham Aizenbud, Dmitry Gourevitch, Stephen Rallis, and Gérard Schiffmann. Multiplicity one theorems. *Ann. of Math. (2)*, 172(2):1407–1434, 2010.

[AS98] Tomoyuki Arakawa and Takeshi Suzuki. Duality between $\mathfrak{sl}_n(\mathbf{C})$ and the degenerate affine Hecke algebra. *J. Algebra*, 209(1):288–304, 1998.

[BK93] Colin J. Bushnell and Philip C. Kutzko. *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.

[BK09] Jonathan Brundan and Alexander Kleshchev. Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras. *Invent. Math.*, 178(3):451–484, 2009.

[Bor76] Armand Borel. Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup. *Invent. Math.*, 35:233–259, 1976.

[BW00] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive p-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[CG97] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Birkhäuser Boston, Inc., Boston, MA, 1997.

[Cha22] Kei Yuen Chan. Restriction for general linear groups: the local non-tempered Gan-Gross-Prasad conjecture (non-Archimedean case). *J. Reine Angew. Math.*, 783:49–94, 2022.

[FLO12] Brooke Feigon, Erez Lapid, and Omer Offen. On representations distinguished by unitary groups. *Publ. Math. Inst. Hautes Études Sci.*, pages 185–323, 2012.

[GGP12] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad. Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups. Number 346, pages 1–109. 2012. Sur les conjectures de Gross et Prasad. I.

[GGP20] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad. Branching laws for classical groups: the non-tempered case. *Compos. Math.*, 156(11):2298–2367, 2020.

[GL21] Maxim Gurevich and Erez Lapid. Robinson-Schensted-Knuth correspondence in the representation theory of the general linear group over a non-Archimedean local field. *Represent. Theory*, 25:644–678, 2021. With an appendix by Mark Shimozono.

[GM21] Maxim Gurevich and Alberto Mínguez. Cyclic representations of general linear p-adic groups. *J. Algebra*, 585:25–35, 2021.

[Gur15] Maxim Gurevich. On a local conjecture of Jacquet, ladder representations and standard modules. *Math. Z.*, 281(3-4):1111–1127, 2015.

[Gur19] Maxim Gurevich. Decomposition rules for the ring of representations of non-Archimedean GL_n. *International Mathematics Research Notices*, 2020(20):6815–6855, 02 2019.

[Gur21a] Maxim Gurevich. Graded Specht modules as Bernstein-Zelevinsky derivatives of the RSK model. *arXiv preprint arXiv:2110.11381, to appear in IMRN*, 2021.

[Gur21b] Maxim Gurevich. Quantum invariants for decomposition problems in type A rings of representations. *Journal of Combinatorial Theory, Series A*, 180:105431, 2021.

[Gur21c] Maxim Gurevich. Simple modules for quiver hecke algebras and the robinson-schensted-knuth correspondence. *arXiv preprint arXiv:2106.03120*, 2021.

[Gur22] Maxim Gurevich. On restriction of unitarizable representations of general linear groups and the non-generic local Gan-Gross-Prasad conjecture. *J. Eur. Math. Soc. (JEMS)*, 24(1):265–302, 2022.

[Hei11] Volker Heiermann. Opérateurs d'entrelacement et algèbres de Hecke avec paramètres d'un groupe réductif p-adique: le cas des groupes classiques. *Selecta Math. (N.S.)*, 17(3):713–756, 2011.

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika. Rankin-Selberg convolutions. *Amer. J. Math.*, 105(2):367–464, 1983.

[Kam22] Joel Kamnitzer. Perfect bases in representation theory: three mountains and their springs. *arXiv preprint arXiv:2201.02289*, 2022.

[KK19] Masaki Kashiwara and Myungho Kim. Laurent phenomenon and simple modules of quiver hecke algebras. *Compositio Mathematica*, 155(12):2263–2295, 2019.

[KKKO15] Seok-Jin Kang, Masaki Kashiwara, Myungho Kim, and Se-jin Oh. Simplicity of heads and socles of tensor products. *Compos. Math.*, 151(2):377–396, 2015.

[KKKO18] Seok-Jin Kang, Masaki Kashiwara, Myungho Kim, and Se-jin Oh. Monoidal categorification of cluster algebras. *J. Amer. Math. Soc.*, 31(2):349–426, 2018.

[KL09] Mikhail Khovanov and Aaron D. Lauda. A diagrammatic approach to categorification of quantum groups. I. *Represent. Theory*, 13:309–347, 2009.

[KR10] Alexander Kleshchev and Arun Ram. Homogeneous representations of Khovanov-Lauda algebras. *J. Eur. Math. Soc. (JEMS)*, 12(5):1293–1306, 2010.

[KS97] Masaki Kashiwara and Yoshihisa Saito. Geometric construction of crystal bases. *Duke Math. J.*, 89(1):9–36, 1997.

[Lec03] B. Leclerc. Imaginary vectors in the dual canonical basis of $U_q(\mathfrak{n})$. *Transform. Groups*, 8(1):95–104, 2003.

[LM14] Erez Lapid and Alberto Mínguez. On a determinantal formula of Tadić. *Amer. J. Math.*, 136(1):111–142, 2014.

[LM16] Erez Lapid and Alberto Mínguez. On parabolic induction on inner forms of the general linear group over a non-archimedean local field. *Selecta Math. (N.S.)*, 22(4):2347–2400, 2016.

[LM18] Erez Lapid and Alberto Mínguez. Geometric conditions for $\square$-irreducibility of certain representations of the general linear group over a non-archimedean local field. *Adv. Math.*, 339:113–190, 2018.

[LM20] Erez Lapid and Alberto Mínguez. Conjectures and results about parabolic induction of representations of $\mathrm{GL}_n(F)$. *Invent. Math.*, 222(3):695–747, 2020.

[Lus90] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3(2):447–498, 1990.

[Lus91] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.*, 4(2):365–421, 1991.

[Mat21] Nadir Matringe. Gamma factors of intertwining periods and distinction for inner forms of $GL(n)$. *J. Funct. Anal.*, 281(10):Paper No. 109223, 70, 2021.

[Mit19] Arnab Mitra. On the relationship between distinction and irreducibility of parabolic induction. *C. R. Math. Acad. Sci. Paris*, 357(11-12):827–831, 2019.

[Off17] Omer Offen. On parabolic induction associated with a p-adic symmetric space. *J. Number Theory*, 170:211–227, 2017.

[Roc02] Alan Roche. Parabolic induction and the Bernstein decomposition. *Compositio Math.*, 134(2):113–133, 2002.

[Rou08] Raphaël Rouquier. 2-Kac-Moody algebras. *arXiv preprint arXiv:0812.5023*, 2008.

[Rou12] Raphaël Rouquier. Quiver Hecke algebras and 2-Lie algebras. *Algebra Colloq.*, 19(2):359–410, 2012.

[Suz00] Takeshi Suzuki. Representations of degenerate affine Hecke algebra and $\mathfrak{gl}_n$. In *Combinatorial methods in representation theory (Kyoto, 1998)*, volume 28 of *Adv. Stud. Pure Math.*, pages 343–372. Kinokuniya, Tokyo, 2000.

[Suz21] Miyu Suzuki. Classification of standard modules with linear periods. *J. Number Theory*, 218:302–310, 2021.

[Zel80] A. V. Zelevinsky. Induced representations of reductive p-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup. (4)*, 13(2):165–210, 1980.

[Zel81] A. V. Zelevinskiĭ. The p-adic analogue of the Kazhdan-Lusztig conjecture. *Funktsional. Anal. i Prilozhen.*, 15(2):9–21, 96, 1981.

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