LOCAL THETA CORRESPONDENCES FOR QUATERNIONIC DUAL PAIRS AND LANGLANDS PARAMETERS

HIROTAKA KAKUHAMA
OSAKA CENTRAL ADVANCED MATHEMATICAL INSTITUTE
OSAKA METROPOLITAN UNIVERSITY

Abstract. This article is an announcement of a result of the author, which is based on the talk at the conference "Automorphic form, automorphic L-functions and related topics" at Research Institute for Mathematical Science on 26 January 2022. We describe local theta correspondence for quaternionic dual pair of almost equal rank in terms of Langlands parameter when the base field is $\mathbb{R}$ and representations are discrete series.

1. Introduction

Let F be a local field, and let (G, H) be a reductive dual pair over F. Then, the local theta correspondence is a map

$$\theta(-, G) \colon \mathrm{Irr}(H(F)) \to \mathrm{Irr}(G(F)) \cup \{0\}.$$

An important property of the local theta correspondence is a simple relation of the L-parameters of representations corresponding by the map. Assume that the pair (G, H) is either $(\mathrm{Sp}_{2n}, \mathrm{SO}_{2n})$, $(\mathrm{SO}_{2n+2}, \mathrm{Sp}_{2n})$, $(U_n(E), U_n(E))$ or $(U_{n+1}(E), U_n(E))$, and take an L-embedding $\xi \colon {}^L H \to {}^L G$ of the L-groups of H into that of G. Let σ be a tempered irreducible smooth representation of $(H(F))$ having L-parameter ϕ'. Then, we have a sufficient condition to not vanishing of $\theta(\sigma, G)$ in terms of L-parameters, and we have $\theta(\sigma, G)$ has the L-parameter $\xi \circ \phi'$ if it is non-zero (c.f.[GI14, Appendix C]). This is a part of the Adams' conjecture [Ada89] [HKS96, Conjecture 7.2] (See also [Mg11], [GI14, §15.1]).

Now we consider an L-packet $\Pi_{\phi'_1}(H(F))$ for a tempered L-parameter ϕ'_1. The local Langlands correspondence provides an injective map $\iota \colon \Pi_{\phi'}(H(F)) \to \mathrm{Irr}(\pi_0(C_{\phi'_1}))$. Here $C_{\phi'_1}$ is the Centralizer of $\mathrm{Im}\phi'_1$ in the Langlands dual $\widehat{H}$ and $\pi_0(C_{\phi'_1})$ is its component group. For $\sigma_2 \in \mathrm{Irr}(H(F))$, we call (ϕ'_2, η'_2) the Langlands parameter of σ_2 if ϕ'_2 is the L-parameter of σ_2 and $\eta'_2 = \iota(\sigma_2)$. Then, it is natural to ask how η' and η are related where (ϕ', η') is the Langlands parameter of σ and $(\xi \circ \phi', \eta)$ is that of $\theta(\sigma, G)$. Prasad had conjectured it [Pra93] [Pra00] and Gan-Ichino [GI16], Atobe [Ato18] proved them (see also [AG17]).

In this article, we discuss extending the above results to quaternionic dual pairs, which contains the following topics.

- We use the theory of Kaletha, which formulates the local Langlands correspondence for each rigid inner twist of a quasi-split reductive group. Since the local theta correspondence is not well-defined in the isomorphism classes of irreducible representations with the rigid inner twist, we need to classify the rigid inner twists associated with a fixed rigid inner form. To do this, we introduce the "orientations of tori". Moreover, for a quaternionic dual pair (U_+, U_-), we discuss the natural "orientations of tori" associated with the embedding $U_+ \times U_- \to \mathrm{Sp}(\mathbb{W})$.
- In the case $F = \mathbb{R}$, the local theta correspondence has been described in terms of Harish-Chandra parameters [LPTZ03]. By translating the Harish-Chandra parameter into the

Langlands parameter, we describe the local theta correspondence in terms of Langlands parameters.

- Finally, we will discuss the prospection of the non-Archimedean case.

2. Local Langlands correspondence

2.1. **Rigid inner twists.** Let F be a local field of characteristic zero. Kaletha defined the multiplicative pro-algebraic group u over F, and proved that

$$H^2(\Gamma, u) = \begin{cases} \widehat{\mathbb{Z}} & F \text{ is non-Archimedean,} \\ \mathbb{Z}/2\mathbb{Z} & F = \mathbb{R} \end{cases}$$

([Kal16]). Then, there exists a group $\mathcal{W}$ equipped with the exact sequence

$$1 \to u(\overline{F}) \to \mathcal{W} \to \Gamma \to 1$$

associated with the cohomology class $-1 \in H^2(\Gamma, u)$. Let G be a connected reductive group over F and let Z be a central finite sub-group of G defined over F. Then, we define

$$Z^1(u \to \mathcal{W}, Z \to G)$$

the set of the 1-cocycles $f \in Z^1(\mathcal{W}, G)$ so that $f(u(\overline{F})) \subset Z(\overline{F})$ and $f|_{u(\overline{F})} \colon u(\overline{F}) \to Z(\overline{F})$ is a homomorphism. Moreover we define

$$H^1(u \to \mathcal{W}, Z \to G) = Z^1(u \to \mathcal{W}, Z \to G)/B^1(\mathcal{W}, G)$$

where $B^1(\mathcal{W}, G)$ is the set of the 1 co-boundaries. A rigid inner twist of G is a pair (z, φ) where z is a 1 co-cycle belonging to $Z^1(u \to \mathcal{W}, Z \to G)$ and φ is an isomorphism over $\overline{F}$ from G to a reductive group G_1 over F satisfying

$$\varphi^{-1} \circ w \circ \varphi \circ w^{-1} = \operatorname{Int} z(w) \quad (w \in \mathcal{W}).$$

Fact 2.1. *(*[Kal16, Corollary 3.8]*) If Z contains the center $Z(G_{der})$ of the derived group of G, then the natural map*

$$H^1(u \to \mathcal{W}, Z \to G) \to H^1(\Gamma, G/Z(G))$$

is surjective.

2.2. **Orientations of tori.** Let G be a reductive group over F, and let $S \subset G$ be a torus defined over F. In this article, by the orientation of S we mean a basis $\partial = (\partial_1, \dots, \partial_r)$ of the lattice $X^*(S)$ of the algebraic characters of S. Let S' be another torus of G defined over F, and let $\partial' = (\partial'_1, \dots, \partial'_r)$ be an orientation of S'. Then we say (S, ∂) and (S', ∂') are *rationally equivalent* if there exists an element $g \in G(F)$ so that $S' = gSg^{-1}$ and

$$(\operatorname{Int} g)^* \partial'_1 = \partial_1, \dots, (\operatorname{Int} g)^* \partial'_r = \partial_r.$$

As explained in the introduction, the orientations of tori have an important role when we study the inner twists.

Let $G^{\#}$ be a quasi-split inner form of G defined over F, let Z be a central finite subgroup of $G^{\#}$, and let $S^{\#}$ be a torus of $G^{\#}$ defined over F. We denote by $X^*(S^{\#})$ the lattice of algebraic characters of $S^{\#}$. The torus $S^{\#}$ is said to be fundamental if the $\mathbb{Z}$-rank of $X^*(S^{\#})^{\Gamma}$ is as small as possible.

Fact 2.2. *(*[Kal16, Corollary 3.7]*) If $S^{\#}$ is fundamental, then the natural map*

$$H^1(u \to \mathcal{W}, Z \to S^{\#}) \to H^1(u \to \mathcal{W}, Z \to G^{\#})$$

is surjective.

Definition 2.3. *Let* $(z,\varphi),(z',\varphi')$ *be rigid inner twists such that* z,z' *are contained in* $Z^1(u\to\mathcal{W},Z\to S^{\#})$ *and* φ,φ' *are isomorphism from* $G^{\#}$ *to the same group* G*. In this article, we say the two rigid inner twists* (z,φ) *and* (z',φ') *are* $S^{\#}$-rational equivalent *if there exist* $a\in S^{\#}(\overline{F})$ *and* $g\in G(F)$ *such that* $\varphi'=\operatorname{Int} g\circ\varphi\circ\operatorname{Int} a$ *and* $z'(w)=a^{-1}z(w)w(a)$ *for all* $w\in\mathcal{W}$*.*

Let (z,φ) be a rigid inner twist where z is contained in $Z^1(u\to\mathcal{W},Z\to S^{\#})$ and φ is an isomorphism $G^{\#}\to G$. Then, one can show that $\varphi(S^{\#})$ is a torus defined over F. Moreover, if we fix an orientation $\partial^{\#}$ of $S^{\#}$, then $(\varphi^{-1})^*\partial^{\#}$ is an orientation of S. If (z',φ') is another rigid inner twist which is $S^{\#}$-rational equivalent to (z,φ), then $(\varphi'(S^{\#}),({\varphi'}^{-1})^*\partial^{\#})$ and $(\varphi(S^{\#}),(\varphi^{-1})^*\partial^{\#})$ are rationally equivalent. Conversely, we have the following lemma.

Lemma 2.4. *Let* $(z,\varphi),(z',\varphi')$ *be rigid inner twists as in Definition 2.3. If* $(\varphi'(S^{\#}),({\varphi'}^{-1})^*\partial^{\#})$ *and* $(\varphi(S^{\#}),(\varphi^{-1})^*\partial^{\#})$ *are rationally equivalent, then* (z,φ) *and* (z',φ') *are* $S^{\#}$*-rationally equivalent.*

2.3. **Splittings and Whittaker data.** We use the setting of §2.2. Let $T_0^{\#}$ be a maximal split torus of $G^{\#}$ defined over F, let $T^{\#}$ be a maximal torus over F which contains $T_0^{\#}$, and let $B^{\#}$ be a Borel subgroup containing $T^{\#}$. Note that $B^{\#}$ is not defined over F in general. Take a root vector X_α for each $\alpha\in\Delta^0_{B^{\#}}$. Then, the triple $(T^{\#},B^{\#},\{X_\alpha\}_{\alpha\in\Delta^0_{B^{\#}}})$ consists a splitting of G. Moreover, take a non-trivial unitary character $\psi\colon F\to\mathbb{C}^\times$. Once the splitting $(T^{\#},B^{\#},\{X_\alpha\}_{\alpha\in\Delta^0_{B^{\#}}})$ and the character ψ are given, we obtain the Whittaker datum $\mathfrak{a}=(B^{\#},\psi\circ\lambda)$. Here, λ is a character on the unipotent radical of $B^{\#}$ so that $\lambda(X_\alpha)=1$ for all $\alpha\in\Delta^0_{B^{\#}}$.

Fact 2.5. *(*[Ato18, §2]*) If* $G^{\#}$ *is either a symplectic group or a quasi-split special orthogonal group, then the equivalence classes of Whittaker data are parametrized by* $c\in F^\times/{F^\times}^2$ *and* ψ*.*

2.4. **A map $l_{\mathfrak{a}}$.** We use the setting of §2.3. But, in this section, we assume that $F=\mathbb{R}$. See also §4 below for this assumption. Moreover, we assume that $G^{\#}$ is either Sp_{2m} or $\mathrm{SO}(2n,\mathrm{sgn}^n)$, which possesses an anisotropic maximal torus over $\mathbb{R}$. We denote it by $S^{\#}$. Let Z be the center of G, let $T_0^{\#}$ be a maximal split torus of $G^{\#}$ defined over $\mathbb{R}$, and let $T^{\#}$ be a maximal torus over $\mathbb{R}$ which contains $T_0^{\#}$. Set

$$\mathfrak{I}(T^{\#},S^{\#})=\{g\in G^{\#}(\overline{F})\mid gS^{\#}g^{-1}=T^{\#}\}.$$

Then we define the map

$$l_{\mathfrak{a}}\colon\mathfrak{I}(T^{\#},S^{\#})\to H^1(\Gamma,S^{\#})$$

as follows. Take $g\in\mathfrak{I}(T^{\#},S^{\#})$. We denote by $\omega_{S^{\#}}(\tau)\in W(G^{\#},T^{\#})$ the Weyl element defined by $\operatorname{Ind} g\tau(g)^{-1}$. Then, as in [LS87, (2.3)] we obtain the 1-cocycle $\tau\mapsto m_{S^{\#}}(\tau)$. Then, we define $l_{\mathfrak{a}}(g)$ by

$$l_{\mathfrak{a}}(g)(\tau)=g^{-1}m_{S^{\#}}(\tau)\tau(g)\times\begin{cases}(-1)^{\epsilon_\psi/\sqrt{-1}} & G^{\#}=\mathrm{Sp}(2m)\\ 1 & G^{\#}=\mathrm{SO}(2n,\mathrm{sgn}^n),\end{cases}$$

Here, ϵ_ψ denotes $\epsilon(1/2,\mathrm{sgn},\psi)$.

2.5. **Local Langlands correspondence.** We use the setting of §2.4. Let $B^{\#}$ be a Borel subgroup containing $T^{\#}$, and let $\Delta^0_{B^{\#}}$ be a basis of the positive system $\Delta_{B^{\#}}$. Note that $B^{\#}$ is not defined over $\mathbb{R}$ in general. Let $\widehat{G}$ be the Langlands dual group of $G^{\#}$ equipped with a splitting $(\mathcal{T},\mathcal{B},\{Y_{\beta'}\}_{\beta'\in\Delta^0_{\mathcal{B}}})$. More precisely, they are characterized by the following properties.

- The group $\widehat{G}$ is a complex connected reductive group, $\mathcal{B}$ is a Borel subgroup of $\widehat{G}$, $\mathcal{T}$ is a maximal torus contained in $\mathcal{B}$, $Y_{\beta'}$ is a root vector associated with a root β' in a basis $\Delta^0_{\mathcal{B}}$ of the positive system $\Delta_{\mathcal{B}}$ in $R(\widehat{G},\mathcal{T})$.

- There are isomorphisms $\mathfrak{D}_1 : X^*(T^\#) \to X_*(\mathcal{T})$ and $\mathfrak{D}_2 : X^*(\mathcal{T}) \to X_*(T^\#)$ so that $\mathfrak{D}_1(\Delta^0_{B^\#}) = (\Delta^0_{\mathcal{B}})^\vee$, $\mathfrak{D}_2(\Delta^0_{\mathcal{B}}) = (\Delta^0_{B^\#})^\vee$, and $\mathfrak{D}_2(\mathfrak{D}_1(\alpha)^\vee) = \alpha^\vee$ for $\alpha \in R(G^\#, T^\#)$.

There is a Γ-action on $\widehat{G}$ such that it preserves $\mathcal{T}$ and $\mathfrak{D}_1$, $\mathfrak{D}_2$ are Γ-isomorphisms. We denote by ${}^LG$ the semi-product $\widehat{G} \rtimes \Gamma$ by this action.

Take an orientation $\partial_{T^\#} = (\partial_{T^\# 1}, \ldots, \partial_{T^\# r})$ of $T^\#$ that is positive with respect to $\mathcal{B}$. Fix a Whittaker datum $\mathfrak{a}$, and take $g_0 \in \mathcal{I}(T^\#, S^\#)$ so that $l_{\mathfrak{a}}(g_0) = 1$. We denote by $S_{T^\#}$ the new $\mathbb{R}$-structure of $T^\#$ so that $\operatorname{Int} g_0 : S^\# \to S_{T^\#}$ is defined over $\mathbb{R}$. Then, we take a 1-cocycle $z_{T^\#} \in Z^1(u \to \mathcal{W}, Z \to S_{T^\#})$, and isomorphism $\varphi : G^\# \to G$ so that $(\operatorname{Int} g_0^{-1} \circ z_{T^\#}, \varphi)$ is a rigid inner twist of $G^\#$.

Finally, let ϕ be a discrete parameter for $G^\#$. Then, we define

$$C_\phi = \varinjlim_{f \in \phi} \operatorname{Cent}_{\widehat{G}}(\operatorname{Im} f).$$

Moreover, we define

$$S_\phi^+ = \varinjlim_{f \in \phi} p^{-1}(\operatorname{Cent}_{\widehat{G}}(\operatorname{Im} f))$$

where p is the canonical covering $\widehat{G/Z} \to \widehat{G}$. Since ϕ is a discrete parameter, C_ϕ and S_ϕ^+ are finite groups. We denote by $\iota[z_{T^\#}, g_0, \varphi, \mathfrak{a}]$ the injective map

$$\Pi_\phi(G(F)) \to \operatorname{Irr}(S_\phi^+)$$

of [Kal16, (5.7)], which is associated with the rigid inner twist $(\operatorname{Int} g_0^{-1} \circ z_{T^\#}, \varphi)$ and the Whittaker datum $\mathfrak{a}$.

Proposition 2.6. *Take another $z'_{T^\#} \in Z^1(u \to \mathcal{W}, Z \to S_{T^\#})$, $g'_0 \in \mathcal{I}(T^\#, S^\#)$, and $\varphi' : G^\# \to G$ so that $(\operatorname{Int} {g'_0}^{-1} \circ z'_{T^\#}, \varphi')$ consists a rigid inner twist. Let S be an anisotropic maximal torus of G over F, and let ∂ be an orientation on S.*

(1) Suppose that $z_{T^\#} = z'_{T^\#}$ and both $(\varphi(S^\#), (\operatorname{Int} g_0^{-1} \circ \varphi^{-1})^* \partial_{T^\#})$ and $(\varphi'(S^\#), (\operatorname{Int} {g'_0}^{-1} \circ {\varphi'}^{-1})^* \partial_{T^\#})$ are rationally equivalent to (S, ∂). Then, we have

$$\iota[z_{T^\#}, g_0, \varphi, \mathfrak{a}] = \iota[z_{T^\#}, g'_0, \varphi', \mathfrak{a}].$$

Hence, we denote by $\iota[z_{T^\#}, \partial, \mathfrak{a}]$ instead of by $\iota[z_{T^\#}, g_0, \varphi, \mathfrak{a}]$.

(2) If $z_{T^\#}$ and $z'_{T^\#}$ represent the same cohomology class, then we have

$$\iota[z_{T^\#}, \partial, \mathfrak{a}] = \iota[z'_{T^\#}, \partial, \mathfrak{a}].$$

3. Reductive dual pairs (Archimedean case)

In this section, we assume $F = \mathbb{R}$.

3.1. **Local theta correspondence.** Let

$$\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + ij\mathbb{R}$$

be the skew field of Hamilton's quaternions. Here the symbols i and j satisfy the relations

$$i^2 = -1,\ j^2 = -1,\ \text{and } ij + ji = 0.$$

We denote by $V = V_{p,q}$ the $m = p + q$ dimensional $\mathbb{H}$-vector space of column vectors equipped with the Hermitian form $(\ ,\ ) : V \times V \to \mathbb{H}$ given by

$$\left(\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \right) = \sum_{k=1}^{p} x_k^* \cdot y_k - \sum_{l=p+1}^{m} x_l^* \cdot y_l.$$

We denote by $W = W_n$ the n dimensional $\mathbb{H}$-vector space of row vectors equipped with the skew-Hermitian form $\langle\ ,\ \rangle\colon W \times W \to \mathbb{H}$ given by

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum_{k=1}^{n} x_k \cdot j \cdot y_k^*.$$

Then, $\mathbb{W} = V \otimes_{\mathbb{H}} W$ is a symplectic space with the symplectic form given by

$$\langle\langle x \otimes y, x' \otimes y' \rangle\rangle = T_{\mathbb{H}}((x, x') \cdot \langle y, y' \rangle^*)$$

for $x, x' \in V$ and $y, y' \in W$. Here, $T_{\mathbb{H}}$ denotes the reduced trace of $\mathbb{H}$ over $\mathbb{R}$. Let us denote by U_+ the unitary group of V, and by U_- the unitary group of W. Then, the action of $U_+ \times U_-$ on $\mathbb{W}$ by

$$(h, g) \cdot x \otimes y = h^{-1} x \otimes yg$$

induces a homomorphism $U_+ \times U_- \to \mathrm{Sp}(\mathbb{W})$. In this article, we consider the polar decomposition $\mathbb{W} = \mathbb{X} + \mathbb{Y}$ obtained in the following way. Let V' (resp. V'') be the subspace of V consisting of the vectors

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \qquad (x_1, \dots, x_m \in \mathbb{H})$$

with $x_{p+1} = \cdots = x_m = 0$ (resp. with $x_1 = \cdots = x_p = 0$). Then, we have the decomposition $\mathbb{W} = \mathbb{W}' + \mathbb{W}''$ where

$$\mathbb{W}' = V' \otimes W, \text{ and } \mathbb{W}'' = V'' \otimes W.$$

For both spaces, choose a polar decompositions $\mathbb{W}' = \mathbb{X}' + \mathbb{Y}'$ and $\mathbb{W}'' = \mathbb{X}'' + \mathbb{Y}''$. Then, putting $\mathbb{X} = \mathbb{X}' + \mathbb{X}''$ and $\mathbb{Y} = \mathbb{Y}' + \mathbb{Y}''$, we have the polar decomposition $\mathbb{W} = \mathbb{X} + \mathbb{Y}$. Then, one can construct a 2-cocycle $c_{\mathbb{Y},\psi} \in Z^2(\mathrm{Sp}(\mathbb{W}), \mathbb{C}^1)$ ([Per81], [RR93]) and the extension

$$1 \to \mathbb{C}^1 \to \mathrm{Mp}(\mathbb{W}, c_{\mathbb{Y},\psi}) \to \mathrm{Sp}(\mathbb{W}) \to 1$$

associated with $c_{\mathbb{Y},\psi}$. Moreover, the homomorphism $U_+ \times U_- \to \mathrm{Sp}(\mathbb{W})$ lifts to an embedding $U_+ \times U_- \to \mathrm{Mp}(\mathbb{W}, c_{\mathbb{Y},\psi})$. Such liftings are not unique, but we use the explicit lifting given by Kudla [Kud94]. Let π be an irreducible representation of $U_-(F)$. Then, we define $\Theta_\psi(\pi, V)$ as the largest quotient module

$$(\omega_{\mathbb{Y},\psi} \otimes \pi^\vee)_{U_+}$$

of $\omega_{\mathbb{Y},\psi} \otimes \pi^\vee$ on which $G(W)$ acts trivially. Then, as a consequence of Howe duality (c.f. [GS17]), $\Theta_\psi(\pi, V)$ has the unique non-zero irreducible quotient if it is non-zero. We denote it by $\theta_\psi(\pi, V)$ if $\Theta_\psi(\pi, V)$ is non-zero. If $\Theta_\psi(\pi, V) = 0$, we define $\theta_\psi(\pi, V) = 0$.

3.2. **Associated orientations.** Put $U_+^\# = \mathrm{Sp}_{2m}$ and $U_-^\# = \mathrm{SO}(2n, \mathrm{sgn}^n)$. Then, it is known that they are the quasi-split inner form of U_+ and U_-. Now, we apply the theory of §2 to them. Let $T_+^\#$ be a split maximal torus of $U_+^\#$, let $T_-^\#$ be a centralizer of a maximal split torus in $U_-^\#$, let $S_+^\#$ be an anisotropic maximal torus of $U_+^\#$, let $S_-^\#$ be an anisotropic maximal torus of $U_-^\#$, and let S_- be an anisotropic maximal torus of U_-. Finally, we take an anisotropic maximal torus S_+ of U_+ as follows. Denote by U_+' (resp. U_+'') the unitary group of V' (resp. V''). Let S_+' (resp. S_+'') be an anisotropic maximal torus of U_+' (resp. U_+''). Then we put $S_+ = S_+' \times S_+'' \subset U_+$.

Lemma 3.1. *There is the unique anisotropic maximal torus $S_{\mathbb{W}}$ over $\mathbb{R}$ containing the image of $S_+ \times S_-$. Moreover, putting $S_{\mathbb{W}'} = S_{\mathbb{W}} \cap \mathrm{Sp}(\mathbb{W}')$ and $S_{\mathbb{W}''} = S_{\mathbb{W}} \cap \mathrm{Sp}(\mathbb{W}'')$, we have $S_{\mathbb{W}} = S_{\mathbb{W}'} \times S_{\mathbb{W}''}$.*

Let $(u_1,\dots,u_{2pn},u^*_{2pn},\dots,u^*_1)$ be a symplectic basis of $\mathbb{W}'=\mathbb{X}'+\mathbb{Y}'$, and let $(u_{2pn+1},\dots,u_{2mn},u^*_{2mn},\dots,u^*_{2pn+1})$ be a symplectic basis of $\mathbb{W}''=\mathbb{X}''+\mathbb{Y}''$. Then we define $T_{\mathbb{X}'}$ (resp. $T_{\mathbb{X}''}$) as the torus of diagonal elements of $\mathrm{Sp}(\mathbb{W}')$ (resp. $\mathrm{Sp}(\mathbb{W}'')$) which is regarded a subgroup of GL_{2np} (resp. GL_{2nq}) via the basis. We choose the Borel subgroup $B_{\mathbb{X}}$, (resp. $B_{\mathbb{X}'}$, $B_{\mathbb{X}''}$) of $\mathrm{Sp}(\mathbb{W})$ (resp. $\mathrm{Sp}(\mathbb{W}')$, $\mathrm{Sp}(\mathbb{W}'')$) as the set of the upper triangle matrices via the basis. Let $\mathfrak{a}_{\mathbb{W}}$, (resp. $\mathfrak{a}_{\mathbb{W}'}$, $\mathfrak{a}_{\mathbb{W}''}$) be the Whittaker datum of $\mathrm{Sp}(\mathbb{W})$ (resp. $\mathrm{Sp}(\mathbb{W}')$, $\mathrm{Sp}(\mathbb{W}'')$) which is parametrized by $c=1$ and ψ above (c.f. Fact 2.5). Take $h'_0\in\mathcal{I}(T_{\mathbb{X}'},S_{\mathbb{W}'})$ and $h''_0\in\mathcal{I}(T_{\mathbb{X}''},S_{\mathbb{W}''})$ such that $l_{\mathfrak{a}_{\mathbb{W}'}}(h'_0)=1$ and $l_{\mathfrak{a}_{\mathbb{W}''}}(h''_0)=-1$. Then, we have $l_{\mathfrak{a}_{\mathbb{W}}}(h_0)=1$ for $h_0=(h'_0,h''_0)\in U'_+\times U''_+\subset U_+$. Now we explain the natural orientations on S_+ and S_-, which we call the *associated orientations*.

Proposition 3.2. *(1) There exists an orientation $\partial_+=(\partial_{+,1},\dots,\partial_{+,m})$ on S_+ such that*

$$\mathbb{X}'\otimes\mathbb{C}\cdot h_0^{-1}=(\partial_{+,1}^{\oplus 2n})\oplus\cdots\oplus(\partial_{+,p}^{\oplus 2n}),$$
$$\mathbb{X}''\otimes\mathbb{C}\cdot h_0^{-1}=(\partial_{+,p+1}^{\oplus 2n})\oplus\cdots\oplus(\partial_{+,m}^{\oplus 2n}),$$
$$\mathbb{Y}'\otimes\mathbb{C}\cdot h_0^{-1}=({\partial_{+,1}^{-1}}^{\oplus 2n})\oplus\cdots\oplus({\partial_{+,p}^{-1}}^{\oplus 2n}),\ \text{and}$$
$$\mathbb{Y}''\otimes\mathbb{C}\cdot h_0^{-1}=({\partial_{+,p+1}^{-1}}^{\oplus 2n})\oplus\cdots\oplus({\partial_{+,m}^{-1}}^{\oplus 2n})$$

as algebraic representations of S_+.

(2) There exists an orientation $\partial_-=(\partial_{-,1},\dots,\partial_{-,n})$ on S_- such that

$$\mathbb{X}\otimes\mathbb{C}\cdot h_0^{-1}=(\partial_{-,1}^{\oplus 2m})\oplus\cdots\oplus(\partial_{+,n}^{\oplus 2m}),\ \text{and}$$
$$\mathbb{Y}'\otimes\mathbb{C}\cdot h_0^{-1}=({\partial_{-,1}^{-1}}^{\oplus 2m})\oplus\cdots\oplus({\partial_{-,n}^{-1}}^{\oplus 2m})$$

as algebraic representations of S_-.

(3) The orientation (S_+,∂_+) (resp. (S_-,∂_-)) satisfying (1) *(resp.* (2)*) is determined uniquely up to rationally equivalence.*

3.3. **Correspondence of Langlands parameters.** In this subsection, we assume either $n=m$ or $n=m+1$. Let $\mathcal{T}_+$ (resp. $\mathcal{T}_-$) be the torus, and let $\mathcal{B}_+$ (resp. $\mathcal{B}_-$) be the Borel subgroup of $\widehat{U_+}$ (resp. $\widehat{U_-}$) as in §2.5. In particular, there are fixed Γ-isomorphisms

$$\mathfrak{D}_1^+\colon X^*(T_+^{\#})\to X_*(\mathcal{T}_+),\quad \mathfrak{D}_1^-\colon X^*(T_-^{\#})\to X_*(\mathcal{T}_-)$$

$$\mathfrak{D}_2^+\colon X^*(\mathcal{T}_+)\to X_*(T_+^{\#}),\quad \mathfrak{D}_2^-\colon X^*(\mathcal{T}_-)\to X_*(T_-^{\#}).$$

We take the Whittaker data $\mathfrak{a}_+,\mathfrak{a}_-$ of U_+,U_- respectively so that both are parametrized by the same $c\in F^\times/{F^\times}^2$ and ψ. We take an L-embedding

$$\begin{cases}\xi\colon {}^LU_-\to{}^LU_+ & \text{in the case } n=m,\\ \xi\colon {}^LU_+\to{}^LU_- & \text{in the case } n=m+1\end{cases}$$

so that $\xi^{-1}(\mathcal{B}_+)=\mathcal{B}_-$ or $\xi^{-1}(\mathcal{B}_-)=\mathcal{B}_+$ respectively. Moreover, we assume that

$$\begin{cases}\xi^{-1}(\partial_{T_+^{\#}})=\partial_{T_-^{\#}} & m=n,\\ \xi^{-1}(\partial_{T_-^{\#}})=\partial_{T_+^{\#}} & n=m+1.\end{cases}$$

Let ϕ be a discrete parameter for ${}^LU_-$ (resp. ${}^LU_+$) when $n=m$ (resp. when $n=m+1$). The L-embedding ξ induces the homomorphism of S-groups

$$\overline{\xi}\colon S_\phi^+\to S_{\xi\circ\phi}^+.$$

Moreover, ξ induces $t_\xi\colon T_{S_+^\#} \to T_{S_-^\#}$ (resp. $t_\xi\colon T_{S_-^\#} \to T_{S_+^\#}$) if $n=m$ (resp. if $n=m+1$), and induces

$$\begin{cases} t_\xi\colon H^1(u\to\mathcal{W}, Z\to T_{S_+^\#}) \to H^1(u\to\mathcal{W}, Z\to T_{S_-^\#}) & \text{if } n=m,\\ t_\xi\colon H^1(u\to\mathcal{W}, Z\to T_{S_-^\#}) \to H^1(u\to\mathcal{W}, Z\to T_{S_+^\#}) & \text{if } n=m+1.\end{cases}$$

Now, we state the main result.

Theorem 3.3. *Let $z_{T_+^\#} \in H^1(u\to\mathcal{W}, Z\to T_{S_+^\#})$, let $z_{T_+^\#} \in H^1(u\to\mathcal{W}, Z\to T_{S_-^\#})$, let $g_+ \in \mathcal{I}(T_+^\#, S_+^\#)$ so that $l_{\mathfrak{a}_+}(g_+)=1$, and let $g_- \in \mathcal{I}(T_-^\#, S_-^\#)$ so that $l_{\mathfrak{a}_-}(g_-)=1$. Assume the quaternionic unitary groups U_+ and U_- are associated with $(\operatorname{Int} g_+)^* z_{T_+^\#}$ and $(\operatorname{Int} g_-)^* z_{T_-^\#}$ respectively. Let π be an irreducible discrete series representation belonging to $\Pi_\phi(U_-(\mathbb{R}))$ (resp. $\Pi_\phi(U_+(\mathbb{R}))$) if $n=m$ (resp. if $n=m+1$). If $\sigma=\theta_\psi(\pi,V)\neq 0$ (resp. $\sigma=\theta_\psi(\pi,W)\neq 0$) and $z_{T_-^\#}=t_\xi(z_{T_+^\#})$ (resp. $z_{T_+^\#}=t_\xi(z_{T_-^\#})$), then we have*

$$\begin{cases} \iota[z_{T_+^\#}, \partial_+, \mathfrak{a}_+](\sigma)(\overline{\xi}(s)) = \iota[z_{T_-^\#}, \partial_-, \mathfrak{a}_-](\pi)(s) & \text{if } n=m,\\ \iota[z_{T_-^\#}, \partial_-, \mathfrak{a}_-](\sigma)(\overline{\xi}(s)) = \iota[z_{T_+^\#}, \partial_+, \mathfrak{a}_+](\pi)(s) & \text{if } n=m+1\end{cases}$$

for all $s\in S_\phi^+$. Here, ∂_+ and ∂_- are as in §3.2.

3.4. **Sketch of the proof.** For simplicity, we consider only the case $n=m$ and $\epsilon_\psi=\sqrt{-1}$. We identify the Weil group of $\mathbb{R}$ with $\mathbb{C}^\times \cup j\mathbb{C}^\times \subset \mathbb{H}^\times$. Take $f\in\phi$ so that $f(\mathbb{C}^\times)\subset \mathcal{T}_-$ and $f(j)$ normalizes $\mathcal{T}_-$. Then we define $\mu_f \in X^*(T_-^\#)$ so that

$$\chi(f(re^{i\theta})) = e^{2i\theta\langle\chi,\mu_f\rangle} \quad (r\in\mathbb{R}_{>0}, \theta\in\mathbb{R})$$

for all $\chi\in X^*(\mathcal{T})$. Then one can show that μ_f is regular. Replacing f with f^w for some Weyl element w if necessary, we may assume that f is positive with respect to $\mathcal{B}_-$. First, there exists a surjective map

$$\mathcal{I}(T_-^\#, S_-^\#) \to \Pi_\phi(U_-^\#(F))\colon g\mapsto \pi(g)$$

and $\iota[1,\mathfrak{a}](\pi(g))(s) = \langle l_{\mathfrak{a}}(g), (\operatorname{Int} g^+)^*(s)\rangle$ for $s\in C_\phi$. Here g_+ is an element of $\mathcal{I}(T_-^\#, S_-^\#)$ so that $l_{\mathfrak{a}}(g_+)=1$. Then, $\pi(g_+)$ is nothing other than the unique generic representation in $\Pi_\phi(U_-^\#(F))$. Second, there is a surjective map

$$\mathcal{I}(\mathcal{T}_-^\#, S_-^\#) \to \Pi_\phi(U_-(F))\colon g\mapsto \pi(g)$$

so that the following properties hold.

- The Harish-Chandra parameter of $\pi(g)$ with respect to (S_-,∂_-) is $((\varphi_+^{-1})^*\circ(\operatorname{Int} g)^*)(\mu_f)$. Here φ_+ is the inner twist so that $((\varphi_+^{-1})^*\circ(\operatorname{Int} g)^*)(\partial_{T_-^\#})=\partial_-$.
- We have

  $$\iota[z_{T_-^\#}, \partial_-, \mathfrak{a}_-](\pi(g))(s) = \langle \operatorname{inv}(\pi(g^+),\pi(g)), (\operatorname{Int} g^+)^*(s)\rangle$$

  for all $s\in S_\phi^+$. Here, the 1-cocycle $\operatorname{inv}(\pi(g^+),\pi(g))$ is defined by

  $$\operatorname{inv}(\pi(g^+),\pi(g))(w) = \gamma_+^{-1} z_{p,q}(w) w(\gamma_+) \quad (w\in\mathcal{W})$$

  where $\gamma_+ = g^{-1}g_+$ and $z_{p,q} = (\operatorname{Int} g_+)\circ z_{T_-^\#}$.

See [Kal16, §5.6] and [She82] for more discussions. In terms of the Harish-Chandra parameters, the description of the local Langlands correspondence had been established [LPTZ03]. Denote by $\xi_S\colon X^*(S_-) \to X^*(S_+)$ the isomorphism satisfying $\xi_S(\partial_-) = \partial_+$. Then, their result implies that the following diagram is commutative.

$$
\begin{array}{ccccccc}
X^*(T_+^\#) & \xrightarrow{(\operatorname{Int} g_+)^*} & X^*(S_+^\#) & \xleftarrow{(\operatorname{Int} \gamma_+)^*} & X^*(S_+^\#) & \xleftarrow{\varphi_+^*} & X^*(S_+) \\
\uparrow{\scriptstyle \xi^*} & & & & & & \uparrow{\scriptstyle \varepsilon_{p,q}\circ\xi_S} \\
X^*(T_-^\#) & \xrightarrow[(\operatorname{Int} g_-)^*]{} & X^*(S_-^\#) & \xleftarrow[(\operatorname{Int} \gamma_-)^*]{} & X^*(S_-^\#) & \xleftarrow[\varphi_-^*]{} & X^*(S_-)
\end{array}
\tag{3.1}
$$

Here, $\varepsilon_{p,q}$ is the isomorphism $X^*(S_+) \to X^*(S_+)$ given by

$$
\varepsilon_{p,q}(\partial_{+,k}) = \begin{cases} \partial_{+,k} & k = 1, \ldots, p, \\ -\partial_{+,2p+q+1-k} & k = p+1, \ldots, m, \end{cases}
$$

which is realized as $\operatorname{Int} g_1$ for certain $g_1 \in U_+(\mathbb{R})$. We put $\gamma_1 = \varphi_+^{-1}(g_1) \cdot \gamma$. On the other hand, the following diagram is commutative by definitions.

$$
\begin{array}{ccccc}
X^*(S_+) & \xrightarrow{\varphi_+^*} & X^*(S_+^\#) & \xleftarrow{(\operatorname{Int} g_+)^*} & X^*(T_+^\#) \\
\uparrow{\scriptstyle \xi_S} & & & & \uparrow{\scriptstyle \xi^*} \\
X^*(S_-) & \xrightarrow{\varphi_-^*} & X^*(S_-^\#) & \xleftarrow{(\operatorname{Int} g_-)^*} & X^*(T_-^\#)
\end{array}
\tag{3.2}
$$

Combining (3.1) and (3.2), we have

$$
(\operatorname{Int} \gamma_-) \circ \mathfrak{p} = \mathfrak{p} \circ (\operatorname{Int} g_1 \gamma_+)
$$

by denoting $\mathfrak{p} = ((\operatorname{Int} g_-^{-1}) \circ \xi \circ (\operatorname{Int} g_+))$. Hence, we have

$$
\begin{aligned}
\mathfrak{p}_*(\operatorname{inv}(\pi, \pi_{\mathfrak{a}}))(w) &= \mathfrak{p}(\gamma_-^{-1} z(w) w(\gamma_-)) \\
&= \mathfrak{p}(\gamma_-^{-1} z(w) \gamma_- \cdot \gamma_-^{-1} w(\gamma_-)) \\
&= \gamma_1^{-1} z_{p,q}(w) \gamma_1 \cdot \gamma_1^{-1} w(\gamma_1) \\
&= \gamma_1^{-1} z_{p,q}(w) w(\gamma_1) \\
&= \gamma_+^{-1} z_{p,q}(w) w(\gamma_+) \\
&= \operatorname{inv}(\sigma, \sigma_{\mathfrak{a}})(w).
\end{aligned}
$$

Here, $z = (\operatorname{Int} g_-) \circ z_{T_-^\#}$. Note that we use the fact that $\mathfrak{p}(\gamma_-^{-1} w(\gamma_-))$ is cohomologous to $\gamma_1^{-1} w(\gamma_1)$, which is a consequence of voluminous and routine computations. Therefore we have

$$
\iota[z_{T_+^\#}, \partial_+, \mathfrak{a}_+](\sigma)(\overline{\xi}(s)) = \iota[z_{T_-^\#}, \partial_-, \mathfrak{a}_-](\pi)(s)
$$

and we have the theorem.

4. Non-Archimedean cases

Finally, we explain some prospects for the non-Archimedean cases. It will be possible to complete the formulation if we find the right definition of the map $l_{\mathfrak{a}}\colon \mathcal{I}(T^\#, S^\#) \to H^1(\Gamma, S^\#)$ in the non-Archimedean cases (see §2.4). More precisely, the arguments of §§2.5–3.2 are still available with a few modifications if the definition of $l_{\mathfrak{a}}$ is determined. Then, by using the compatibility between the local theta correspondence and the localization of the global theta correspondence, we will have a lot of examples of the descriptions of the local theta correspondences in terms of Langlands parameters.

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