ON IWAHORI–HECKE ALGEBRAS AND LOCAL L-FACTORS OF UNRAMIFIED REPRESENTATIONS: ANNOUNCEMENT

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This note is an announcement of a recent result of the author obtained in a joint work with Ryotaro Sakamoto and Hiroyoshi Tamori. The details are presented in a preprint [OST21].

Let $\mathbf{G}$ be a connected reductive group over a non-archimedean local field F. (Although our result holds for any connected reductive group, we assume that $\mathbf{G}$ is split over F in this note for simplicity.) We fix a split maximal torus $\mathbf{T}$ of $\mathbf{G}$ and a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. Let $\hat{\mathbf{G}}$ be the Langlands dual group over $\mathbb{C}$. We also fix a split maximal torus $\hat{\mathbf{T}}$ of $\hat{\mathbf{G}}$ and a Borel subgroup $\hat{\mathbf{B}}$ containing $\hat{\mathbf{T}}$. Furthermore, we fix a hyperspecial open compact subgroup K of $G := \mathbf{G}(F)$ so that K is consistent with the choices of $\mathbf{T}$ and $\mathbf{B}$.

Recall that we say that an irreducible smooth representation of G is *unramified* (or *K-spherical*) if π has a nonzero K-fixed vector. The unramified representations of G are one of the most fundamental classes in representation theory of the group G. Their importance can be explained in relation to the global theory, that is, almost all local components of automorphic representations are unramified.

One fundamental result on unramified representations is the construction of the local L-factors. Recall that the structure of the *spherical Hecke algebra* $\mathcal{H}_K := C_c^\infty(G/\!\!/K)$ is described via the *Satake isomorphism* $\mathcal{H}_K \cong \mathbb{C}[X_*(\mathbf{T})]^W$, where $X_*(\mathbf{T})$ is the cocharacter group of $\mathbf{T}$ and W is the Weyl group of $\mathbf{T}$ in $\mathbf{G}$. For any irreducible unramified representation π of G, the spherical Hecke algebra acts on the space π^K of K-fixed vectors and defines a simple $\mathcal{H}_K$-module. Thus, by the Satake isomorphism, we see that π^K is 1-dimensional and its isomorphism class is given by an element of $\mathrm{Hom}(X_*(\mathbf{T}), \mathbb{C}^\times)/W$ (called *Satake parameter of* π, say $s(\pi)$). Through an identification of the dual torus of $\mathbf{T}$ with $\hat{\mathbf{T}}$, we get an isomorphism

$$\mathrm{Hom}(X_*(\mathbf{T}), \mathbb{C}^\times)/W \cong \hat{\mathbf{T}}/W.$$

Then, for any finite-dimensional representation $r\colon \hat{\mathbf{G}} \to \mathrm{GL}_{\mathbb{C}}(V)$ of $\hat{\mathbf{G}}$, we can define the associated *local L-factor* $L(s, \pi, r)$ by

$$L(s, \pi, r) := \det(1 - q^{-s} \cdot r(s(\pi)) \mid V)^{-1},$$

where q denotes the cardinality of the residue field of F.

In order to state our main result, we introduce a few more notations. Let $\{u_\alpha\colon \mathbb{G}_{\mathrm{a}} \cong \mathbf{U}_\alpha\}_{\alpha\in\Phi}$ be the Chevalley basis of $\mathbf{G}$ corresponding to K, where Φ denotes the set of roots of $\mathbf{T}$ in $\mathbf{G}$ and $\mathbf{U}_\alpha$ denotes the root subgroup of $\alpha \in \Phi$. For a dominant cocharacter $\mu \in X_*(\mathbf{T})$ (i.e., $\langle\alpha, \mu\rangle \geq 0$ for any positive root α), we define an open compact subgroup J_μ of K by

$$J_\mu := \left\langle \mathbf{T}(\mathcal{O}_F), u_\alpha(\mathcal{O}_F), u_\beta(\mathfrak{p}_F) \;\middle|\; \begin{array}{l} \alpha \in \Phi; \langle \alpha, \mu \rangle \geq 0 \\ \beta \in \Phi; \langle \beta, \mu \rangle < 0 \end{array} \right\rangle,$$

where $\mathcal{O}_F$ is the ring of integers in F and $\mathfrak{p}_F$ is its maximal ideal. We put $\mathcal{H}_{J_\mu} := C_c^\infty(G /\!\!/ J_\mu)$ and

$$\mathbb{1}_\mu := \mathrm{vol}(J_\mu)^{-1} \cdot \mathbb{1}_{J_\mu \mu(\varpi) J_\mu} \in \mathcal{H}_{J_\mu}$$

by fixing a Haar measure of G and a uniformizer ϖ of F. Recall that, when the Satake parameter of an irreducible unramified representation π is given by $s(\pi)$, we may regard $s(\pi)$ also as an unramified character of $\mathbf{T}(F)$ via the isomorphism

$$\mathrm{Hom}(X_*(\mathbf{T}), \mathbb{C}^\times) \cong \mathrm{Hom}(\mathbf{T}(F)/\mathbf{T}(\mathcal{O}_F), \mathbb{C}^\times).$$

Then, by writing χ for this character, π is realized as a subquotient of the principal series representation $(I_\chi := \text{n-Ind}_{\mathbf{B}(F)}^{\mathbf{G}(F)} \chi, V_\chi)$.

Theorem 1 (Main result of [OST21])**.** *For any irreducible unramified representation π of G, with the above notations, the following identity holds:*

$$L(s, \pi, r) = \prod_{\mu \in \mathcal{P}^+(r)} \det(1 - q^{-s - \langle \rho, \mu \rangle} \cdot I_\chi(\mathbb{1}_\mu) \mid V_\chi^{J_\mu})^{-m_\mu}.$$

Here,

- *$\mathcal{P}^{(+)}(r)$ is the set of (dominant) weights of $\hat{\mathbf{T}}$ in r,*
- *ρ is the half sum of positive roots of $\mathbf{T}$ in $\mathbf{G}$,*
- *for each $\mu \in \mathcal{P}^{(+)}(r)$, m_μ denotes its multiplicity in r.*

Remark 2. As stated in the beginning, our result holds for any connected reductive group. Note that there does not exist a hyperspecial open compact subgroup unless $\mathbf{G}$ is unramified (i.e., quasi-split and split over an unramified extension). Therefore, when $\mathbf{G}$ is not unramified, we consider the class of irreducible representations having a nonzero vector fixed by a *parahoric* subgroup ("*parahoric-spherical*" representations). In [Hai15, Hai17], Haines associated Satake parameters to such representations. Thus, we can define the associated local L-factors (precisely, "*semisimple L-factors*") by using Haines' Satake parameters. Our result in [OST21] is stated in terms of the the parahoric-spherical representations and the local L-factors defined in this way.

We remark that the above formula becomes quite simple when r is *minuscule* (that is, $\mathcal{P}^+(r)$ is a singleton) as in the following examples.

Example 3. Let $\mathbf{G} := \mathrm{GL}_2$. We take r to be the standard representation Std of the Langlands dual group $\mathrm{GL}_2(\mathbb{C})$ of GL_2. Then $\mathcal{P}^+(r)$ is a singleton $\{\mu\}$ and we get

$$L(s, \pi, \mathrm{Std}) = \det\bigl(1 - q^{-(s+1/2)} I_\chi(\mathbb{1}_\mu) \bigm| V_\chi^{J_\mu}\bigr)^{-1}.$$

Here J_μ is given by

$$J_\mu = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \;\middle|\; c \in \mathfrak{p}_F \right\}.$$

Example 4. Let

$$\mathbf{G} := \mathrm{GSp}_4 := \left\{ g \in \mathrm{GL}_4 \;\middle|\; {}^t g \begin{pmatrix} & -J_2 \\ J_2 & \end{pmatrix} g = x \begin{pmatrix} & -J_2 \\ J_2 & \end{pmatrix} \text{ for some } x \in \mathbb{G}_{\mathrm{m}} \right\},$$

where J_2 denotes the anti-diagonal matrix whose anti-diagonal entries are one. We take r to be the spin representation Spin of the Langlands dual group $\mathrm{GSpin}_5(\mathbb{C})$ of GSp_4. Then $\mathcal{P}^+(r)$ is a singleton $\{\mu\}$ and we get

$$L(s, \pi, \mathrm{Spin}) = \det\bigl(1 - q^{-(s+3/2)} I_\chi(\mathbb{1}_\mu) \bigm| V_\chi^{J_\mu}\bigr)^{-1},$$

where J_μ is given by

$$J_\mu = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_4(F) \;\middle|\; A, D \in \mathrm{GL}_2(\mathcal{O}_F), B \in M_2(\mathcal{O}_F), C \in M_2(\mathfrak{p}_F) \right\},$$

In fact, this identity is nothing but the one found by Taylor in his thesis [Tay88, Section 2.4] (see also [LSZ17, Section 3.4.2]).

The interesting point of the formula in Theorem 1 is as follows. Recall that, in the definition of the local L-factors for unramified representations, we utilize the Satake parameters determined by the Satake isomorphism. This amounts to looking at the action of the whole spherical Hecke algebra on the subspace of spherical vectors, which is 1-dimensional. On the other hand, in the formula of Theorem 1, the local L-factor is expressed by the characteristic polynomial of the action of only one test function on the subspace whose dimension is the same as the degree of the local L-factor. For instance, in Example 3, the local L-factor $L(s, \pi, \mathrm{Std})$ is described by the action of a single test function $\mathbb{1}_\mu$ on the subspace $V_\chi^{J_\mu}$, which is 2-dimensional.

We explain the outline of the proof of Theorem 1. The key in our proof is that the action of $I_\chi(\mathbb{1}_\mu)$ on the space $V_\chi^{J_\mu}$ can be triangulated with respect to an ordered basis of $V_\chi^{J_\mu}$. To explain this, we assume that μ is strictly dominant for simplicity. In this case, J_μ is an Iwahori subgroup, hence let us simply write I for J_μ. Then we can find an explicit basis $\{v_w^\vee\}_{w \in W}$ of the subspace V_χ^I of I-fixed vectors in V_χ, which is labelled by the elements of the Weyl group W. With respect to this ordered basis of V_χ^I, we have the following:

Proposition 5. *For any $w \in W$, there exists a family $\{c_{w'}\}_{w' \in W, w' < w}$ of complex numbers satisfying*

$$I_\chi(\mathbb{1}_\mu) \cdot v_w^\vee = q^{\langle \rho, \mu \rangle} \cdot \chi\bigl(w(\mu)(\varpi)\bigr) \cdot v_w^\vee + \sum_{\substack{w' \in W \\ w' > w}} c_{w'} \cdot v_{w'}^\vee.$$

Once this proposition is proved, we immediately get a description of the characteristic polynomial of the action of $\mathbb{1}_\mu$ on V_χ^I. Then we obtain Theorem 1 by tracking the construction of the Satake parameter and rewriting the local L-factor $L(s, \pi, r)$ in terms of the weights of the representation r.

The outline of the proof of Proposition 5 is as follows. We let $\mathcal{H}_I$ denote the *Iwahori–Hecke algebra* $C_c^\infty(G /\!\!/ I)$. For any $\mu \in X_*(\mathbf{T})$, we put $\Theta_\mu \in \mathcal{H}_I$ to be the *Bernstein function* with respect to μ. For example, when μ is dominant, we have $\Theta_\mu = q^{-\langle \rho, \mu \rangle} \cdot \mathbb{1}_\mu$. (See [OST21, Definition 2.10] for the definition of Θ_μ.) Then, we can check that Proposition 5 is equivalent to the following proposition:

Proposition 6. *For any $w \in W$, there exists a family $\{c_{w'}\}_{w' \in W, w' < w}$ of complex numbers satisfying*

$$v_w * \Theta_\mu = \chi\bigl(w(\mu)(\varpi)\bigr)^{-1} \cdot v_w + \sum_{\substack{w' \in W \\ w' < w}} c_{w'} \cdot v_{w'}.$$

Here, $*$ on the left-hand side denotes the convolution product and $\{v_w\}_{w\in W}$ on the right-hand side is an explicit basis of the subspace $V^I_{\chi^{-1}}$ of I-fixed vectors in $V_{\chi^{-1}}$ which is defined in a similar way to $\{v^\vee_w\}_{w\in W}$.

The point here is that the ring structure of $\mathcal{H}_I$ and its action on the space of unramified principal series are well-investigated, especially in the works of Haines–Kottwitz–Prasad (split case, [HKP10]) and Rostami (general case, [Ros15]). By using several basic relations of the Iwahori–Hecke algebra such as the *Bernstein relation*, we can prove Proposition 6 by a simple induction argument on the length of $w \in W$.

Finally, we would like to add a remark about our proof. Originally, we proved Proposition 5 by making full use of the Chevalley basis by assuming that our group $\mathbf{G}$ is split. By utilizing various relations of the Chevalley basis, we carried out the induction on the length of $w \in W$; then the problem is essentially reduced to the case of SL_2.

However, after we released the first version of our paper ([OST19]), Thomas Haines told the authors that Proposition 5 can be proved in a more sophisticated way if we appeal to the theory of the Iwahori–Hecke algebra. Furthermore, he also explained that his approach naturally enables us to prove if for any general (i.e., possibly non-split) connected reductive group $\mathbf{G}$. Hence we decided to follow his idea and present his simplified version of the proof in the second version of our paper ([OST21]).

It seems that our computations in the previous version of the proof are essentially encoded in the various identities in the theory of the Iwahori–Hecke algebra. In this sense, the core of the new proof presented in this paper is not totally different from our original proof. Nevertheless, we would like to emphasize that most of the arguments are drastically simplified and our main result is far more generalized by following the formulation suggested by Haines.

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