NONINTEGRABILITY OF DYNAMICAL SYSTEMS NEAR DEGENERATE EQUILIBRIA

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ABSTRACT. We prove that general three- or four-dimensional systems are real-analytically nonintegrable near degenerate equilibria in the Bogoyavlenskij sense under additional weak conditions when the Jacobian matrices have a zero and pair of purely imaginary eigenvalues or two incommensurate pairs of purely imaginary eigenvalues at the equilibria. For this purpose, we reduce their integrability to that of the corresponding Poincaré-Dulac normal forms and further to that of simple planar systems, and use a novel approach for proving the analytic nonintegrability of planar systems. Our result also implies that general three- and four-dimensional systems exhibiting fold-Hopf and double-Hopf codimension-two bifurcations, respectively, are real-analytically nonintegrable under the weak conditions. To demonstrate these results, we give two examples for the Rössler system and coupled van der Pol oscillators.

1. INTRODUCTION

In this paper we study the nonintegrability of systems of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $n = 3$ or 4 and $f(x)$ is analytic. We assume that $x = 0$ is an equilibrium, i.e., $f(0) = 0$, and the Jacobian matrix $\mathrm{D}f(0)$ of $f(x)$ at $x = 0$ has (I) a zero and pair of purely imaginary eigenvalues, $\lambda = 0, \pm i\omega$ ($\omega > 0$), for $n = 3$ or (II) two pairs of purely imaginary eigenvalues, $\pm i\omega_j$ ($\omega_j > 0$), $j = 1, 2$, with $\omega_1/\omega_2 \notin \mathbb{Q}$ for $n = 4$. Here we adopt the following concept of integrability in the Bogoyavlenskij sense [6].

Definition 1.1 (Bogoyavlenskij). *For any integer $n \geq 1$, the n-dimensional system* (1.1) *is called* $(m, n-m)$-*integrable or simply* integrable *for some integer $m \in [1, n]$ if there exist m vector fields $f_1(x)(:= f(x)), f_2(x), \ldots, f_m(x)$ and $n-m$ scalar-valued functions $F_1(x), \ldots, F_{n-m}(x)$ such that the following two conditions hold:*

(i) *$f_1(x), \ldots, f_m(x)$ are linearly independent almost everywhere and commute with each other, i.e., $[f_j, f_k](x) := \mathrm{D}f_k(x)f_j(x) - \mathrm{D}f_j(x)f_k(x) \equiv 0$ for $j, k = 1, \ldots, m$, where $[\cdot, \cdot]$ denotes the Lie bracket;*

(ii) *The derivatives $\mathrm{D}F_1(x), \ldots, \mathrm{D}F_{n-m}(x)$ are linearly independent almost everywhere and $F_1(x), \ldots, F_{n-m}(x)$ are first integrals of $f_1, \ldots, f_m$, i.e., $\mathrm{D}F_k(x)^{\mathrm{T}} f_j(x) \equiv 0$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n-m$, where the superscript 'T' represents the transpose operator.*

We say that the system is analytically (*resp.* meromorphically) integrable *if the first integrals and commutative vector fields are analytic* (*resp. meromorphic*).

If an ℓ-degree-of-freedom Hamiltonian system with $\ell \geq 1$ is integrable in the Liouville sense [3, 26], then so is it in the Bogoyavlenskij sense, since it has not only ℓ functionally independent first integrals but also ℓ linearly independent commutative (Hamiltonian) vector fields generated by the first integrals. Thus, the Bogoyavlenskij-integrability in Definition 1.1 is considered as a generalization of Liouville-integrability for Hamiltonian systems.

Under our assumptions, by power series changes of coordinates, the system (1.1) is formally transformed to

$$
\begin{aligned}
\dot{x}_1 &= -\omega x_2 + g_1(x_1^2 + x_2^2, x_3)x_1 - g_2(x_1^2 + x_2^2, x_3)x_2, \\
\dot{x}_2 &= \omega x_1 + g_2(x_1^2 + x_2^2, x_3)x_1 + g_1(x_1^2 + x_2^2, x_3)x_2, \\
\dot{x}_3 &= g_3(x_1^2 + x_2^2, x_3)
\end{aligned}
\tag{1.2}
$$

with $x = (x_1, x_2, x_3)$ for case (I), and to

$$
\begin{aligned}
\dot{x}_1 &= -\omega_1 x_2 + h_1(x_1^2 + x_2^2, x_3^2 + x_4^2)x_1 - h_2(x_1^2 + x_2^2, x_3^2 + x_4^2)x_2, \\
\dot{x}_2 &= \omega_1 x_1 + h_2(x_1^2 + x_2^2, x_3^2 + x_4^2)x_1 + h_1(x_1^2 + x_2^2, x_3^2 + x_4^2)x_2, \\
\dot{x}_3 &= -\omega_2 x_4 + h_3(x_1^2 + x_2^2, x_3^2 + x_4^2)x_3 - h_4(x_1^2 + x_2^2, x_3^2 + x_4^2)x_4, \\
\dot{x}_4 &= \omega_2 x_3 + h_4(x_1^2 + x_2^2, x_3^2 + x_4^2)x_3 + h_3(x_1^2 + x_2^2, x_3^2 + x_4^2)x_4
\end{aligned}
\tag{1.3}
$$

with $x = (x_1, x_2, x_3, x_4)$ for case (II), where $g_j(y_1, y_2)$, $j = 1, 2, 3$, and $h_j(y_1, y_2)$, $j = 1, 2, 3, 4$, are formal power series of y_1 and y_2, which may not be convergent, such that $g_j(0, 0), \mathrm{D}_{x_3}g_3(0, 0) = 0$, $j = 1, 2, 3$, and $h_j(0, 0) = 0$, $j = 1, 2, 3, 4$. See, e.g., Lemmas 1.12 and 1.15 in Section 3.1 of [15] for the derivation of (1.2) and (1.3). Equations (1.2) and (1.3) are, respectively, represented as

$$
\begin{aligned}
\dot{x}_1 &= -\omega x_2 + \alpha_1 x_1 x_3 - \alpha_2 x_2 x_3, \\
\dot{x}_2 &= \omega x_1 + \alpha_2 x_1 x_3 + \alpha_1 x_2 x_3, \\
\dot{x}_3 &= \alpha_3(x_1^2 + x_2^2) + \alpha_4 x_3^2
\end{aligned}
\tag{1.4}
$$

up to $O(|x|^2)$ for case (I), and as

$$
\begin{aligned}
\dot{x}_1 &= -\omega_1 x_2 + (\alpha_1(x_1^2 + x_2^2) + \alpha_2(x_3^2 + x_4^2))x_1 \\
&\quad - (\beta_1(x_1^2 + x_2^2) + \beta_2(x_3^2 + x_4^2))x_2, \\
\dot{x}_2 &= \omega_1 x_1 + (\beta_1(x_1^2 + x_2^2) + \beta_2(x_3^2 + x_4^2))x_1 \\
&\quad + (\alpha_1(x_1^2 + x_2^2) + \alpha_2(x_3^2 + x_4^2))x_2, \\
\dot{x}_3 &= -\omega_2 x_4 + (\alpha_3(x_1^2 + x_2^2) + \alpha_4(x_3^2 + x_4^2))x_3 \\
&\quad - (\beta_3(x_1^2 + x_2^2) + \beta_4(x_3^2 + x_4^2))x_4, \\
\dot{x}_4 &= \omega_2 x_3 + (\beta_3(x_1^2 + x_2^2) + \beta_4(x_3^2 + x_4^2))x_3 \\
&\quad + (\alpha_3(x_1^2 + x_2^2) + \alpha_4(x_3^2 + x_4^2))x_4
\end{aligned}
\tag{1.5}
$$

up to $O(|x|^3)$ for case (II), where $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, \ldots, 4$. Such simplification is also one of the standard techniques, especially for bifurcations, in dynamical systems. See Chapter 3 of [15] and Chapters 3 and 8 of [18] for the details. The above cases are rather standard applications of the technique, as mentioned below.

Our main results are stated as follows:

Theorem 1.2. *Let $n = 3$ and suppose that the system* (1.1) *is transformed to* (1.4) *up to $O(|x|^2)$. If one of the following conditions holds, then the system* (1.1) *is not real-analytically integrable in the Bogoyavlenskij sense near the origin*:

(i) $\alpha_1\alpha_4 > 0$;
(ii) $\alpha_1\alpha_4 < 0$ *and* $\alpha_4/\alpha_1 \notin \mathbb{Q}$.

Theorem 1.3. *Let $n = 4$ and suppose that the system* (1.1) *is transformed to* (1.5) *up to $O(|x|^3)$. If $\alpha_1 \neq \alpha_3$, $\alpha_2 \neq \alpha_4$, and one of the following conditions holds, then the system* (1.1) *is not real-analytically integrable in the Bogoyavlenskij sense near the origin*:

(i) $\alpha_1\alpha_3$ *or* $\alpha_2\alpha_4 > 0$;
(ii) $\alpha_1\alpha_3$, $\alpha_2\alpha_4 < 0$ *and* α_1/α_3, $\alpha_2/\alpha_4 \notin \mathbb{Q}$.

We prove these theorems in Section 4. The unfoldings of (1.4) and (1.5),

$$
\begin{aligned}
\dot{x}_1 &= \nu x_1 - \omega x_2 + \alpha_1 x_1 x_3 - \alpha_2 x_2 x_3, \\
\dot{x}_2 &= \omega x_1 + \nu x_2 + \alpha_2 x_1 x_3 + \alpha_1 x_2 x_3, \\
\dot{x}_3 &= \mu + \alpha_3(x_1^2 + x_2^2) + \alpha_4 x_3^2
\end{aligned}
\tag{1.6}
$$

and

$$
\begin{aligned}
\dot{x}_1 = {}& -\omega_1 x_2 + (\nu + \alpha_1(x_1^2 + x_2^2) + \alpha_2(x_3^2 + x_4^2))x_1 \\
& - (\beta_1(x_1^2 + x_2^2) + \beta_2(x_3^2 + x_4^2))x_2, \\
\dot{x}_2 = {}& \omega_1 x_1 + (\beta_1(x_1^2 + x_2^2) + \beta_2(x_3^2 + x_4^2))x_1 \\
& + (\nu + \alpha_1(x_1^2 + x_2^2) + \alpha_2(x_3^2 + x_4^2))x_2, \\
\dot{x}_3 = {}& -\omega_2 x_4 + (\mu + \alpha_3(x_1^2 + x_2^2) + \alpha_4(x_3^2 + x_4^2))x_3, \\
& - (\beta_3(x_1^2 + x_2^2) + \beta_4(x_3^2 + x_4^2))x_4, \\
\dot{x}_4 = {}& \omega_2 x_3 + (\beta_3(x_1^2 + x_2^2) + \beta_4(x_3^2 + x_4^2))x_3 \\
& + (\mu + \alpha_3(x_1^2 + x_2^2) + \alpha_4(x_3^2 + x_4^2))x_4
\end{aligned}
\tag{1.7}
$$

represent normal forms of fold-Hopf and double-Hopf bifurcations, respectively, where $\mu, \nu \in \mathbb{R}$ are the control parameters: At $(\mu, \nu) = (0, 0)$, fold (saddle-node) and Hopf bifurcation curves meet for the former, and two Hopf bifurcation curves for the latter. Such codimension-two bifurcations are fundamental and interesting phenomena in dynamical systems and have been studied extensively since the seminal papers of Arnold [2] and Takens [39]. See, e.g., [14, 15, 18] for the details. In [1, 42], the nonintegrability of the normal forms (1.6) and (1.7) in the Bogoyavlenskij sense were discussed: They were shown to be meromorphically nonintegrable for almost all parameter values of α_j, β_j, $j = 1, 2, 3, 4$, near the x_3-axis and the (x_1, x_2)- or (x_3, x_4)-plane, respectively, when $(\mu, \nu) \neq (0, 0)$, while it was not determined whether they are nonintegrable or not when $(\mu, \nu) = (0, 0)$. (A special case of (1.7) in which $\beta_j = 0$, $j = 1, 2, 3, 4$, was actually considered in [1] but their values do not affect the conclusion, as in Theorem 1.3.) Our results show that not only the normal forms (1.6) and (1.7) with $(\mu, \nu) = (0, 0)$ but also the full system (1.1) is real-analytically nonintegrable if the hypotheses of Theorems 1.2 or 1.3 hold when it is transformed to (1.2) or (1.3) having the $O(|x|^2)$- or $O(|x|^3)$-truncation (1.4) or (1.5).

We provide further backgrounds and related work. For a while, we consider the system (1.1) in a more general situation in which $n \neq 3, 4$ is allowed but $x = 0$ is still an equilibrium.

Definition 1.4 (Poincaré-Dulac normal form). *Change the coordinates in* (1.1) *such that* $\mathrm{D}f(0)$ *is in Jordan normal form. The system* (1.1) *is called a* Poincaré-Dulac (PD) normal form *if* $[Sx, f] = 0$, *where S is the semisimple part of* $\mathrm{D}f(0)$, *i.e.,* $S = \mathrm{diag}\lambda_j$, *where λ_j, $j = 1, \ldots, n$, are the eigenvalues of* $\mathrm{D}f(0)$.

We easily see that the systems (1.2) and (1.3) can be written in PD normal form for (1.1) under our assumptions although they may not be convergent. Actually, Eqs. (1.2) and (1.3) become

$$
\begin{aligned}
\dot{z}_1 &= i\omega z_1 + (g_1(z_1 z_2, x_3) + i g_2(z_1 z_2, x_3))z_1, \\
\dot{z}_2 &= -i\omega z_2 + (g_1(z_1 z_2, x_3) - i g_2(z_1 z_2, x_3))z_2, \\
\dot{x}_3 &= g_3(z_1 z_2, x_3)
\end{aligned}
\tag{1.8}
$$

and

$$
\begin{aligned}
\dot{z}_1 &= i\omega_1 z_1 + (h_1(z_1 z_2, z_3 z_4) + i h_2(z_1 z_2, z_3 z_4))z_1, \\
\dot{z}_2 &= -i\omega_1 z_2 + (h_1(z_1 z_2, z_3 z_4) - i h_2(z_1 z_2, z_3 z_4))z_2, \\
\dot{z}_3 &= i\omega_2 z_3 + (h_3(z_1 z_2, z_3 z_4) + i h_4(z_1 z_2, z_3 z_4))z_3, \\
\dot{z}_4 &= -i\omega_2 z_4 + (h_3(z_1 z_2, z_3 z_4) - i h_4(z_1 z_2, z_3 z_4))z_4,
\end{aligned}
\tag{1.9}
$$

respectively, where

$$
z_1 = x_1 + i x_2, \quad z_2 = x_1 - i x_2, \quad z_3 = x_3 + i x_4, \quad z_4 = x_3 - i x_4.
$$

We easily see that the systems (1.8) and (1.9) are PD normal forms. Henceforth we also refer to (1.2) and (1.3) as PD normal forms.

Let λ_j, $j = 1, \ldots, n$, be eigenvalues of $\mathrm{D}f(0)$, and let

$$
\mathbb{Z}_j^n = \{p = (p_1, \ldots, p_n) \in \mathbb{Z}^n \mid p_j \geq -1, \, p_l \geq 0, \, l \neq j, \, p \neq 0\}
$$

for $j = 1, \ldots, n$.

Definition 1.5 (Resonance set and degree). *Let*

$$
\mathscr{R}_j = \left\{ p \in \mathbb{Z}_j^n \, \middle| \, \sum_{l=1}^n \lambda_j p_j = 0 \right\}, \quad j = 1, \ldots, n,
$$

and let

$$
\mathscr{R} = \bigcup_{j=1}^n \mathscr{R}_j.
$$

We refer to $\mathscr{R}$ as the resonance set *of* (1.1) *and to*

$$
\gamma_{\mathrm{R}} = \dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}} \mathscr{R}
$$

as the resonance degree *of* (1.1).

For the PD normal forms (1.2) and (1.3) we easily see that the resonance sets are given by

$$
\mathscr{R} = \mathrm{span}_{\mathbb{N}}\{(1, 0, 0), (0, 1, 1)\} \quad \text{and} \quad \mathscr{R} = \mathrm{span}_{\mathbb{N}}\{(1, 1, 0, 0), (0, 0, 1, 1)\},
$$

respectively, and the resonance degrees are $\gamma_{\mathrm{R}} = 2$. Yamanaka [48] proved the following result for the general case.

Theorem 1.6 (Yamanaka). *If the resonance degree γ_{R} is less than two, then the PD normal form is analytically integrable. Moreover, there exists an n-dimensional, analytically nonintegrable PD normal form with $n = \gamma_{\mathrm{R}} + 1$ for $\gamma_{\mathrm{R}} \geq 2$.*

Similar results for Hamiltonian systems are found in [9,10,49,55]. The above result does not exclude the analytic nonintegrability of (1.2) and (1.3). He also gave a necessary and sufficient condition for (1.2) to be analytically $(1,2)$-integrable in [48]. For example, if the system (1.4) is analytically $(1,2)$-integrable, then $\alpha_1, \alpha_3, \alpha_4 = 0$.

On the other hand, Zung [54] proved the following remarkable result on analytically integrable PD normal forms.

Theorem 1.7 (Zung). *Let $n \geq 1$ be any integer. If the system (1.1) is analytically integrable near $x = 0$ in the Bogoyavlenskij sense, then there exists an analytic change of coordinates under which it is transformed to a PD normal form.*

A similar result for Hamiltonian systems was obtained by Zung [55]. Theorem 1.7 also implies that the corresponding PD normal form is convergent and analytically integrable if the system (1.1) is analytically integrable. Hence, the system (1.1) is analytically nonintegrable if the corresponding PD normal form is divergent or analytically nonintegrable. So we only have to prove the analytic nonintegrability of (1.2) and (1.3) for the proofs of Theorems 1.2 and 1.3. In their proofs, we assume that the system (1.1) is analytically integrable and that the power series in (1.2) and (1.3) are convergent, and show that these assumptions yield contradictions.

For the problem on nonintegrability of dynamical systems, the Morales-Ramis theory [26,28] and its extension [4,30] were developed and have produced numerous remarkable results. See, e.g., [24,27,29] for such examples. Recently, the author and his coworker also applied the techniques and obtained several results on the problem for nearly integrable systems in [31,43,47], for the restricted three-body problems in [44,45] and for an epidemic model in [46]. Here we use a different approach without relying on the techniques. In particular, a useful relation between first integral and commutative vector fields for proving the analytic nonintegrability of planar systems is provided.

The outline of this paper is as follows: In Section 2 we reduce the nonintegrability of (1.2) and (1.3) to that of simple planar systems. For this purpose, we use Proposition 2.1 of [1], which enables us to reduce a special class of systems, including (1.2) and (1.3), to planar systems, along with a simple but clever trick. In Section 3 we provide the useful relation on first integrals and commutative vector fields. In Section 4 we prove the main theorems using the results of Sections 2 and 3. Finally, to demonstrate our results, we give two examples for the Rössler system [7,18,20,25,51] and coupled van der Pol oscillators [8,17,23,32,33,37,38] in Section 5.

2. Reduction to Simple Planar Systems

In this section we reduce the nonintegrability of (1.2) and (1.3) to that of simple planar systems.

Using the change of coordinate $(x_1, x_2) = (r \cos\theta, r \sin\theta)$, we transform (1.4) to

$$\dot{r} = g_1(r^2, x_3)r, \quad \dot{x}_3 = g_3(r^2, x_3), \quad \dot{\theta} = \omega + g_2(r^2, x_3), \tag{2.1}$$

of which the (r_1, r_2)-components are independent of θ. Using the change of coordinates $(x_1, x_2) = (r_1 \cos\theta_1, r_1 \sin\theta_1)$ and $(x_3, x_4) = (r_2 \cos\theta_2, r_2 \sin\theta_2)$, we also

transform (1.5) to

$$\dot{r}_1 = h_1(r_1^2, r_2^2)r_1, \quad \dot{r}_2 = h_3(r_1^2, r_2^2)r_2,$$
$$\dot{\theta}_1 = \omega_1 + h_2(r_1^2, r_2^2), \quad \dot{\theta}_2 = \omega_2 + h_4(r_1^2, r_2^2) \tag{2.2}$$

of which the (r_1, r_2)-components are independent of θ_1 and θ_2. So we expect that one can reduce the nonintegrability of (1.4) and (1.5) to that of the (r, x_3)-components of (2.1),

$$\dot{r} = g_1(r^2, x_3)r, \quad \dot{x}_3 = g_3(r^2, x_3), \tag{2.3}$$

and the (r_1, r_2)-components of (2.2),

$$\dot{r}_1 = h_1(r_1^2, r_2^2)r_1, \quad \dot{r}_2 = h_3(r_1^2, r_2^2)r_2, \tag{2.4}$$

respectively. This is true in a more general situation as follows.

Let $m > 0$ be an integer and consider $m + 2$-dimensional systems of the form

$$\dot{x} = f_x(x, y), \quad \dot{y} = f_y(x, y), \quad (x, y) \in D, \tag{2.5}$$

where $D \subset \mathbb{C}^2 \times \mathbb{C}^m$ is a region containing the m-dimensional y-plane $\{(0, y) \in \mathbb{C}^2 \times \mathbb{C}^m \mid y \in \mathbb{C}^m\}$, and $f_x : D \to \mathbb{C}^2$ and $f_y : D \to \mathbb{C}^m$ are analytic. Assume that by the change of coordinates $x = (x_1, x_2) = (r \cos \theta, r \sin \theta)$, Eq. (2.5) is transformed to

$$\dot{r} = R(r, y), \quad \dot{y} = \tilde{f}_y(r, y), \quad \dot{\theta} = \Theta(r, y), \quad (r, y, \theta) \in \tilde{D} \times \mathbb{C}, \tag{2.6}$$

where $\tilde{D} \subset \mathbb{C} \times \mathbb{C}^m$ is a region containing the m-dimensional y-plane, and $R : \tilde{D} \to \mathbb{C}$, $\tilde{f}_y : \tilde{D} \to \mathbb{C}^m$ and $\Theta : \tilde{D} \to \mathbb{R}$ are analytic. Note that $\tilde{f}_y(r, y) = f_y(r \cos \theta, r \sin \theta, y)$. We are especially interested in the (r, y)-components of (2.6),

$$\dot{r} = R(r, y), \quad \dot{y} = \tilde{f}_y(r, y), \tag{2.7}$$

which are independent of θ. In this situation we have the following proposition.

Proposition 2.1.

(i) *Suppose that Eq. (2.5) has a meromorphic (resp. analytic) first integral $F(x_1, x_2, y)$ near $(x_1, x_2) = (0, 0)$, and let $\tilde{F}(r, \theta, y) = F(r \cos \theta, r \sin \theta, y)$. If $\tilde{f}_{yj}(0, y) \neq 0$ for almost all $y \in \tilde{D}$ for some $j = 1, \ldots, m$, then*

$$G(r, y) = \tilde{F}(r, \tilde{\theta}_j(y_j), y)$$

is a meromorphic (resp. analytic) first integral of (2.7) near $r = 0$, where y_j and $\tilde{f}_{yj}(r, y)$ are the j-th components of y and $\tilde{f}_y(r, y)$, respectively, and $\tilde{\theta}_j(y_j)$ represents the θ-component of a solution to

$$\frac{dr}{dy_j} = \frac{R(r, y)}{\tilde{f}_{yj}(r, y)}, \quad \frac{dy_\ell}{dy_j} = \frac{\tilde{f}_{y\ell}(r, y)}{\tilde{f}_{yj}(r, y)}, \quad \frac{d\theta}{dy_j} = \frac{\Theta(r, y)}{\tilde{f}_{yj}(r, y)}, \quad \ell \neq j.$$

(ii) *Suppose that Eq. (2.5) has a meromorphic (resp. analytic) commutative vector field*

$$v(x_1, x_2, y) := \begin{pmatrix} v_1(x_1, x_2, y) \\ v_2(x_1, x_2, y) \\ v_y(x_1, x_2, y) \end{pmatrix}$$

with $v_1, v_2 : D \to \mathbb{C}$ and $v_y : D \to \mathbb{C}^m$ near $(x_1, x_2) = (0, 0)$. If $\Theta(0, y) \neq 0$ for almost all $y \in \tilde{D}$, then

$$\begin{pmatrix} \tilde{v}_r(r, \theta, y) \\ \tilde{v}_y(r, \theta, y) \end{pmatrix}$$
$$= \begin{pmatrix} v_1(r\cos\theta, r\sin\theta, y)\cos\theta + v_2(r\cos\theta, r\sin\theta, y)\sin\theta \\ v_y(r\cos\theta, r\sin\theta, y) \end{pmatrix}$$

is independent of θ and it is a meromorphic (resp. analytic) commutative vector field of (2.7) near $r = 0$.

See Proposition 2.1 of [1] for the proof. Only the case in which first integrals and commutative vector fields are meromorphic was treated there but its reduction to the case in which they are analytic is obvious. Using Proposition 2.1 for (1.4) and (1.5) (once for the former and twice for the latter), we immediately obtain the following propositions.

Proposition 2.2. *If the complexification of (1.1) in case (I) is analytically integrable near $(x_1, x_2) = (0, 0)$, then so is the system (2.3) near $r = 0$.*

Proposition 2.3. *If the complexification of (1.1) in case (II) is analytically integrable near $(x_1, x_2) = (0, 0)$ and near $(x_3, x_4) = (0, 0)$, then so is the system (2.4) near $r_1 = 0$ and near $r_2 = 0$, respectively.*

Here we remark that the systems (2.3) and (2.4) only need to have one first integral or commutative vector field for their integrability, since they are of dimension two. Moreover, the converse statements of Propositions 2.2 and 2.3 do not hold in general.

We turn to systems of the general form (1.1) with $n \geq 2$ but $f(0) = 0$, $\mathrm{D}f(0) = 0, \ldots, \mathrm{D}^{k-1}f(0) = 0$ and $\mathrm{D}^k f(0) \neq 0$ for some $k \in \mathbb{N}$. Since $f(x)$ is analytic near $x = 0$, we have

$$f(x) = \sum_{j=k}^{\infty} f_j(x), \tag{2.8}$$

where the elements of $f_j(x)$ are jth-order homogeneous polynomials of x. Letting $x = \varepsilon y$ and changing the time variable as $t \to \varepsilon^k t$, we rewrite (1.1) as

$$\dot{y} = \sum_{j=0}^{\infty} \varepsilon^j f_{j+k}(y).$$

We prove the following result.

Theorem 2.4. *Suppose that $f(x)$ has the form (2.8) for some $k \in \mathbb{N}$. If the system (1.1) is analytically integrable in the Bogoyavlenskij sense, then so is the truncated system*

$$\dot{y} = f_k(y). \tag{2.9}$$

Proof. Let $F(x)$ be an analytic first integral of (1.1) near $x = 0$, and let

$$F(x) = \sum_{j=\ell}^{\infty} F_j(x) \tag{2.10}$$

for some $\ell \in \mathbb{N}$, where $F_j(x)$ is a jth-order homogeneous polynomial of x. Here we have assumed that $F(0) \equiv 0$ without loss of generality. Then we have

$$\mathrm{D}F(\varepsilon y)^{\mathrm{T}} f(\varepsilon y) = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \varepsilon^{k+\ell+j+l} \mathrm{D}F_{\ell+j}(y)^{\mathrm{T}} f_{k+l}(y) \equiv 0,$$

in particular,

$$\mathrm{D}F_\ell(y)^{\mathrm{T}} f_k(y) \equiv 0,$$

which means that $F_\ell(y)$ is an analytic first integral of the truncated system (2.9).

On the other hand, let $v(x)$ be an analytic commutative vector field of (1.1) near $x = 0$. Let

$$v(\varepsilon y) = \sum_{j=0}^{\infty} \varepsilon^j v_j(y),$$

where the elements of $v_j(x)$ are jth-order homogeneous polynomials of x, and assume that $v_j(y) \equiv 0$, $j = 0, \ldots, \ell - 1$, and $v_\ell(y) \not\equiv 0$ for some $\ell \in \mathbb{N}$. Then we have

$$\begin{aligned}
[v, f](\varepsilon y) =& \mathrm{D}f(\varepsilon y)v(\varepsilon y) - \mathrm{D}v(\varepsilon y)f(\varepsilon y) \\
=& \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \varepsilon^{k+\ell+j+l} \mathrm{D}f_{k+l}(y)v_{\ell+j}(y) - \mathrm{D}v_{\ell+j}(y)f_{k+l}(y) \equiv 0,
\end{aligned}$$

in particular,

$$\mathrm{D}f_k(y)v_\ell(y) - \mathrm{D}v_\ell(y)f_k(y) \equiv 0,$$

which means that $v_\ell(y)$ is an analytic commutative vector field of the truncated system (2.9).

Suppose that the system (1.1) is analytically integrable. Then we can choose the analytic first integrals (resp. commutative vector fields) such that their leading terms are linearly independent almost everywhere in a neighborhood of $x = 0$, by taking their linear combinations if necessary. Actually, for instance, if $F(x)$ and $G(x)$ are linearly independent first integrals with (2.10) and

$$G(x) = \sum_{j=\ell}^{\infty} G_j(x),$$

where $G_j(x)$ is a jth-order homogeneous polynomials of x, and for some $m > \ell$,

$$\sum_{j=\ell}^{m-1} (c_1 \mathrm{D}F_j(x) + c_2 \mathrm{D}G_j(x)) = 0$$

for some $(c_1, c_2) \neq (0, 0)$ but

$$\sum_{j=\ell}^{m} (\tilde{c}_1 \mathrm{D}F_j(x) + \tilde{c}_2 \mathrm{D}G_j(x)) \neq 0$$

for any $(\tilde{c}_1, \tilde{c}_2) \neq (0, 0)$, then one may take $F(x)$ and

$$\tilde{G}(x) = c_1 F(x) + c_2 G(x),$$

for which the leading term is

$$c_1 \mathrm{D}F_m(x) + c_2 \mathrm{D}G_m(x),$$

as two new linearly independent first integrals. So we show that the leading terms of first integrals and commutative vector fields satisfy conditions (i) and (ii) of Definition 1.1 for (2.9), along with the above observations. Thus, we obtain the desired result. □

Remark 2.5.

(i) *In contrast to Theorem 2.4, the truncated system*

$$\dot{y} = \sum_{j=k}^{k+m} f_j(x)$$

with $m \geq 1$ may not be analytically integrable in general even if the full system (2.8) *is analytically integrable. Actually, Yoshida [50] showed that the truncation of the three-particle Toda lattice [40] with $k = 1$ is analytically nonintegrable at any order $m \geq 1$ although the Toda lattice is analytically integrable as well known [11, 16].*

(ii) *An argument similar to that of the above proof was used for a three-degree-of-freedom Hamiltonian system in Section 2 of [36].*

Applying Theorem 2.4 to (2.3) and (2.4) and using Propositions 2.2 and 2.3, we obtain the following.

Proposition 2.6. *If the complexification of* (1.1) *in case* (I) *is analytically integrable near the origin $x = 0$, then so is the truncated system*

$$\dot{r} = \alpha_1 r x_3, \quad \dot{x}_3 = \alpha_3 r^2 + \alpha_4 x_3^2 \tag{2.11}$$

near $(r, x_3) = (0, 0)$.

Proposition 2.7. *If the complexification of* (1.1) *in case* (II) *is analytically integrable near the origin $x = 0$, then so is the truncated system*

$$\dot{r}_1 = (\alpha_1 r_1^2 + \alpha_2 r_2^2) r_1, \quad \dot{r}_2 = (\alpha_3 r_1^2 + \alpha_4 r_2^2) r_2 \tag{2.12}$$

near $(r_1, r_2) = (0, 0)$.

Remark 2.8. *If the system* (1.1) *is real-analytically integrable near $x = 0$, then its complexification is also analytically integrable near $x = 0$. So we only have to prove that the complexifications of* (2.11) *and* (2.12) *are analytically nonintegrable near $x = 0$ for the proofs of Theorems* 1.2 *and* 1.3.

3. PLANAR VECTOR FIELDS

In this section we give a useful relation between first integrals and commutative vector fields for proving the analytic nonintegrability of such planar systems as (2.11) and (2.12).

Consider planar vector fields of the form

$$\dot{z} = p(z), \quad z \in \mathbb{C}^2, \tag{3.1}$$

where $p(z)$ is analytic in z. We prove the following.

Proposition 3.1. *Let $D \subset \mathbb{C}$ be a region that is covered by nonconstant solutions to* (3.1) *almost everywhere. Suppose that the system* (3.1) *has a first integral $Q(z)$ and commutative vector field $q(x)$ in D. Let*

$$\Delta(z) = \det(p(z), q(z)) = p_1(z)q_2(z) - p_2(z)q_1(z),$$

where $q_j(z)$ and $p_j(z)$ are the jth-elements of $q(z)$ and $p(z)$, respectively. Then there exists a function $\chi : \mathbb{C} \to \mathbb{C}$ such that

$$\Delta(z)\mathrm{D}Q(z) = \chi(Q(z)) \begin{pmatrix} -p_2(z) \\ p_1(z) \end{pmatrix}. \tag{3.2}$$

Proof. Let $z = \varphi(t)$ be a nonconstant particular solution to (3.1). We begin with the following lemmas.

Lemma 3.2. *If the planar system* (3.1) *has a commutative vector field* $q(z)$ (*resp. a first integral* $Q(z)$), *then* $\xi = q(\varphi(t))$ *is a solution to the variational equation* (VE) *of* (3.1) *along* $\varphi(t)$,

$$\dot{\xi} = \mathrm{D}p(\varphi(t))\xi \tag{3.3}$$

(*resp. then* $\eta = \mathrm{D}Q(\varphi(t))$ *is a solution to the adjoint variational equation* (AVE) *of* (3.1) *along* $\varphi(t)$,

$$\dot{\eta} = -\mathrm{D}p(\varphi(t))^{\mathrm{T}}\eta \ ). \tag{3.4}$$

Proof. Let $q(z)$ be a commutative vector field of (3.1). Then

$$\mathrm{D}q(z)p(z) - \mathrm{D}p(z)q(z) = 0,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}q(\varphi(t)) = \mathrm{D}p(\varphi(t))q(\varphi(t)).$$

Hence, $\xi = q(\varphi(t))$ is a solution to (3.3).

On the other hand, let $Q(z)$ be a first integral of (3.1). Then

$$p(z)^{\mathrm{T}}\mathrm{D}Q(z) = 0,$$

so that

$$\mathrm{D}(p(z)^{\mathrm{T}}\mathrm{D}Q(z)) = \mathrm{D}p(z)^{\mathrm{T}}\mathrm{D}Q(z) + \mathrm{D}^2Q(z)p(z) = 0.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}Q(\varphi(t)) = \mathrm{D}^2Q(\varphi(t))p(\varphi(t)) = -\mathrm{D}p(\varphi(t))^{\mathrm{T}}\mathrm{D}Q(\varphi(t)),$$

which means that $\eta = \mathrm{D}Q(\varphi(t))$ is a solution to (3.4). $\qquad\square$

Lemma 3.3. *Let* $\Phi(t)$ *and* $\Psi(t)$ *be fundamental matrices to the VE* (3.3) *and AVE* (3.4), *respectively. Then*

$$\Phi(t)^{\mathrm{T}}\Psi(t) = const.$$

Proof. We easily compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(t)^{\mathrm{T}}\Psi(t)) =& \dot{\Phi}(t)^{\mathrm{T}}\Psi(t) + \Phi(t)^{\mathrm{T}}\dot{\Psi}(t) \\
=& \Phi(t)^{\mathrm{T}}\mathrm{D}p(\varphi(t))^{\mathrm{T}}\Psi(t) - \Phi(t)^{\mathrm{T}}\mathrm{D}p(\varphi(t))^{\mathrm{T}}\Psi(t) = 0,
\end{aligned}$$

which yields the desired result. $\qquad\square$

We return to the proof of Proposition 3.1. By Lemma 3.2 $\xi = q(\varphi(t))$ and $\eta = \mathrm{D}Q(\varphi(t))$ are solutions to the VE (3.3) and AVE (3.4), respectively. Let $\tilde{\eta}(t)$ be another linearly independent solution to (3.4). Noting that $\xi = p(\varphi(t)) = \dot{\varphi}(t)$ is another linearly independent solution to (3.3), we see by Lemma 3.3 that

$$\begin{pmatrix} p(\varphi(t))^{\mathrm{T}} \\ q(\varphi(t))^{\mathrm{T}} \end{pmatrix} (\mathrm{D}Q(\varphi(t)) \ \tilde{\eta}(t)) = \mathrm{const.}, \tag{3.5}$$

so that

$$\begin{pmatrix} p(z)^{\mathrm{T}} \\ q(z)^{\mathrm{T}} \end{pmatrix} \mathrm{D}Q(z) = \begin{pmatrix} 0 \\ C(Q(z)) \end{pmatrix} \tag{3.6}$$

holds almost everywhere in D, where $C(Q(z)) \neq 0$ is a constant only depending on the value of $Q(z)$, since Eq. (3.5) holds at any point $z = \varphi(t)$ on the non-constant solution for the same constant matrix in its right hand side. Note that $\mathrm{D}Q(z)^{\mathrm{T}} p(z) = p(z)^{\mathrm{T}} \mathrm{D}Q(z) = 0$ since $Q(z)$ is a first integral of (3.1). The matrix

$$\begin{pmatrix} p(z)^{\mathrm{T}} \\ q(z)^{\mathrm{T}} \end{pmatrix}$$

is nonsigular and its inverse matrix is given by

$$\frac{1}{\Delta(z)} \begin{pmatrix} q_2(z) & -p_2(z) \\ -q_1(z) & p_1(z) \end{pmatrix}.$$

From (3.6) we obtain

$$\mathrm{D}Q(z) = \frac{1}{\Delta(z)} \begin{pmatrix} -C(Q(z))p_2(z) \\ C(Q(z))p_1(z) \end{pmatrix},$$

which yields (3.2) with $\chi(Q) = C(Q)$. $\qquad\square$

4. Proofs of the Main Theorems

We are now in a position to prove Theorems 1.2 and 1.3.

4.1. **Proof of Theorem 1.2.** By Proposition 2.6 and Remark 2.8, Theorem 1.2 immediately follows from the following proposition.

Proposition 4.1. *If one of the following conditions holds, then the truncated system* (2.11) *is analytically nonintegrable.*

(i) $\alpha_1 \alpha_4 > 0$;
(ii) $\alpha_1 \alpha_4 < 0$ *and* $\alpha_4/\alpha_1 \notin \mathbb{Q}$.

Proof. Assume that condition (i) or (ii) holds. We easily see that the system (2.11) has no constant solution except for $(r, x_3) = (0, 0)$ and a first integral

$$Q(r, x_3) = r^{-2\alpha_4/\alpha_1}(\alpha_3 r^2 + (\alpha_4 - \alpha_1)x_3^2),$$

for which

$$\mathrm{D}_r Q(r, x_3) = \frac{2(\alpha_1 - \alpha_4)}{\alpha_1} r^{-2\alpha_4/\alpha_1 - 1}(\alpha_3 r^2 + \alpha_4 x_3^2),$$

$$\mathrm{D}_{x_3} Q(r, x_3) = -2(\alpha_1 - \alpha_4) r^{-2\alpha_4/\alpha_1} x_3.$$

Obviously, $Q(r, x_3)$ is not analytic. We have the following lemma.

Lemma 4.2. *When* $\alpha_1 \neq 0$, *the system* (2.11) *has an analytic first integral near the origin if and only if* $\alpha_1 \alpha_4 \leq 0$ *and* $\alpha_4/\alpha_1 \in \mathbb{Q}$.

Proof. We easily show the sufficiency. Actually, if $\alpha_1 \neq 0$, $\alpha_1 \alpha_4 \leq 0$ and $2\alpha_4/\alpha_1 = -\ell/m \in \mathbb{Q}$, where $\ell, m \in \mathbb{N}$ are relatively prime, then

$$r^m(\alpha_3 r^2 + (\alpha_4 - \alpha_1)x_3^2)^{\ell}$$

is an analytic first integral.

We turn to the necessity. If $\alpha_1 \alpha_4 > 0$, then

$$\lim_{r \to 0} \lim_{x_3 \to 0} Q(r, x_3) \to 0, \quad \lim_{x_3 \to 0} \lim_{r \to 0} Q(r, x_3) \to \infty,$$

so that $\phi(Q(r, x_3))$ is not analytic for any function $\phi(s)$. We next assume that $\alpha_1 \neq 0$, $\alpha_1\alpha_4 < 0$, $\alpha_4/\alpha_1 \notin \mathbb{Q}$, and $\phi(Q(r, x_3))$ is analytic for some function $\phi(s)$. Let

$$\phi(Q(r, x_3)) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi_{jk} r^j x_3^k,$$

where ϕ_{jk}, $j, k \in \mathbb{Z}_{\geq 0} := \{l \in \mathbb{Z} \mid l \geq 0\}$, are constants and the right hand side is convergent near the origin. Here we can set $\phi_{00} = 0$ without loss of generality. Differentiating the above relation with respect to r and x_3 yields

$$2\phi'(Q(r, x_3))\left(\frac{\alpha_3(\alpha_1 - \alpha_4)}{\alpha_1} r^{1-2\alpha_4/\alpha_1} + \frac{\alpha_4(\alpha_1 - \alpha_4)}{\alpha_1} r^{-2\alpha_4/\alpha_1 - 1} x_3^2\right))$$
$$= \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} j\phi_{jk} r^{j-1} x_3^k$$

and

$$-2\phi'(Q(r, x_3))(\alpha_1 - \alpha_4) r^{-2\alpha_4/\alpha_1} x_3 = \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} k\phi_{jk} r^j x_3^{k-1},$$

respectively, where $\phi'(s) = \mathrm{d}\phi(s)/\mathrm{d}s$. Hence,

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} j\phi_{jk}\, r^j x_3^{k+2} = -\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} k\phi_{jk}\left(\frac{\alpha_3}{\alpha_1} r^{j+2} x_3^k + \frac{\alpha_4}{\alpha_1} r^j x_3^{k+2}\right),$$

which yields

$$\phi_{0k}, \phi_{1k} = 0, \quad \left(\frac{k\alpha_4}{\alpha_1} + j + 2\right)\phi_{j+2,k} = -\frac{(k+2)\alpha_3}{\alpha_1}\phi_{j,k+2}, \quad j, k \in \mathbb{Z}_{\geq 0}.$$

Since

$$\frac{k\alpha_4}{\alpha_1} + j + 2 \neq 0 \quad \text{for any } j \geq 0 \text{ and } k \geq 1,$$

we have

$$\phi_{2j,k} = -\frac{(k+2)\alpha_3}{\alpha_1}\left(\frac{k\alpha_4}{\alpha_1} + 2j\right)^{-1}\phi_{2(j-1),k+2} = \cdots$$
$$= (-1)^j \frac{(k+2)\alpha_3}{\alpha_1} \cdots \frac{(k+2j)\alpha_3}{\alpha_1}$$
$$\times \left(\frac{k\alpha_4}{\alpha_1} + 2j\right)^{-1} \cdots \left(\frac{(k+2(j-1))\alpha_4}{\alpha_1} + 2\right)^{-1}\phi_{0,k+2j} = 0,$$
$$\phi_{2j+1,k} = -\frac{(k+2)\alpha_3}{\alpha_1}\left(\frac{k\alpha_4}{\alpha_1} + 2j + 1\right)^{-1}\phi_{2(j-1),k+2} = \cdots$$
$$= (-1)^j \frac{(k+2)\alpha_3}{\alpha_1} \cdots \frac{(k+2j)\alpha_3}{\alpha_1}$$
$$\times \left(\frac{k\alpha_4}{\alpha_1} + 2j + 1\right)^{-1} \cdots \left(\frac{(k+2(j-1))\alpha_4}{\alpha_1} + 3\right)^{-1}\phi_{1,k+2j} = 0$$

for any $j \geq 1$ and $k \geq 0$. Thus, we obtain $\phi_{jk} = 0$, $j, k \in \mathbb{Z}_{\geq 0}$, which yields a contraction. This means the desired result. $\qquad\square$

Assume that the system (2.11) has a commutative vector field $q(r, x_3)$. Let $p(r, x_3)$ denote the vector field of (2.11). We compute

$$\frac{Q(r, x_3)p_2(r, x_3)}{\mathrm{D}_r Q(r, x_3)} = -\frac{Q(r, x_3)p_1(r, x_3)}{\mathrm{D}_{x_3} Q(r, x_3)} = \frac{\alpha_1 r(\alpha_3 r^2 + (\alpha_4 - \alpha_1)x_3^2)}{2(\alpha_1 - \alpha_4)},$$

so that by Proposition 3.1

$$\Delta(r, x_3) = Cr(\alpha_3 r^2 + (\alpha_4 - \alpha_1)x_3^2), \tag{4.1}$$

where $C \neq 0$ is some constant, since $Q(r, x_3)$ is not analytic. We write the Taylor expansion of $q_j(r, x_3)$ around the origin as

$$q_j(r, x_3) = \sum_{k,l=1}^{\infty} q_{jkl} r^k x_3^l,$$

where $q_{jkl} \in \mathbb{C}$, $k, l = 1, 2$, are constants, for $j = 1, 2$. Substituting them into (4.1) and solving the resulting equation about p_{jkl}, we obtain

$$q(r, x_3) = C \begin{pmatrix} r \\ x_3 \end{pmatrix} + O(|r|^2 + |x_3|^2).$$

So we have

$$\mathrm{D}p(r, x_3)q(r, x_3) - \mathrm{D}q(r, x_3)p(z) = C \begin{pmatrix} \alpha_1 r x_3 \\ \alpha_3 r^2 + \alpha_4 x_3^2 \end{pmatrix} + O(|r|^3 + |x_3|^3),$$

which means that $q(r, x_3)$ is not a commutative vector field. Thus, we obtain the desired result. $\square$

4.2. **Proof of Theorem 1.3.** As in Section 4.1, by Proposition 2.7 and Remark 2.8, Theorem 1.3 immediately follows from the following proposition.

Proposition 4.3. *If $\alpha_1 \neq \alpha_3$, $\alpha_2 \neq \alpha_4$, and one of the following conditions holds, then the truncated system (2.12) is analytically nonintegrable near $(r_1, r_2) = (0, 0)$:*

(i) $\alpha_1\alpha_3$ *or* $\alpha_2\alpha_4 > 0$;
(ii) $\alpha_1\alpha_3$, $\alpha_2\alpha_4 < 0$ *and* α_1/α_3, $\alpha_2/\alpha_4 \notin \mathbb{Q}$.

Proof. Assume that $\alpha_1 \neq \alpha_3$, $\alpha_2 \neq \alpha_4$, and condition (i) or (ii) holds. We easily see that the system (2.12) has no constant solution except for $(r, x_3) = (0, 0)$ and a first integral

$$Q(r_1, r_2) = \left(\frac{r_1^{\alpha_4}}{r_2^{\alpha_2}}\right)^{2(\alpha_1 - \alpha_3)} \left(\frac{(\alpha_1 - \alpha_3)r_1^2}{r_2^2} + \alpha_2 - \alpha_4\right)^{\alpha_2\alpha_3 - \alpha_1\alpha_4},$$

for which

$$\mathrm{D}_{r_1}Q(r_1, r_2) = \frac{2(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)(\alpha_3 r_1^2 + \alpha_4 r_2^2)}{r_1((\alpha_1 - \alpha_3)r_1^2 + (\alpha_2 - \alpha_4)r_2^2)}Q(r_1, r_2),$$

$$\mathrm{D}_{r_2}Q(r_1, r_2) = -\frac{2(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)(\alpha_1 r_1^2 + \alpha_2 r_2^2)}{r_2((\alpha_1 - \alpha_3)r_1^2 + (\alpha_2 - \alpha_4)r_2^2)}Q(r_1, r_2).$$

Obviously, $Q(r_1, r_2)$ is not analytic. We have the following.

Lemma 4.4. *If $\alpha_1 \neq \alpha_3$, $\alpha_2 \neq \alpha_4$, and condition* (i) *or* (ii) *in Proposition* 4.3 *holds, then the system* (2.12) *has no analytic first integral.*

Proof. We first assume that $\alpha_2\alpha_3 - \alpha_1\alpha_4 = 0$ and $\alpha_2\alpha_4 \neq 0$ as well as $\alpha_1 \neq \alpha_3$. Then $\alpha_2 \neq \alpha_4$ and $\alpha_1\alpha_3 \neq 0$. Hence, $\alpha_2\alpha_4 > 0$ (resp. < 0) if and only if $\alpha_1\alpha_3 > 0$ (resp. < 0). We see that

$$\tilde{Q}(r_1, r_2) = \frac{r_1}{r_2^{\alpha_2/\alpha_4}}$$

is a first integral, so that conditions (i) and (ii) do not hold if $\phi(\tilde{Q}(r_1, r_2))$ is analytic for some function $\phi(s)$.

We next assume that $\alpha_1 \neq \alpha_3$, $\alpha_2 \neq \alpha_4$, $\alpha_1\alpha_2\alpha_3\alpha_4 \neq 0$, and $\alpha_2\alpha_3 - \alpha_1\alpha_4 \neq 0$. We see that

$$\tilde{Q}(r_1, r_2) = \frac{r_1^{\rho_1} r_2^{\rho_2}}{\alpha_2 - \alpha_4} + \frac{r_1^{\rho_1'} r_2^{\rho_2'}}{\alpha_1 - \alpha_3},$$

where

$$\begin{aligned}
\rho_1 &= 2\alpha_3(\alpha_2 - \alpha_4)/(\alpha_2\alpha_3 - \alpha_1\alpha_4), \\
\rho_2 &= -2\alpha_1(\alpha_2 - \alpha_4)/(\alpha_2\alpha_3 - \alpha_1\alpha_4), \\
\rho_1' &= 2\alpha_4(\alpha_1 - \alpha_3)/(\alpha_2\alpha_3 - \alpha_1\alpha_4), \\
\rho_2' &= -2\alpha_2(\alpha_1 - \alpha_3)/(\alpha_2\alpha_3 - \alpha_1\alpha_4),
\end{aligned}$$

is a first integral. We have

$$\tilde{Q}(r_1, r_2) \to 0 \quad \text{or} \quad 1/\tilde{Q}(r_1, r_2) \to 0 \tag{4.2}$$

as $(r_1, r_2) \to (0,0)$ only if $\alpha_1\alpha_3, \alpha_3\alpha_4 < 0$. Hence, if condition (i) holds, then there is no function $\phi(s)$ such that $\phi(\tilde{Q}(r_1, r_2))$ is analytic, since if so, then Eq. (4.2) must hold.

We additionally assume that condition (ii) holds and $\phi(\tilde{Q}(r_1, r_2))$ is analytic for some function $\phi(s)$. By assumption, ρ_j and ρ_j, $j = 1, 2$, are of the same sign, and

$$\frac{\rho_1}{\rho_2}, \frac{\rho_1'}{\rho_2'} \notin \mathbb{Q}. \tag{4.3}$$

Let

$$\phi(\tilde{Q}(r_1, r_2)) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \phi_{jk} r_1^j r_2^l,$$

where ϕ_{jk}, $j, k \in \mathbb{Z}_{\geq 0}$, are constants, $\phi_{00} = 0$, and the right hand side is convergent near the origin. Differentiating the above relation with respect to r_1 and r_2 yields

$$\frac{2\phi'(\tilde{Q}(r_1, r_2))}{\alpha_2\alpha_3 - \alpha_1\alpha_4}\left(\alpha_3 r_1^{\rho_1-1} r_2^{\rho_2} + \alpha_4 r_1^{\rho_1'-1} r_2^{\rho_2'}\right) = \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} j\phi_{jk} r_1^{j-1} r_2^k$$

and

$$-\frac{2\phi'(\tilde{Q}(r_1, r_2))}{\alpha_2\alpha_3 - \alpha_1\alpha_4}\left(\alpha_1 r_1^{\rho_1} r_2^{\rho_2-1} + \alpha_2 r_1^{\rho_1'} r_2^{\rho_2'-1}\right) = \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} k\phi_{jk} r_1^j r_2^{k-1},$$

respectively, where $\phi'(s) = \mathrm{d}\phi(s)/\mathrm{d}s$. Hence,

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} j\phi_{jk}\left(\alpha_1 r_1^{\rho_1+j-1} r_2^{\rho_2+k-1} + \alpha_2 r_1^{\rho_1'+j-1} r_2^{\rho_2'+k-1}\right)$$

$$= -\sum_{j=0}^{\infty}\sum_{k=1}^{\infty} k\phi_{jk}\left(\alpha_3 r_1^{\rho_1+j-1} r_2^{\rho_2+k-1} + \alpha_4 r_1^{\rho_1'+j-1} r_2^{\rho_2'+k-1}\right). \tag{4.4}$$

We consider the following two cases separately: (i) $\rho_1 - \rho_1'$ or $\rho_2 - \rho_2' \notin \mathbb{Z}$; and (ii) $\rho_1 - \rho_1', \rho_2 - \rho_2' \in \mathbb{Z}$.

Case (i): From (4.4) we have $\phi_{j0}, \phi_{0k} = 0$, and

$$(j\alpha_1 + k\alpha_3)\phi_{jk} = (j\alpha_2 + k\alpha_4)\phi_{jk} = 0,$$

i.e., $\phi_{jk} = 0$, for $j, k \in \mathbb{N}$.

Case (ii): We assume that $\rho_1 - \rho_1' = j_0 \in \mathbb{Z}_{\geq 0}$ and $\rho_2 - \rho_2' = k_0 \in \mathbb{Z}_{\geq 0}$. The other cases can be treated similarly. If $j_0 = 0$ and $k_0 = 0$, then $\rho_1 - \rho_1' = 0$ and $\rho_2 - \rho_2' = 0$, respectively, so that $\alpha_2\alpha_3 - \alpha_1\alpha_4 = 0$. Hence, $j_0, k_0 > 0$. From (4.4) we have $\phi_{j0}, \phi_{0k} = 0$ for $j, k \in \mathbb{N}$ and

$$(j\alpha_2 + k\alpha_4)\phi_{jk} = 0 \quad \text{for } 0 < j < j_0 \text{ or } 0 < k < k_0,$$
$$(j\alpha_1 + k\alpha_3)\phi_{jk} + ((j + j_0)\alpha_2 + (k + k_0)\alpha_4)\phi_{j+j_0,k+k_0} = 0$$
$$\text{for } j \geq j_0 \text{ and } k \geq k_0,$$

which yields $\phi_{jk} = 0$ for $j, k \in \mathbb{N}$.

Thus, we have a contraction for both cases. This completes the proof. $\square$

Assume that the system (2.12) has a commutative vector field $q(r_1, r_2)$. Let $p(r_1, r_2)$ denote the vector field of (2.12). We have

$$\frac{Q(r_1, r_2)p_2(r_1, r_2)}{\mathrm{D}_{r_1}Q(r_1, r_2)} = -\frac{Q(r_1, r_2)p_1(r_1, r_2)}{\mathrm{D}_{r_2}Q(r_1, r_2)} = \tfrac{1}{2}r_1 r_2 \left(\frac{r_1^2}{\alpha_2 - \alpha_4} + \frac{r_1^2}{\alpha_1 - \alpha_3} \right),$$

so that by Proposition 3.1

$$\Delta(r_1, r_2) = C r_1 r_2 \left(\frac{r_1^2}{\alpha_2 - \alpha_4} + \frac{r_2^2}{\alpha_1 - \alpha_3} \right), \tag{4.5}$$

where $C \neq 0$ is some constant, since $Q(r_1, r_2)$ is not analytic. We write the Taylor expansion of $q_j(r_1, r_2)$ around the origin as

$$q_j(r_1, r_2) = \sum_{k,l=1}^{\infty} q_{jkl} r_1^k r_2^l,$$

where $q_{jkl} \in \mathbb{C}$, $k, l = 1, 2$, are constants, for $j = 1, 2$. Substituting them into (4.1) and solving the resulting equation about q_{jkl}, we obtain

$$q(r_1, r_2) = -\frac{C}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + O(|r|_1^2 + |r_2|^2).$$

So we have

$$\mathrm{D}p(r_1, r_2)q(r_1, r_2) - \mathrm{D}q(r_1, r_2)p(r_1, r_2)$$
$$= -\frac{2C}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)} \begin{pmatrix} (\alpha_1 r_1^2 + \alpha_2 r_2^2)r_1 \\ (\alpha_3 r_1^2 + \alpha_4 r_2^2)r_2 \end{pmatrix} + O\left((|r|^4 + |x_3|^4) \right),$$

which means that $q(r_1, r_2)$ is not a commutative vector field. Thus, we obtain the desired result. $\square$

5. EXAMPLES

As stated in Section 1, Theorems 1.2 and 1.3 imply that three- or four-dimensional systems exhibiting fold-Hopf and double-Hopf bifurcations are analytically nonintegrable under the weak conditions. In this section we give two such examples.

5.1. **Rössler system.** We first consider the three-dimensional system

$$\dot{x}_1 = -(x_2 + x_3), \quad \dot{x}_2 = x_1 + ax_2, \quad \dot{x}_3 = bx_1 + x_3(x_1 - c), \qquad (5.1)$$

where a, b, c are constants. The Rössler system

$$\dot{x}_1 = -(x_2 + x_3), \quad \dot{x}_2 = x_1 + a_0 x_2, \quad \dot{x}_3 = b_0 + x_3(x_1 - c_0), \qquad (5.2)$$

which was originally proposed by Rössler [35] and has been extensively studied, e.g., in [5, 12, 13, 19, 21, 22, 34, 41, 52, 53], is transformed to (5.1) with

$$a = a_0, \quad b = \frac{c_0 \pm \sqrt{c_0^2 - 4a_0 b_0}}{2a_0}, \quad c = \tfrac{1}{2}\left(c_0 \pm \sqrt{c_0^2 - 4a_0 b_0}\right),$$

by a change of coordinates

$$x \mapsto x - \left(x_{10}, -\frac{x_{10}}{a_0}, \frac{x_{10}}{a_0}\right), \quad x_{10} = \tfrac{1}{2}\left(c_0 \mp \sqrt{c_0^2 - 4a_0 b_0}\right)$$

if $c_0^2 > 4a_0 b_0$ and $a_0 \neq 0$, where the upper or lower sign is taken simultaneously, and with

$$a = 0, \quad b = \frac{b_0}{c_0}, \quad c = c_0,$$

by a change of coordinates

$$x \mapsto x - \left(0, -\frac{b_0}{c_0}, \frac{b_0}{c_0}\right)$$

if $a_0 = 0$ and $c_0 \neq 0$. The system (5.1) has also been referred to as the Rössler system in some references. Periodic orbits, invariant tori, chaos and fold-Hopf bifurcations in (5.1) were studied in [7, 20, 25, 51]. Moreover, the $(1, 2)$- or $(2, 1)$-integrability of (5.1) and/or (5.2) was discussed in [21, 22, 52] and the following results were obtained:

(i) The systems (5.1) and (5.2) are analytically $(1, 2)$-integrable when $a, b, c = 0$ and $a_0, b_0, c_0 = 0$, respectively;
(ii) The system (5.2) is neither analytically $(1, 2)$- nor $(2, 1)$-integrable near $x = 0$ when $a_0 \neq 0$.

The second statement above means that the system (5.1) is neither analytically $(1, 2)$- nor $(2, 1)$-integrable near any point on the line

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = -ax_1, x_3 = ax_1\},$$

especially near the origin when $a \neq 0$ and $c = 2ab \neq 0$. However, we cannot deny from the statement that the systems (5.1) and (5.2) may be analytically $(3, 0)$-integrable near the origin even when $a_0 \neq 0$.

When $b = 1$ and $c = a \in (-\sqrt{2}, \sqrt{2})$, the system (5.1) satisfies condition (I) with $\omega = \sqrt{2 - a^2}$. We compute the coefficients in (1.4) as

$$\alpha_1 = -\frac{a^3}{2\omega^2}, \quad \alpha_2 = \frac{a^2 + 1}{2\omega}, \quad \alpha_3 = \frac{2a}{\omega^2}, \quad \alpha_4 = \frac{a}{\omega^2}. \qquad (5.3)$$

See Appendix A.1 for the derivation of (5.3). Applying Theorem 1.2, we obtain the following.

Proposition 5.1. *When $b = 1$, $c = a \in (-\sqrt{2}, \sqrt{2})$ and $a^2 \notin \mathbb{Q}$, the Rössler system* (5.1) *is not real-analytically integrable near the origin.*

5.2. **Coupled van der Pol oscillators.** We turn to the second example, the coupled van der Pol oscillators

$$\ddot{u}_1 - (\delta_1 - a_1 u_1^2)\dot{u}_1 + u_1 = b_1 u_2,$$
$$\ddot{u}_2 - (\delta_2 - a_2 u_2^2)\dot{u}_2 + cu_2 = b_2 u_1,$$

or as a first-order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + (\delta_1 - a_1 x_1^2)x_2 + b_1 x_3,$$
$$\dot{x}_3 = x_4, \quad \dot{x}_4 = -cx_3 + (\delta_2 - a_2 x_3^2)x_4 + b_2 x_1, \tag{5.4}$$

where $x = (u_1, \dot{u}_1, u_2, \dot{u}_2)$, and $\delta_j, a_j, b_j, c \in \mathbb{R}$, $j = 1, 2$, and $c > 0$ are constants. The coupled van der Pol oscillators such as (5.4) have attracted much attention in the field of dynamical systems and for instance, their dynamics and bifurcations with $a_j = \delta_j > 0$, $j = 1, 2$, or $a_j = \delta_1 > 0$ were studied in [8, 17, 23, 32, 33, 37, 38]. For simplicity we assume that $a_j, b_j > 0$, $j = 1, 2$, and $c > 1$.

When $\delta_j = 0$, $j = 1, 2$, and $b_1 b_2 < c$, the system (5.4) satisfies condition (II) with

$$\omega_1 = \sqrt{\frac{(c+1) - \sqrt{(c-1)^2 + 4b_1 b_2}}{2}}, \quad \omega_2 = \sqrt{\frac{(c+1) + \sqrt{(c-1)^2 + 4b_1 b_2}}{2}}$$

if $\omega_1/\omega_2 \notin \mathbb{Q}$. We easily see that $\omega_1 < 1 < \omega_2$ and that

$$\omega_1^2 + \omega_2^2 = c + 1, \quad \omega_1^2 \omega_2^2 = c - b_1 b_2, \quad (\omega_1^2 - 1)(\omega_2^2 - 1) = -b_1 b_2. \tag{5.5}$$

We compute the coefficients in (1.5) as

$$\alpha_1 = \frac{a_1 b_1(\omega_2^2 - 1)^2 + a_2 b_2(\omega_1^2 - 1)^2}{2b_2 \omega_1^2(\omega_1^2 - 1)(\omega_2^2 - \omega_1^2)}, \quad \alpha_2 = \frac{(\omega_1^2 - 1)(a_1 b_1 + a_2 b_2)}{b_2 \omega_2^2(\omega_2^2 - \omega_1^2)},$$
$$\alpha_3 = -\frac{(\omega_2^2 - 1)(a_1 b_1 + a_2 b_2)}{b_2 \omega_1^2(\omega_2^2 - \omega_1^2)}, \quad \alpha_4 = -\frac{a_1 b_1(\omega_1^2 - 1)^2 + a_2 b_2(\omega_2^2 - 1)^2}{2b_2 \omega_2^2(\omega_2^2 - 1)(\omega_2^2 - \omega_1^2)}, \tag{5.6}$$
$$\beta_j = 0, \quad j = 1, 2, 3, 4.$$

See Appendix A.2 for the derivation of (5.6). Using (5.5), we see that the conditions $\alpha_1 \neq \alpha_3$ and $\alpha_2 \neq \alpha_4$ become

$$a_1 b_1((\omega_2^2 - 1)^2 - 2b_1 b_2) + a_2 b_2((\omega_1^2 - 1)^2 - 2b_1 b_2) \neq 0 \tag{5.7}$$

and

$$a_1 b_1(\omega_1^2 - 1)(\omega_1^2 + 2\omega_2^2 - 3) + a_2 b_2(\omega_2^2 - 1)(2\omega_1^2 + \omega_2^2 - 3) \neq 0, \tag{5.8}$$

respectively. Moreover, the conditions $\alpha_1 \alpha_3 > 0$ and $\alpha_2 \alpha_4 > 0$ become

$$a_1 b_1(\omega_2^2 - 1)^2 + a_2 b_2(\omega_1^2 - 1)^2 > 0$$

and

$$a_1 b_1(\omega_1^2 - 1)^2 + a_2 b_2(\omega_2^2 - 1)^2 > 0,$$

respectively, and both of them always hold since $\omega_1 < 1 < \omega_2$ and $a_j, b_j > 0$, $j = 1, 2$. Applying Theorem 1.3, we obtain the following.

Proposition 5.2. *When $\delta_j = 0$, $a_j, b_j > 0$, $j = 1, 2$, $b_1 b_2 < c$ and $\omega_1/\omega_2 \notin \mathbb{Q}$, the coupled van der Pol oscillators (5.4) are not real-analytically integrable near the origin if conditions (5.7) and (5.8) hold.*

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DATA AVAILABILITY

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

CONFLICT OF INTEREST STATEMENT

The author declares that he has no conflict of interest.

APPENDIX A. DERIVATION OF (5.3) AND (5.6)

We compute the coefficients in (1.4) and (1.5) for the examples in Sections 5.1 and 5.2, and derive (5.3) and (5.6). For the reader's convenience, we also give general formulas for computing these coefficients. See Sections 8.7.5 and 8.7.6 of [18] for the details. We write the Taylor expansion of $f(x)$ around $x = 0$ as

$$f(x) = Ax + \tfrac{1}{2}B(x, x) + \tfrac{1}{6}C(x, x, x) + O(|x|^4),$$

where $A = \mathrm{D}f(0)$, and $B(\xi, \eta)$ and $C(\xi, \eta, \zeta)$ are the bilinear and trilinear vector-values functions with components

$$B_j(\xi, \eta) = \sum_{k,l=1}^{n} \frac{\partial^2 f_j}{\partial x_k \partial x_l}(0)\xi_k\eta_l, \quad C_j(\xi, \eta, \zeta) = \sum_{k,l,m=1}^{n} \frac{\partial^3 f_j}{\partial x_k \partial x_l \partial x_m}(0)\xi_k\eta_l\zeta_m$$

for $j = 1, \ldots, n$ with $n = 3$ or 4.

A.1. **Derivation of** (5.3). Assume that $f(x)$ satisfies condition (I). Let $v_0 \in \mathbb{R}^3$ and $v_1 \in \mathbb{C}^3$ be eigenvectors of A corresponding to the eigenvalues $\lambda = 0$ and $i\omega$, respectively,

$$Av_0 = 0, \quad Av_1 = i\omega v_1,$$

and let $u_0 \in \mathbb{R}^3$ and $u_1 \in \mathbb{C}^3$ be eigenvectors of A^{T} corresponding to the eigenvalues $\lambda = 0$ and $-i\omega$, respectively,

$$A^{\mathrm{T}}u_0 = 0, \quad A^{\mathrm{T}}u_1 = -i\omega u_1,$$

such that $\langle u_0, v_0 \rangle = \langle u_1, v_1 \rangle = 1$, where $\langle \cdot, \cdot \rangle$ represents the inner product in $\mathbb{C}^n$. Then we can transform (1.1) to

$$\begin{aligned} \dot{w}_0 &= \kappa_{01}w_0^2 + \kappa_{02}|w_1|^2 + O(|w_0|^3 + |w_1|^3), \\ \dot{w}_1 &= i\omega w_1 + \kappa_{11}w_0w_1 + O(|w_0|^3 + |w_1|^3), \end{aligned} \tag{A.1}$$

where $w_0 = \langle u_0, x \rangle \in \mathbb{R}$, $w_1 = \langle u_1, x \rangle \in \mathbb{C}$ and

$$\begin{aligned} \kappa_{01} &= \tfrac{1}{2}\langle u_0, B(v_0, v_0) \rangle \in \mathbb{R}, \quad \kappa_{02} = \langle u_0, B(v_1, v_1^*) \rangle \in \mathbb{R}, \\ \kappa_{11} &= \langle u_1, B(u_0, v_1) \rangle \in \mathbb{C} \end{aligned} \tag{A.2}$$

with the superscript '$*$' denoting complex conjugate. Letting $x_1 = \operatorname{Re} w_1$, $x_2 = \operatorname{Im} w_1$ and $x_3 = w_0$, we rewrite (A.1) as (1.4) with

$$\alpha_1 = \operatorname{Re} \kappa_{11}, \quad \alpha_2 = \operatorname{Im} \kappa_{11}, \quad \alpha_3 = \kappa_{02}, \quad \alpha_4 = \kappa_{01}$$

up to $O(|x|^2)$.

We now compute the coefficients α_j, $j = $ 1-4, for (5.4) when $b = 1$ and $c = a \in (-\sqrt{2}, \sqrt{2})$. We have

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 1 & 0 & -a \end{pmatrix}, \quad B(\xi, \eta) = \begin{pmatrix} 0 \\ 0 \\ \xi_1 \eta_3 + \xi_3 \eta_1 \end{pmatrix}$$

and

$$v_0 = \begin{pmatrix} a \\ -1 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} a + i\omega \\ 1 - a^2 - i\omega a \\ 1 \end{pmatrix},$$

$$u_0 = \frac{1}{\omega^2} \begin{pmatrix} -a \\ -1 \\ 1 \end{pmatrix}, \quad u_1 = \frac{1}{2\omega^2(a^2 - 1 - i\omega a)} \begin{pmatrix} a - i\omega \\ a^2 - 1 - i\omega a \\ -1 \end{pmatrix}.$$

By (A.2) we obtain

$$\kappa_{01} = \frac{a}{\omega^2}, \quad \kappa_{02} = \frac{2a}{\omega^2}, \quad \kappa_{11} = \frac{-a^3 + i\omega(a^2 + 1)}{2\omega^2}$$

which yields (5.3).

A.2. **Derivation of** (5.6). Assume that $f(x)$ satisfies condition (II). For simplicity we also assume that $B(x, x) \equiv 0$. For $j = 1, 2$, let $v_j \in \mathbb{C}^4$ be an eigenvector of A corresponding to the eigenvalue $\lambda = i\omega_j$,

$$A v_j = i\omega_j v_j,$$

and let $u_j \in \mathbb{C}^4$ be an eigenvector of A^{T} corresponding to the eigenvalue $-i\omega_j$,

$$A^{\mathrm{T}} u_j = -i\omega_j u_j,$$

such that $\langle u_j, v_j \rangle = 1$. Then we can transform (1.1) to

$$\begin{aligned} \dot{w}_1 &= i\omega_1 w_1 + \kappa_{11} w_1 |w_1|^2 + \kappa_{12} w_1 |w_2|^2 + O(|w_1|^4 + |w_1|^4), \\ \dot{w}_2 &= i\omega_2 w_2 + \kappa_{21} w_2 |w_1|^2 + \kappa_{22} w_2 |w_2|^2 + O(|w_1|^4 + |w_1|^4), \end{aligned} \tag{A.3}$$

where $w_j = \langle u_j, x \rangle \in \mathbb{C}$, $j = 1, 2$, and

$$\begin{aligned} \kappa_{11} &= \tfrac{1}{2}\langle u_1, C(v_1, v_1, v_1^*) \rangle, \quad \kappa_{12} = \langle u_1, C(v_1, v_2, v_2^*) \rangle, \\ \kappa_{21} &= \langle u_2, C(v_1, v_1^*, v_2) \rangle, \quad \kappa_{22} = \tfrac{1}{2}\langle u_2, C(v_2, v_2, v_2^*) \rangle \end{aligned} \tag{A.4}$$

are also complex. Letting $x_1 = \operatorname{Re} w_1$, $x_2 = \operatorname{Im} w_1$, $x_3 = \operatorname{Re} w_2$ and $x_4 = \operatorname{Im} w_2$, we rewrite (A.3) as (1.3) with

$$\begin{aligned} \alpha_1 &= \operatorname{Re} \kappa_{11}, \quad \alpha_2 = \operatorname{Re} \kappa_{12}, \quad \alpha_3 = \operatorname{Re} \kappa_{21}, \quad \alpha_4 = \operatorname{Re} \kappa_{22}, \\ \beta_1 &= \operatorname{Im} \kappa_{11}, \quad \beta_2 = \operatorname{Im} \kappa_{12}, \quad \beta_3 = \operatorname{Im} \kappa_{21}, \quad \beta_4 = \operatorname{Im} \kappa_{22} \end{aligned}$$

up to $O(|x|^3)$.

We now compute the coefficients $\alpha_j, \beta_j, j = 1\text{-}4$, for (5.1) with $\delta_j = 0$, $j = 1, 2$, and $b_1 b_2 < c$. Note that $B(\xi, \eta) \equiv 0$. We have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & b_1 & 0 \\ 0 & 0 & 0 & 1 \\ b_2 & 0 & -c & 0 \end{pmatrix}, \quad C(\xi, \eta, \zeta) = \begin{pmatrix} 0 \\ -2a_1(\xi_1\eta_1\zeta_2 + \xi_1\eta_2\zeta_1 + \xi_2\eta_1\zeta_1) \\ 0 \\ -2a_2(\xi_3\eta_3\zeta_4 + \xi_3\eta_4\zeta_3 + \xi_4\eta_3\zeta_3) \end{pmatrix}$$

and

$$v_1 = \begin{pmatrix} ib_1/(\omega_1(\omega_1^2 - 1)) \\ -b_1/(\omega_1^2 - 1) \\ -i/\omega_1 \\ 1 \end{pmatrix}, \quad u_1 = \frac{\omega_1^2 - 1}{2(\omega_1^2 - \omega_2^2)} \begin{pmatrix} -i\omega_1(\omega_2^2 - 1)/b_1 \\ (\omega_2^2 - 1)/b_1 \\ -i\omega_1 \\ 1 \end{pmatrix},$$

$$v_2 = \begin{pmatrix} ib_1/(\omega_2(\omega_2^2 - 1)) \\ -b_1/(\omega_2^2 - 1) \\ -i/\omega_2 \\ 1 \end{pmatrix}, \quad u_2 = \frac{\omega_2^2 - 1}{2(\omega_2^2 - \omega_1^2)} \begin{pmatrix} -i\omega_2(\omega_1^2 - 1)/b_1 \\ (\omega_1^2 - 1)/b_1 \\ -i\omega_2 \\ 1 \end{pmatrix}.$$

Using (5.5) and (A.4), we obtain

$$\kappa_{11} = \frac{a_1 b_1^2(\omega_2^2 - 1) + a_2 b_2(\omega_1^2 - 1)^2}{2b_2\omega_1^2(\omega_1^2 - 1)(\omega_2^2 - \omega_1^2)},$$

$$\kappa_{12} = \frac{(\omega_1^2 - 1)(a_1 b_1 + a_2 b_2)}{b_2\omega_2^2(\omega_2^2 - \omega_1^2)},$$

$$\kappa_{21} = \frac{(\omega_2^2 - 1)(a_1 b_1 + a_2 b_2)}{b_2\omega_1^2(\omega_2^2 - \omega_1^2)},$$

$$\kappa_{22} = -\frac{a_1 b_1(\omega_1^2 - 1)^2 + a_2 b_2(\omega_2^2 - 1)^2}{2b_2\omega_2^2(\omega_2^2 - 1)(\omega_2^2 - \omega_1^2)},$$

which yields (5.6).

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