Painlevé equations on weighted projective spaces

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Abstract

Weighted degrees of Hamiltonian functions of the Painlevé equations are investigated. A tuple of positive integers, called a regular weight, satisfying certain conditions related to singularity theory is classified. In particular, for 2 and 4-dim cases, it is shown that there exists a Painlevé equation associated with each regular weight.
Keywords: Painlevé equations; weights

1 Introduction

A differential equation defined on a complex region is said to have the Painlevé property if any movable singularity of any solution is a pole. Painlevé and his group classified second order ODEs having the Painlevé property and found new six differential equations called the Painlevé equations. Nowadays, it is known that they are written in Hamiltonian forms

$$(\mathrm{P_J}) : \frac{dq}{dz} = \frac{\partial H_J}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial H_J}{\partial q}, \quad J = \mathrm{I}, \cdots, \mathrm{VI}. \tag{1.1}$$

Among six Painlevé equations, the Hamiltonian functions of the first, second and fourth Painlevé equations are polynomials in both of the independent variable z and the dependent variables (q, p). They are given by

$$\begin{aligned} H_{\mathrm{I}} &= \frac{1}{2}p^2 - 2q^3 - zq, \\ H_{\mathrm{II}} &= \frac{1}{2}p^2 - \frac{1}{2}q^4 - \frac{1}{2}zq^2 - \alpha q, \\ H_{\mathrm{IV}} &= -pq^2 + p^2 q - 2pqz - \alpha p + \beta q, \end{aligned}$$

respectively, where $\alpha, \beta \in \mathbb{C}$ are arbitrary parameters.

In general, a polynomial $H(x_1, \cdots, x_n)$ is called a quasihomogeneous polynomial if there are positive integers $a_1, \cdots, a_n$ and h such that

$$H(\lambda^{a_1} x_1, \cdots, \lambda^{a_n} x_n) = \lambda^h H(x_1, \cdots, x_n) \tag{1.2}$$

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for any $\lambda \in \mathbb{C}$. A polynomial H is called a semi-quasihomogeneous if H is decomposed into two polynomials as $H = H^P + H^N$, where H^P satisfies (1.2) and H^N satisfies

$$H^N(\lambda^{a_1}x_1, \cdots, \lambda^{a_n}x_n) \sim o(\lambda^h), \quad |\lambda| \to \infty.$$

The integer $\deg(H) := h$ is called the weighted degree of H with respect to the weight $\deg(x_1, \cdots, x_n) := (a_1, \cdots, a_n)$. H^P and H^N are called the principle part and the non-principle part of H, respectively. The weight of H is determined by the Newton diagram. Plot all exponents $(r_1, \cdots, r_n)$ of monomials $x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$ included in H^P on the integer lattice in $\mathbb{R}^n$. If they lie on a unique hyperplane $a_1x_1+\cdots+a_nx_n = h$, then $\deg(H^P) = h$ with respect to the weight $(a_1, \cdots, a_n)$. Exponents of monomials included in H^N should be on the lower side of the hyperplane.

The above Hamiltonian functions are semi-quasihomogeneous. If we define degrees of variables by $\deg(q, p, z) = (2, 3, 4)$ for H_{I}, $\deg(q, p, z) = (1, 2, 2)$ for H_{II} and $\deg(q, p, z) = (1, 1, 1)$ for H_{IV}, then Hamiltonian functions have the weighted degrees $6, 4$ and 3, respectively, (Table 1) with $H_{\mathrm{I}}^N = 0, H_{\mathrm{II}}^N = -\alpha q$ and $H_{\mathrm{IV}}^N = -\alpha p + \beta q$.

Higher dimensional Painlevé equations have not been classified yet, however, a lot of such equations have been reported in the literature. A list of four dimensional Painlevé equations derived from the monodromy preserving deformation is given in [14, 12]. Lie-algebraic approach is often employed to find new Painlevé equations [8, 9, 10, 16]. Several Painlevé hierarchies, which are hierarchies of $2n$-dimensional Painlevé equations, are obtained by the similarity reductions of soliton equations such as the KdV equation. Among them, it is known that Hamiltonian functions of the the first Painlevé hierarchy $(\mathrm{P_I})_n$[13, 15, 18], the second-first Painlevé hierarchy $(\mathrm{P_{II\text{-}1}})_n$[5, 6, 13, 15], the second-second Painlevé hierarchy $(\mathrm{P_{II\text{-}2}})_n$ and the fourth Painlevé hierarchy $(\mathrm{P_{IV}})_n$[11, 13] can be expressed as polynomials with respect to both of the dependent variables and the independent variables. They are Hamiltonian PDEs of the form

$$\left\{ \begin{array}{l} \dfrac{\partial q_j}{\partial z_i} = \dfrac{\partial H_i}{\partial p_j}, \; \dfrac{\partial p_j}{\partial z_i} = -\dfrac{\partial H_i}{\partial q_j}, \quad j = 1, \cdots, n; \; i = 1, \cdots, n \\ H_i = H_i(q_1, \cdots, q_n, p_1, \cdots, p_n, z_1, \cdots, z_n) \end{array} \right. \tag{1.3}$$

consisting of n Hamiltonians $H_1, \cdots, H_n$ with n independent variables $z_1, \cdots, z_n$. When $n = 1$, $(\mathrm{P_I})_1$ and $(\mathrm{P_{IV}})_1$ are reduced to the first and fourth Painlevé equations, respectively. Both of $(\mathrm{P_{II\text{-}1}})_1$ and $(\mathrm{P_{II\text{-}2}})_1$ coincide with the second Painlevé equation, while they are different systems for $n \geq 2$. When $n = 2$, Hamiltonians of $(\mathrm{P_I})_2$, $(\mathrm{P_{II\text{-}1}})_2$, $(\mathrm{P_{II\text{-}2}})_2$ and $(\mathrm{P_{IV}})_2$ are given by

$$(\mathrm{P_I})_2 \left\{ \begin{array}{l} H_1 = 2p_2p_1 + 3p_2^2q_1 + q_1^4 - q_1^2q_2 - q_2^2 - z_1q_1 + z_2(q_1^2 - q_2), \\ H_2 = p_1^2 + 2p_2p_1q_1 - q_1^5 + p_2^2q_2 + 3q_1^3q_2 - 2q_1q_2^2 \\ \qquad +z_1(q_1^2 - q_2) + z_2(z_2q_1 + q_1q_2 - p_2^2), \end{array} \right. \tag{1.4}$$

$$(\mathrm{P_{II\text{-}1}})_2 \left\{ \begin{array}{l} H_1 = 2p_1p_2 - p_2^3 - p_1q_1^2 + q_2^2 - z_1p_2 + z_2p_1 + 2\alpha q_1, \\ H_2 = -p_1^2 + p_1p_2^2 + p_1p_2q_1^2 + 2p_1q_1q_2 \\ \qquad +z_1p_1 + z_2(z_2p_1 - p_1q_1^2 + p_1p_2) - \alpha(2p_2q_1 + 2q_2 + 2z_2q_1), \end{array} \right. \tag{1.5}$$

$$
(\mathrm{P}_{\text{II-2}})_2 \begin{cases} H_1 = p_1p_2 - p_1q_1^2 - 2p_1q_2 + p_2q_1q_2 + q_1q_2^2 + q_2z_1 + z_2(q_1q_2 - p_1) + \alpha q_1, \\ H_2 = p_1^2 - p_1p_2q_1 + p_2^2q_2 - 2p_1q_1q_2 - p_2q_2^2 + q_1^2q_2^2 \\ \qquad\qquad +z_1(q_1q_2 - p_1) - z_2(p_1q_1 + q_2^2 + q_2z_2) + \alpha p_2, \end{cases} \tag{1.6}
$$

$$
(\mathrm{P}_{\mathrm{IV}})_2 \begin{cases} H_1 = p_1^2 + p_1p_2 - p_1q_1^2 + p_2q_1q_2 - p_2q_2^2 - z_1p_1 + z_2p_2q_2 + \alpha q_2 + \beta q_1, \\ H_2 = p_1p_2q_1 - 2p_1p_2q_2 - p_2^2q_2 + p_2q_1q_2^2 \\ \qquad +p_2q_2z_1 + z_2(p_1p_2 - p_2q_2^2 + p_2q_2z_2) + (p_1 - q_1q_2 + q_2z_2)\alpha - \beta p_2, \end{cases} \tag{1.7}
$$

respectively, with arbitrary parameters $\alpha, \beta \in \mathbb{C}$. The weighted degrees of these hierarchies determined by the Newton diagrams are shown in Table 2 (see also Table 3). From Table 1, 2 and the equations, we deduce the following properties.

- $\deg(q_i) + \deg(p_i) = \deg(H_1) - 1$.
- $\deg(z_1) = \deg(H_1) - 2$.
- $\deg(z_i) + \deg(H_i)$ is independent of $i = 1, \cdots, n$.
- $\min_{1\le i\le n} \{\deg(q_i), \deg(p_i)\} = 1$ or 2.
- The equation (1.3) is invariant under the $\mathbb{Z}_s$-action

  $$(q_i, p_i, z_i) \mapsto (\omega^{\deg(q_i)}q_i,\ \omega^{\deg(p_i)}p_i,\ \omega^{\deg(z_i)}z_i),$$

  where $s := \deg(H_1) - 1$ and $\omega := e^{2\pi i/s}$.
- The symplectic form $\sum_{i=1}^{n} dq_i \wedge dp_i + \sum_{i=1}^{n} dz_i \wedge dH_i$ is also invariant under the same $\mathbb{Z}_s$-action, for which $H_i \mapsto \omega^{\deg(H_i)}H_i$.

We decompose the Hamiltonian function H_i into the principle part H_i^P and the non-principle part H_i^N. Then, we further deduce

- H_i^N consists of monomials including arbitrary parameters.
- $\deg(H_i^N) = \deg(H_i) - \deg(H_1) + 1$.
- The variety defined by

  $$H_1^P(q_1, \cdots, q_n, p_1, \cdots, p_n, 0, \cdots, 0) = 0$$

  in $\mathbb{C}^{2n}$ has a unique singularity at the origin.

For $(\mathrm{P_I})$, $(\mathrm{P_{II}})$ and $(\mathrm{P_{IV}})$, we have

$$
\begin{aligned}
H_{\mathrm{I}}^P(q,p,0) &= \frac{1}{2}p^2 - 2q^3 = 0, \\
H_{\mathrm{II}}^P(q,p,0) &= \frac{1}{2}p^2 - \frac{1}{2}q^4 = 0, \\
H_{\mathrm{IV}}^P(q,p,0) &= -pq^2 + p^2q = 0.
\end{aligned}
$$

They define A_2, A_3 and D_4 singularities at the origin, respectively. In singularity theory, it is known that if a singularity defined by a quasihomogeneous polynomial $H(x_1, \cdots, x_n)$ is isolated, then the rational function

$$\chi(T) := \frac{(T^{h-a_1}-1)\cdots(T^{h-a_n}-1)}{(T^{a_1}-1)\cdots(T^{a_n}-1)} \tag{1.8}$$

becomes a polynomial (Poincaré polynomial), where $\deg(x_i) = a_i$ and $\deg(H) = h$.

Motivated by these observation, we classify weights $(a_1, \cdots, a_n, b_1, \cdots, b_n; h)$ satisfying certain conditions in Section 2. In particular, for $n = 1$ and 2, we will show that there is a corresponding Painlevé equation for each weight such that $\deg(q_i) = a_i, \deg(p_i) = b_i$ and $\deg(H) = h$. In Section 3, a Hamiltonian system, whose Hamiltonian function satisfies certain assumptions on the quasihomogeneity, will be considered. Then, some of the above properties of weights will be proved. A list of Kovalevskaya exponents of 4-dim Painlevé equations are also given.

The Hamiltonian functions of the third, fifth and sixth Painlevé equations are not polynomials in z, and their weights include nonpositive integers. They are not treated in this paper, while the analysis of them using weighted projective spaces is given in [4].

	$\deg(q,p,z)$	$\deg(H)$	κ
$\mathrm{P_I}$	$(2,3,4)$	6	6
$\mathrm{P_{II}}$	$(1,2,2)$	4	4
$\mathrm{P_{IV}}$	$(1,1,1)$	3	3
$\mathrm{P_{III(D_8)}}$	$(-1,2,4)$	2	2
$\mathrm{P_{III(D_7)}}$	$(-1,2,3)$	2	2
$\mathrm{P_{III(D_6)}}$	$(0,1,2)$	2	2
$\mathrm{P_V}$	$(1,0,1)$	2	2
$\mathrm{P_{VI}}$	$(1,0,0)$	2	2

Table 1: $\deg(H)$ denotes the weighted degree of the Hamiltonian function with respect to the weight $\deg(q,p,z)$. κ denotes the Kovalevskaya exponent, which is one of the invariants of the equations and they satisfy $\deg(H) = \kappa$. See Chiba [2, 3, 4] for the detail.

2 Classification of regular weights

Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ and h be positive integers such that $1 \leq a_i, b_i < h$. Motivated by the observation in Section 1, we suppose the following.

(W1) $\min\limits_{1\leq i\leq n} \{a_i, b_i\} = 1$ or 2.

(W2) $a_i + b_i = h - 1$ for $i = 1, \cdots, n$.

	$\deg(q_j, p_j)$	$\deg(z_i)$	$\deg(H_i)$
$(\mathrm{P_I})_n$	$(2j, 2n+3-2j)$	$2n-2i+4$	$2n+2i+2$
$(\mathrm{P_{II\text{-}1}})_n$	$(2j-1, 2n+2-2j)$	$2n-2i+2$	$2n+2i$
$(\mathrm{P_{II\text{-}2}})_n$	$(j, n+2-j)$	$n-i+2$	$n+i+2$
$(\mathrm{P_{IV}})_n$	$(j, n+1-j)$	$n-i+1$	$n+i+1$

Table 2: Weights for four classes of the Painlevé hierarchies.

	$\{\deg(q_j), \deg(p_j)\}$	$\deg(z_i)$	$\deg(H_i)$
$(\mathrm{P_I})_2$	$(2,3,4,5)$	$6,4$	$8,10$
$(\mathrm{P_I})_3$	$(2,3,4,5,6,7)$	$8,6,4$	$10,12,14$
$(\mathrm{P_{II\text{-}1}})_2$	$(1,2,3,4)$	$4,2$	$6,8$
$(\mathrm{P_{II\text{-}1}})_3$	$(1,2,3,4,5,6)$	$6,4,2$	$8,10,12$
$(\mathrm{P_{II\text{-}2}})_2$	$(1,2,2,3)$	$3,2$	$5,6$
$(\mathrm{P_{II\text{-}2}})_3$	$(1,2,2,3,3,4)$	$4,3,2$	$6,7,8$
$(\mathrm{P_{IV}})_2$	$(1,1,2,2)$	$2,1$	$4,5$
$(\mathrm{P_{IV}})_3$	$(1,1,2,2,3,3)$	$3,2,1$	$5,6,7$

Table 3: Weights for four classes of the Painlevé hierarchies when $n = 2, 3$, where $\deg(q_j), \deg(p_j)$'s are shown in ascending order.

(W3) A function

$$\chi(T) = \frac{(T^{h-a_1}-1)(T^{h-b_1}-1)\cdots(T^{h-a_n}-1)(T^{h-b_n}-1)}{(T^{a_1}-1)(T^{b_1}-1)\cdots(T^{a_n}-1)(T^{b_n}-1)} \tag{2.1}$$

is polynomial.

In Saito[17], a tuple of integers $(a_1, \cdots, a_n, b_1, \cdots, b_n; h)$ satisfying (W3) is called a regular weight. In this paper, a tuple is called a regular weight if it satisfies (W1) to (W3). In this section, we will classify all regular weights for $n = 1, 2, 3$. In particular, for $n = 1$ and $n = 2$, we will show that there are Hamiltonians of Painlevé equations associated with regular weights such that $\deg(q_i) = a_i, \deg(p_i) = b_i$ and $\deg(H) = h$.

2.1 $n = 1$

Proposition 2.1. When $n = 1$, regular weights satisfying (W1) to (W3) are only

$$(a, b; h) = (2, 3; 6), \quad (1, 2; 4), \quad (1, 1; 3).$$

They coincide with the weights $(\deg(q), \deg(p); \deg(H))$ of $H_{\mathrm{I}}, H_{\mathrm{II}}$ and H_{IV}, respectively, given in Sec.1.

Hence, there is a one to one correspondence between regular weights and the 2-dim Painlevé equations written in polynomial Hamiltonians. Note that $\deg(z)$ is recovered by the rule $\deg(z) = \deg(H) - 2$. Now we show that $H_{\mathrm{I}}, H_{\mathrm{II}}$ and H_{IV} can be reconstructed from the regular weights with the aid of singularity theory.

Step 1. Consider generic polynomials $H(q,p)$ whose weighted degrees are $\deg(q,p;H) = (2,3;6), (1,2;4)$ and $(1,1;3)$. They are given by

$$
\begin{aligned}
H &= c_1p^2 + c_2q^3, \\
H &= c_1^2p^2 + c_2q^2p + c_3q^4, \\
H &= c_1q^3 + c_2pq^2 + c_3p^2q + c_4p^3,
\end{aligned}
$$

respectively.

Step 2. Simplify by symplectic transformations. One of the results are

$$
\begin{aligned}
H &= \frac{1}{2}p^2 - 2q^3, \\
H &= \frac{1}{2}p^2 - \frac{1}{2}q^4, \\
H &= -pq^2 + p^2q,
\end{aligned}
$$

respectively.

Step 3. Consider the versal deformations of them[1]. We obtain

$$
\begin{aligned}
H &= \frac{1}{2}p^2 - 2q^3 + \alpha_4q + \alpha_6, \\
H &= \frac{1}{2}p^2 - \frac{1}{2}q^4 + \alpha_2q^2 + \alpha_3q + \alpha_4, \\
H &= -pq^2 + p^2q + \alpha_1pq + \alpha_2p + \beta_2q + \alpha_3,
\end{aligned}
$$

respectively, where $\alpha_i, \beta_i \in \mathbb{C}$ are deformation parameters. The subscripts i of α_i, β_i denote the weighted degrees of α_i, β_i so that H becomes a quasihomogeneous.

Step 4. Now we use the ansatz $\deg(z) = \deg(H) - 2$ given in Sec.1. If there is a parameter α_i such that $i = \deg(H) - 2$, then replace it by z. The results are

$$
\begin{aligned}
H &= \frac{1}{2}p^2 - 2q^3 + zq + \alpha_6, \\
H &= \frac{1}{2}p^2 - \frac{1}{2}q^4 + zq^2 + \alpha_3q + \alpha_4, \\
H &= -pq^2 + p^2q + zpq + \alpha_2p + \beta_2q + \alpha_3,
\end{aligned}
$$

respectively. They are equivalent to $H_{\mathrm{I}}, H_{\mathrm{II}}$ and H_{IV} up to the scaling of z (constant terms in Hamiltonians such as α_6 do not play a role).

Hence, when $n = 1$, there is a one to one correspondence between the regular weights and 2-dim polynomial Painlevé equations, and we can recover one of them from the other.

2.2 $n = 2$

Proposition 2.2. When $n = 2$, regular weights satisfying (W1) to (W3) are only

$$\begin{aligned}(a_1, a_2, b_2, b_1; h) &= (2, 3, 4, 5; 8),\\ &= (1, 2, 3, 4; 6),\\ &= (2, 2, 3, 3; 6),\\ &= (1, 2, 2, 3; 5),\\ &= (1, 1, 2, 2; 4),\\ &= (1, 1, 1, 1; 3),\end{aligned}$$

where we assume without loss of generality that $a_1 \leq a_2 \leq b_2 \leq b_1$. For each weight, there exists a polynomial Hamiltonian of a 4-dim Painlevé equation (not unique). Explicit forms of Hamiltonian functions are given as follows.

(2,3,4,5;8). The first Hamiltonian H_1 of $(\mathrm{P_I})_2$ shown in Eq.(1.4) has this weight with $\deg(q_1, p_1, q_2, p_2) = (2, 5, 4, 3)$. Another example is

$$H_{\mathrm{Cosgrove}} = -4p_1p_2 - 2p_2^2q_1 - \frac{73}{128}q_1^4 + \frac{11}{8}q_1^2q_2 - \frac{1}{2}q_2^2 - q_1z - \frac{1}{48}\left(q_1 + \frac{\alpha}{6}\right)q_1^2\alpha. \quad (2.2)$$

This Hamiltonian system is derived by a Lie-algebraic method of type B_2 and can be written in Lax form, which will be reported elsewhere. It seems that it does not appear in the list of 4-dim Painlevé equations in [12, 14]. If we rewrite the system as the fourth order single equation of $q_1 = y$, we obtain

$$y'''' = 18yy'' + 9(y')^2 - 24y^3 + 16z + \alpha y(y + \frac{1}{9}\alpha). \quad (2.3)$$

This equation was given in Cosgrove [7], denoted by F-VI. He conjectured that this equation defines a new Painlevé transcendents (i.e. it is not reduced to known equations).

(1,2,3,4;6). The first Hamiltonian H_1 of $(\mathrm{P_{II\text{-}1}})_2$ shown in Eq.(1.5) has this weight $\deg(q_1, p_1, q_2, p_2) = (1, 4, 3, 2)$. Another example is the matrix Painlevé equation of the first type $H_{\mathrm{I}}^{\mathrm{Mat}}$[12, 14] defined by

$$H_{\mathrm{I}}^{\mathrm{Mat}} = \frac{1}{2}p_1^2 - 2q_1^3 - 2p_2^2q_2 + 6q_1q_2 - 2q_1z + 2\alpha p_2, \quad (2.4)$$

with $\deg(q_1, p_1, q_2, p_2) = (2, 3, 4, 1)$.

(2,2,3,3;6). For H_1 and H_2 of $(\mathrm{P_{II\text{-}1}})_2$ shown in Eq.(1.5), perform the symplectic transformation

$$q_1 = -\frac{y_1}{2x_1},\ p_1 = -x_1^2,\ q_2 = \frac{y_2}{2},\ p_2 = 2x_2. \quad (2.5)$$

Then we obtain the Hamiltonians

$$\left\{\begin{array}{l} H_1^{(2,3,2,3)} = -4x_1^2x_2 - 8x_2^3 + \dfrac{y_1^2}{4} + \dfrac{y_2^2}{4} - 2z_1x_2 - z_2x_1^2 - \dfrac{\alpha y_1}{x_1}, \\ H_2^{(2,3,2,3)} = -x_1^4 - 4x_1^2x_2^2 - \dfrac{x_2y_1^2}{2} + \dfrac{x_1y_1y_2}{2} \\ \qquad -z_1x_1^2 - z_2^2x_1^2 - 2z_2x_1^2x_2 + \dfrac{z_2y_1^2}{4} - \dfrac{\alpha z_2y_1}{x_1} + \dfrac{2\alpha x_2y_1}{x_1} - \alpha y_2. \end{array}\right. \tag{2.6}$$

Thus, putting $\alpha = 0$ yields semi-quasihomogeneous Hamiltonians of $\deg(H_1^{(2,3,2,3)}, H_2^{(2,3,2,3)}) = (6,8)$ with respect to $\deg(x_1, y_1, x_2, y_2) = (2,3,2,3)$ and $\deg(z_1, z_2) = (4,2)$. Although this is equivalent to $(\mathrm{P_{II\text{-}1}})_2$ for $\alpha = 0$, they should be distinguished from each other from a view point of a geometric classification of Painlevé equations (i.e. a classification based on the spaces of initial conditions) because the above symplectic transformation is not a one-to-one mapping.

(1,2,2,3;5). The first Hamiltonian H_1 of $(\mathrm{P_{II\text{-}2}})_2$ shown in Eq.(1.6) has this weight with $\deg(q_1, p_1, q_2, p_2) = (1,3,2,2)$.

(1,1,2,2;4). The first Hamiltonian H_1 of $(\mathrm{P_{IV}})_2$ shown in Eq.(1.7) has this weight with $\deg(q_1, p_1, q_2, p_2) = (1,2,1,2)$. Another example is the matrix Painlevé equation of the second type $H_{\mathrm{II}}^{\mathrm{Mat}}$[12, 14] defined by

$$H_{\mathrm{II}}^{\mathrm{Mat}} = \frac{1}{2}p_1^2 - p_1q_1^2 + p_1q_2 - 2p_2^2q_2 - 4p_2q_1q_2 - p_1z + 2\alpha p_2 + 2\beta(p_2 + q_1), \tag{2.7}$$

with $\deg(q_1, p_1, q_2, p_2) = (1,2,2,1)$.

(1,1,1,1;3). The Noumi-Yamada system of type A_4 [14, 16] defined by

$$H_{\mathrm{NY}}^{A_4} = 2p_1p_2q_1 + p_1q_1(p_1 - q_1 - z) + p_2q_2(p_2 - q_2 - z) + \alpha p_1 + \beta q_1 + \gamma p_2 + \delta q_2 \tag{2.8}$$

has the weight $\deg(q_1, p_1, q_2, p_2) = (1,1,1,1)$, where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters.

Remark. The author does not know an example of a 4-dim Painlevé equation whose Hamiltonian function is semi-quasihomogeneous but its degree is different from that in Prop.2.2.

2.3 $n = 3$

To determine all regular weights satisfying (W1) to (W3), the following lemma is useful. Without loss of generality, we assume $a_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_2 \leq b_1$. There exist integers N and $j(1), \cdots, j(N)$ such that

$$\begin{aligned} a_1 = \cdots = a_{j(1)} < a_{j(1)+1} = \cdots = a_{j(2)} < \cdots < a_{j(N)+1} = \cdots = a_n \\ \leq b_n = \cdots = b_{j(N)+1} < \cdots < b_{j(2)} = \cdots = b_{j(1)+1} < b_{j(1)} = \cdots = b_1. \end{aligned}$$

We put $J_l = j(l) - j(l-1)$ $(l = 1, \cdots, N+1)$, where $j(0) = 0$ and $j(N+1) = n$.

Lemma 2.3.
(i) When $N = 0$ (i.e. $a_1 = a_n$), then

$$\begin{aligned}(a_1, \cdots, a_n, b_n, \cdots, b_1; h) &= (1, \cdots, 1, 1, \cdots, 1; 3)\\ &= (1, \cdots, 1, 2, \cdots, 2; 4)\\ &= (2, \cdots, 2, 3, \cdots, 3; 6).\end{aligned}$$

(ii) When $N \geq 1$, the equality $b_{j(i)} = b_{j(i+1)} + 1$ holds for $i = 1, \cdots, N$ and $J_{i+1} \geq J_i$ holds for $i = 1, \cdots, N-1$. If $a_n \neq b_n$, further $b_n = a_n + 1$ and $J_{N+1} \geq J_N$ hold.
(iii) If $a_i < a_{i+1}$ for any $i = 1, \cdots, n-1$, then

$$\begin{aligned}(a_1, \cdots, a_n, b_n, \cdots, b_1; h) &= (1, \cdots, n, n, \cdots, 2n-1; 2n+1)\\ &= (1, \cdots, n, n+1, \cdots, 2n; 2n+2)\\ &= (2, \cdots, n+1, n+2, \cdots, 2n+1; 2n+4).\end{aligned}$$

Proof. Because of (W2), Eq.(2.1) is rewritten as

$$\chi(T) = \frac{(T^{a_1+1}-1)\cdots(T^{a_n+1}-1)(T^{b_n+1}-1)\cdots(T^{b_1+1}-1)}{(T^{a_1}-1)\cdots(T^{a_n}-1)(T^{b_n}-1)\cdots(T^{b_1}-1)}. \tag{2.9}$$

(i) In this case, $a_1 = a_n \leq b_n = b_1$ due to (W2), which implies

$$\chi(T) = \frac{(T^{a_1+1}-1)^n(T^{b_1+1}-1)^n}{(T^{a_1}-1)^n(T^{b_1}-1)^n}.$$

Since it is polynomial, either $b_1 + 1$ or $a_1 + 1$ is a multiple of b_1. If $b_1 m = b_1 + 1$, then $(m, b_1) = (2, 1)$ and we obtain $(a_1, \cdots, a_n, b_n, \cdots, b_1) = (1, \cdots 1, 1, \cdots, 1)$. If $a_1 = b_1$, the same result is obtained. Now suppose that $b_1 m = a_1 + 1 < b_1 + 1$. It is easy to verify that $m = 1$ and $b_1 = a_1 + 1$. Then,

$$\chi(T) = \frac{(T^{a_1+2}-1)^n}{(T^{a_1}-1)^n}.$$

Since $a_1 + 2$ is a multiple of a_1, we have $a_1 m = a_1 + 2$. This provides $a_1 = 1$ or 2 (we need not use (W1)).

(ii) In what follows, we suppose that $b_1 > 1$. In this case, $b_j > 1$ for any $j = 1, \cdots, n$ due to the assumption $1 \leq a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1$ and (W2).

Step 1. Since $\chi(T)$ is polynomial, there is a multiple of $b_{j(1)}$ among exponents $b_{j(l)} + 1$ in the numerator. If $b_{j(1)} m = b_{j(1)} + 1$, then $(m, b_{j(1)}) = (2, 1)$ and it contradicts the assumption $b_{j(1)} = b_1 > 1$.

If $b_{j(1)} m = b_{j(l)} + 1 < b_{j(1)} + 1$ for some $l > 1$, it is easy to verify $m = 1, l = 2$ and $b_{j(1)} = b_{j(2)} + 1$. There are J_1 factors $T^{b_{j(1)}} - 1$ in the denominator. This implies that $2J_2 \geq J_1$ when $N = 1$ and $a_n = b_n$, and $J_2 \geq J_1$ otherwise.

Step 2. Now we assume that $r \leq N$ and $b_{j(i)} = b_{j(i+1)} + 1$ holds for $i = 1, \cdots, r-1$. There exists a multiple of $b_{j(r)}$ among $b_{j(l)} + 1$. If $l \leq r$, we have

$$b_{j(r)}m = b_{j(l)} + 1 = b_{j(l+1)} + 2 = \cdots = b_{j(r)} + r - l + 1,$$

which yields

$$1 < b_{j(r)} \leq r - l + 1 \leq r.$$

This proves $b_{j(r)} = b_n = a_n = r$ (otherwise, a_1 becomes nonpositive). Hence, $r = N + 1$, which contradicts the assumption $r \leq N$.

If $b_{j(r)}m = b_{j(l)} + 1$ for some $l > r$, it is easy to verify $m = 1, l = r + 1$ and $b_{j(r)} = b_{j(r+1)} + 1$. There are J_r factors $T^{b_{j(r)}} - 1$ in the denominator. This implies that $2J_{r+1} \geq J_r$ when $r = N$ and $a_n = b_n$, and $J_{r+1} \geq J_r$ otherwise.

Step 3. By induction, we obtain $b_{j(i)} = b_{j(i+1)} + 1$ for $i = 1, \cdots, N$, and $J_{i+1} \geq J_i$ for $i = 1, \cdots, N-1$. In particular, if $a_n \neq b_n$, $J_{N+1} \geq J_N$ also holds.

Step 4. There exists a multiple of $b_{j(N+1)} = b_n$ among exponents of the numerator. Suppose $b_{j(N+1)}m = b_{j(l)} + 1$ for some $l = 1, \cdots, N+1$. The same argument as Step 2 shows that $a_n = b_n$. Suppose $b_{j(N+1)}m = a_{j(l)} + 1 < b_{j(N+1)} + 1$ for some $l = 1, \cdots, N+1$. Then, we obtain $m = 1, l = N + 1$ and $b_{j(N+1)} = b_n = a_n + 1$. This completes the proof of (ii).

(iii) This is verified by a direct calculation with the aid of (ii). □

Proposition 2.4. When $n = 3$, regular weights satisfying (W1) to (W3) are only

$$\begin{aligned}
(a_1, a_2, a_3, b_3, b_2, b_1; h) &= (2, 3, 4, 5, 6, 7; 10), \\
&= (2, 3, 3, 4, 4, 5; 8), \\
&= (1, 2, 3, 4, 5, 6; 8), \\
&= (1, 2, 3, 3, 4, 5; 7), \\
&= (2, 2, 2, 3, 3, 3; 6), \\
&= (1, 2, 2, 3, 3, 4; 6), \\
&= (1, 1, 2, 2, 3, 3; 5), \\
&= (1, 1, 1, 2, 2, 2; 4), \\
&= (1, 1, 1, 1, 1, 1; 3),
\end{aligned}$$

where we assume without loss of generality that $a_1 \leq a_2 \leq a_3 \leq b_3 \leq b_2 \leq b_1$.

This proposition is easily obtained with the aid of Lemma 2.3. To find corresponding Painlevé equations is a future work.

3 Semi-quasihomogeneous Hamiltonian systems

3.1 Properties of weights

Let us consider the $2n$-dimensional Hamiltonian system

$$\frac{dq_i}{dz} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dz} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \cdots, n, \tag{3.1}$$

with the Hamiltonian function $H(q_1, \cdots, q_n, p_1, \cdots, p_n, z)$. We suppose the following.

(A1) $H = H^P + H^N$ is semi-quasihomogeneous; there exist integers $1 \le a_i, b_i, r < h$ such that

$$H^P(\lambda^a q, \lambda^b p, \lambda^r z) = \lambda^h H^P(q, p, z), \tag{3.2}$$

where $\lambda^a q = (\lambda^{a_1} q_1, \cdots, \lambda^{a_n} q_n)$ and $\lambda^b p = (\lambda^{b_1} p_1, \cdots, \lambda^{b_n} p_n)$.

(A2) The Hamiltonian vector field of H^P satisfies

$$\frac{\partial H^P}{\partial p_i}(\lambda^a q, \lambda^b p, \lambda^r z) = \lambda^{1+a_i} \frac{\partial H^P}{\partial p_i}(q, p, z), \quad \frac{\partial H^P}{\partial q_i}(\lambda^a q, \lambda^b p, \lambda^r z) = \lambda^{1+b_i} \frac{\partial H^P}{\partial q_i}(q, p, z).$$

(A3) $H^N(\lambda^a q, \lambda^b p, \lambda^r z) \sim o(\lambda^h)$ as $|\lambda| \to \infty$.

(A4) The Hamiltonian vector field of $H = H^P + H^N$ is invariant under the $\mathbb{Z}_s$ action

$$(q_j, p_j, z) \mapsto (\omega^{a_j} q_j, \omega^{b_j} p_j, \omega^r z), \tag{3.3}$$

where $s = h - 1$ and $\omega := e^{2\pi i/s}$.

(A5) The symplectic form $\displaystyle\sum_{j=1}^n dq_j \wedge dp_j + dz \wedge dH$ is also invariant under the same $\mathbb{Z}_s$-action, for which $H \mapsto \omega^h H$.

From these assumptions, we will explain some of properties of weights shown in Section 1.

Remark. The assumption (A2) is used to define the Kovalevskaya exponents in the next section. Due to the assumption (A1), it is easy to show that the Hamiltonian vector field of H^P is invariant under the action (3.3). The assumption (A4) requires that the vector field of H^N is also invariant under the action. Then, Eq.(3.1) induces a rational differential equation on the weighted projective space $\mathbb{C}P^{2n+1}(a, b, r, s)$ [2, 3].

In what follows, we assume $h \ge 3$ (if $h \le 2$, Eq.(3.1) is linear).

Proposition 3.1. Suppose that Eq.(3.1) satisfies (A1) to (A5) and $h \ge 3$. Then,
(i) $a_i + b_i = h - 1$ for $i = 1, \cdots, n$,
(ii) $r = h - 2$,
(iii) $\deg(H^N) = 1$,
(iv) if Eq.(3.1) is non-autonomous, $\displaystyle\min_{1 \le i \le n} \{a_i, b_i\} = 1$ or 2.

Proof. The first statement (i) immediately follows from (A1) and (A2).

(ii) Because of (A5), there exists an integer N such that $r + h = N(h-1)$. Since $r < h$, we obtain $0 < r = N(h-1) - h < h$. This yields $h < N/(N-2)$ if $N \neq 2$. This contradicts the assumption $h \ge 3$. Therefore, $N = 2$, which proves $r = h - 2$.

(iii) Let $q_1^{\mu_1} \cdots q_n^{\mu_n} p_1^{\nu_1} \cdots p_n^{\nu_n} z^\eta$ be a monomial included in H^N. Due to (A3), the exponents satisfy

$$0 \le \sum_{i=1}^n (a_i \mu_i + b_i \nu_i) + r\eta \le h - 1.$$

Further, (A4) implies that there exists an integer N such that

$$\sum_{i=1}^{n}(a_i\mu_i + b_i\nu_i) + r\eta - a_j - b_j + r = N(h-1).$$

This and (i),(ii) give

$$\sum_{i=1}^{n}(a_i\mu_i + b_i\nu_i) + r\eta = N(h-1) + 1.$$

Hence, we obtain $0 \le N(h-1)+1 \le h-1$. This proves $N = 0$ and $\sum_{i=1}^{n}(a_i\mu_i + b_i\nu_i) + r\eta = 1$.

(iv) Suppose that H includes z. Since $\deg(H) = h$ and $\deg(z) = h-2$, z is multiplied by a function whose weighted degree is 2. It exists only when $\min_{1\le i\le n}\{a_i, b_i\} = 1$ or 2. □

3.2 Kovalevskaya exponents of 4-dim Painlevé equations

Kovalevskaya exponents are the most important invariants of a quasihomogeneous vector field related to the Painlevé test. Let us consider the system of differential equations

$$\frac{dx_i}{dz} = f_i(x_1,\cdots,x_m,z) + g_i(x_1,\cdots,x_m,z), \quad i = 1,\cdots,m, \tag{3.4}$$

where f_i and g_i are polynomials in $(x_1,\cdots,x_m,z) \in \mathbb{C}^{m+1}$. We suppose that

(K1) $(f_1,\cdots,f_m)$ is a quasi-homogeneous vector field satisfying

$$f_i(\lambda^{a_1}x_1,\cdots,\lambda^{a_m}x_m,\lambda^r z) = \lambda^{1+a_i}f_i(x_1,\cdots,x_m,z) \tag{3.5}$$

for any $\lambda \in \mathbb{C}$ and $i = 1,\cdots,m$, where $(a_1,\cdots,a_m,r) \in \mathbb{Z}_{>0}^{m+1}$.

(K2) $(g_1,\cdots,g_m)$ satisfies

$$g_i(\lambda^{a_1}x_1,\cdots,\lambda^{a_m}x_m,\lambda^r z) = o(\lambda^{a_i+1}), \quad |\lambda| \to \infty.$$

Put $f_i^A(x_1,\cdots,x_m) := f_i(x_1,\cdots,x_m,0)$ and $f_i^{NA} := f_i - f_i^A$ (i.e. f_i^A and f_i^{NA} are autonomous and nonautonomous parts, respectively). We also consider the truncated system

$$\frac{dx_i}{dz} = f_i^A(x_1,\cdots,x_m), \quad i = 1,\cdots,m. \tag{3.6}$$

If the equation

$$-a_ic_i = f_i^A(c_1,\cdots,c_m), \quad i = 1,\cdots,m \tag{3.7}$$

has a root $(c_1,\cdots,c_m) \in \mathbb{C}^m$, $x_i(z) = c_i(z-z_0)^{-a_i}$ is an exact solution of the truncated system for any $z_0 \in \mathbb{C}$.

Definition 3.2. Fix a root $\{c_i\}_{i=1}^m$ of the equation $-a_i c_i = f_i^A(c_1, \cdots, c_m)$. The matrix

$$K = \left\{ \frac{\partial f_i^A}{\partial x_j}(c_1, \cdots, c_m) + a_i \delta_{ij} \right\}_{i,j=1}^m \tag{3.8}$$

and its eigenvalues are called the Kovalevskaya matrix and the Kovalevskaya exponents, respectively, of the system (3.4) associated with $\{c_i\}_{i=1}^m$.

Fact 3.3 (see [3] for the detail.)
(i) -1 is always a Kovalevskaya exponent.
(ii) For a semi-quasihomogeneous Hamiltonian system of $\deg(H) = h$, if κ is a Kovalevskaya exponent, so is μ given by $\kappa + \mu = h - 1$.
(iii) If a given system has the Painlevé property, then there exists a root $\{c_i\}_{i=1}^m$ such that all of the associated Kovalevskaya exponents (except for -1) are positive integers (Painlevé test).
(iv) The Kovalevskaya exponents are invariant under weight preserving diffeomorphisms.

A Hamiltonian system (3.1) satisfying (A1) to (A3) satisfies (K1) and (K2), so that its Kovalevskaya exponents are well-defined. The Kovalevskaya exponents of 2-dim Painlevé equations are shown in Table 1. Because of the above properties (i) and (ii), $\kappa = \deg(H) = h$ always holds for 2-dim systems.

We give a list of Kovalevskaya exponents of 4-dim Painlevé equations shown in Section 2.2. In Table 4, $H_1^{9/2}, H_1^{7/2+1}, H_1^5$ and H_1^{4+1} denote the first Hamiltonians of $(\mathrm{P_I})_2, (\mathrm{P_{II\text{-}1}})_2, (\mathrm{P_{II\text{-}2}})_2$ and $(\mathrm{P_{IV}})_2$, respectively, given in Section 1 (this notation is related to the spectral type of a monodromy preserving deformation [14]). For example, $(-1, 2, 3, 6) \times 2$ in Table 4 implies that there are two roots $\{c_i\}_{i=1}^m$ of the equation $-a_i c_i = f_i^A(c_1, \cdots, c_m)$ for which the associated Kovalevskaya exponents are $\kappa = -1, 2, 3$ and 6. Since Kovalevskaya exponents are invariant under weight preserving diffeomorphisms, we can conclude that two Hamiltonian systems having the same weights are actually different systems if their Kovalevskaya exponents are different from each other.

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	$(a_1, a_2, b_2, b_1; h)$	κ	
$H_1^{9/2}$	$(2, 3, 4, 5; 8)$	$(-1, 2, 5, 8)$	$\times 1$
		$(-3, -1, 8, 10)$	$\times 1$
H_{Cosgrove}	$(2, 3, 4, 5; 8)$	$(-1, 3, 4, 8)$	$\times 1$
		$(-5, -1, 8, 12)$	$\times 1$
$H_1^{7/2+1}$	$(1, 2, 3, 4; 6)$	$(-1, 2, 3, 6)$	$\times 2$
		$(-3, -1, 6, 8)$	$\times 2$
$H_{\text{I}}^{\text{Mat}}$	$(1, 2, 3, 4; 6)$	$(-1, 2, 3, 6)$	$\times 2$
		$(-2, -1, 6, 7)$	$\times 1$
		$(-7, -1, 6, 12)$	$\times 1$
$H_1^{(2,3,2,3)}$	$(2, 2, 3, 3; 6)$	$(-1, 1, 4, 6)$	$\times 1$
		$(-3, -1, 6, 8)$	$\times 2$
H_1^5	$(1, 2, 2, 3; 5)$	$(-1, 1, 3, 5)$	$\times 2$
		$(-2, -1, 5, 6)$	$\times 3$
H_1^{4+1}	$(1, 1, 2, 2; 4)$	$(-1, 1, 2, 4)$	$\times 3$
		$(-2, -1, 4, 5)$	$\times 5$
$H_{\text{II}}^{\text{Mat}}$	$(1, 1, 2, 2; 4)$	$(-1, 1, 2, 4)$	$\times 3$
		$(-2, -1, 4, 5)$	$\times 2$
		$(-5, -1, 4, 8)$	$\times 2$
		$(-1, -1, 4, 4)$	$\times 1$
$H_{\text{NY}}^{A_4}$	$(1, 1, 1, 1; 3)$	$(-1, 1, 1, 3)$	$\times 5$
		$(-1, -1, 3, 3)$	$\times 5$
		$(-3, -1, 3, 5)$	$\times 5$

Table 4: Weights and Kovalevskaya exponents of 4-dim Painlevé equations.

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