有理曲面を成すモーデル・ヴェイユ群が自明な平面曲線束について

北川真也

1 Introduction

切断を持つ有理楕円曲面については, 楕円曲線束が II^* 型の特異ファイバーを持つことが, モーデル・ヴェイユ群が自明となる必要十分条件であることが知られている. 9 年前に前述を, ファイバーの集合体が有理曲面を成す仮定は保持するが, 一般ファイバーを楕円曲線から種数 g の超楕円曲線に一般化した場合を考察した ([7]). 任意の g に対して, 有理曲面のピカール数は $4g+6$ 以下で, その最大値 $4g+6$ をとる場合に制限すれば $g=1$ のときと同様に, 相対極小な超楕円曲線束が II^* 型を一般化した特異ファイバーを持つことが, モーデル・ヴェイユ群が自明となる必要十分条件であることが判明した. 今回も有理曲面を成す仮定は保持するが, 一般ファイバーは平面 d 次曲線である場合を考察する. ピカール数は d^2+1 以下で, その最大値をとる場合に制限すると, モーデル・ヴェイユ群が自明な平面曲線束は，超楕円的なときと同様に, 特殊な特異ファイバーで特徴づけられることを紹介する.

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2 Preliminaries

We briefly review basic notation and results on fibred rational surfaces. Here, a fibred rational surface means a smooth projective rational surface $X/\mathbb{C}$ together with a relatively minimal fibration $f: X \to \mathbb{P}^1$ whose general fibre F is a smooth projective curve of

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genus $g \geq 1$. In particular, any fibre of f is connected and contains no (-1)-curves as components. Since X is rational, the first Betti number of X equals zero. The second Betti number of X is equal to the Picard number $\rho(X)$ since the geometric genus of X is zero. Hence, we see that

$$\rho(X) = 10 - K_X^2 = 4g + 6 - (K_X + F)^2 \tag{2.1}$$

by virtue of Noether's formula. The adjoint divisor $(K_X + F)$ is nef when $g \geq 2$ (See [5, Lemma 1.1]). Thus we have that $\rho(X) \leq 4g + 6$. By means of slope inequalities [9, Corollary 4.4], we also have that $(K_X + F)^2 \geq g - 2$ and $\rho(X) \leq 3g + 8$ if F is non-hyperelliptic (See [13, Proposition 2.2]).

Lemma 2.1 (See [5, Lemma 1.2]). *Let C be an irreducible curve on S such that $(K_S + F).C = 0$. If $(K_S + F)^2 > 0$, then C is a smooth rational curve satisfying one of the following*:

(i) *C is a (-2)-curve contained in a fibre.*

(ii) *C is a (-1)-section, i.e., a (-1)-curve with $F.C = 1$.*

From now on, we assume that $f : S \to \mathbb{P}^1$ is a relatively minimal fibration of genus $g \geq 2$ such that $(K_S+F)^2 > 0$. Suppose that there exists a (-1)-curve E with $(K_S+F).E = 0$ and let $\mu_1 : S \to S_1$ be its contraction. Since $F.E = 1$, $F_1 := (\mu_1)_*F$ is smooth on S_1. Furthermore, we have $\mu_1^*(K_{S_1} + F_1) = K_S + F$. If there exists a (-1)-curve E_1 with $(K_{S_1} + F_1).E_1 = 0$, then, by contracting it, we get the pair (S_2, F_2) with F_2 smooth and $K_{S_2} + F_2$ pulls back to $K_S + F$. We can continue the procedure until we arrive at a pair (S_n, F_n) such that we cannot find a (-1)-curve E_n with $(K_{S_n} + F_n).E_n = 0$. We put $W := S_n$ and $G := F_n$. If $\mu : S \to W$ denotes the natural map, then $\mu^*(K_W+G) = K_S+F$ and $G = \mu_*F$ is a smooth curve isomorphic to F. The original fibration $f : S \to \mathbb{P}^1$ corresponds to a pencil $\Lambda_f \subset |G|$ with at most simple (but not necessarily transversal) base points. From the assumption $(K_S+F)^2 > 0$, K_S+F is nef and big. This implies that, W is the minimal resolution of singularities of the surface $\mathrm{Proj}(R(S, K_S+F))$, which has at most rational double points by Lemma 2.1, where $R(S, K_S+F) = \bigoplus_{n\geq 0} H^0(S, n(K_S+F))$. Therefore, such a model is uniquely determined. We call the pair (W, G) the *reduction* of (S, F).

As a corollary of [6, Theorem 2.3], we have the following.

THEOREM 2.2. *Let S be a smooth rational surface and $f : S \to \mathbb{P}^1$ a relatively minimal fibration whose general fibre F is a smooth plane curve of degree $d \geq 4$. Then*

$$\rho(S) \leq d^2 + 1.$$

Let (W, G) denote the reduction of (S, F). If $\rho(S) = d^2 + 1$, then $W = \mathbb{P}^2$ and G is a curve of degree d. In particular, f has at least one (-1)-section. Furthermore, f has at most d^2 (-1)-sections, which are disjoint from each other.

COROLLARY 2.3. *Let S be a smooth rational surface of $\rho(S) = d^2 + 1$ for any integer $d \geq 3$ and $f : S \to \mathbb{P}^1$ a relatively minimal fibration of plane curve of degree d. Assume that f has no multiple fibres when $d = 3$. Then there exists a birational morphism $\nu : S \to \mathbb{P}^2$ such that the pull-back to S of a (-1)-curve contracted by ν intersects with F at just one point. In particular, $\nu_* F$ is a smooth plane curve of degree d and f has at least one (-1)-section.*

3 Mordell-Weil lattices

Via f, we can regard S as a smooth projective curve of genus g defined over the rational function field $\mathbb{K} = f^*\mathbb{C}(\mathbb{P}^1)$. We assume that it has a $\mathbb{K}$-rational point O. Let $\mathcal{J}_\mathcal{F}/\mathbb{K}$ be the Jacobian variety of the generic fibre $\mathcal{F}/\mathbb{K}$ of f. The Mordell-Weil group of f is the group of $\mathbb{K}$-rational points $\mathcal{J}_\mathcal{F}(\mathbb{K})$. It is a finitely generated Abelian group, since $S/\mathbb{C}$ is a rational surface. The rank $\mathrm{rk}\mathcal{J}_\mathcal{F}(\mathbb{K})$ of the group is called the *Mordell-Weil rank*. There is a formula, often referred as the Shioda-Tate formula, relating the Mordell-Weil rank and the Picard number:

$$\mathrm{rk}\mathcal{J}_\mathcal{F}(\mathbb{K}) = \rho(S) - 2 - \sum_{t \in \mathbb{P}^1} (v_t - 1), \tag{3.2}$$

where v_t denotes the number of irreducible components of the fibre $f^{-1}(t)$. There is a natural one-to-one correspondence between the set of $\mathbb{K}$-rational points $\mathcal{F}(\mathbb{K})$ and the set of sections of f. For $P \in \mathcal{F}(\mathbb{K})$, we denote by (P) the section corresponding to P which is regarded as a horizontal curve on S. In particular, (O) corresponding to the origin O of $\mathcal{J}_\mathcal{F}(\mathbb{K})$ is called the *zero section*. Shioda's main idea in [16] and [19] is to view the free part of $\mathcal{J}_\mathcal{F}(\mathbb{K})$ as a Euclidean lattice with respect to a natural pairing induced by the intersection form on $H^2(S)$. The lattice is called the *Mordell-Weil lattice* of f and is denoted by $\mathrm{MWL}(f)$. In fact, by describing the Néron-Severi group $\mathrm{NS}(S)$, we can explicitly determine the structure of MWL(f) as follows: Let T be the subgroup of NS(S) generated by (O) and the irreducible components of the fibres of f. When we equip NS(X) and T with the bilinear form which is (-1) times of the intersection form, we call them the Néron-Severi lattice NS$(S)^-$ and the trivial lattice T^- respectively. Since S is a rational surface, NS$(S)^-$ is a unimodular lattice, that is, the absolute value of the determinant of the Gram matrix equals one. Then the following holds.

Theorem 3.1 (See [16], [19, Theorem 3]). *Keep the notation and assumptions as above. Then*

$$\mathcal{J}_{\mathcal{F}}(\mathbb{K}) \simeq \mathrm{NS}(S)/T.$$

Let L be the orthogonal complement $(T^-)^\perp \subset \mathrm{NS}(S)^-$. Then the dual lattice

$$L^* = \{\mathfrak{x} \in L \otimes \mathbb{Q} \mid \langle \mathfrak{x}, \mathfrak{y} \rangle_{L\otimes\mathbb{Q}} \in \mathbb{Z}, \quad {}^\forall \mathfrak{y} \in L\}$$

is isomorphic to MWL(f).

4 Main Theorem

Theorem 4.1. *Let S be a smooth rational surface of $\rho(S) = d^2 + 1$ for any integer $d \geq 3$ and $f : S \to \mathbb{P}^1$ a relatively minimal fibration of plane curves of degree d. Assume that f has no multiple fibres when $d = 3$. Then f has at least one (-1)-section, and the following four conditions are equivalent.*

(1) *The Mordell-Weil group of f is trivial.*

(2) *f has a reducible fibre whose dual graph corresponds to the graph as in Figure* 1.

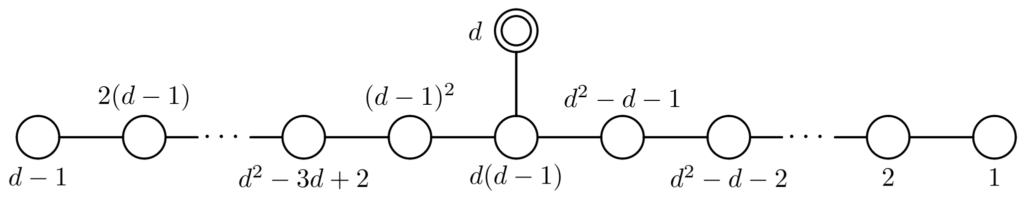

Figure 1.

Here, a double circle denotes a $(-d+1)$-curve and the other circles denote (-2)-curves. The numbers indicated outside the circles denote the multiplicities of components in the degenerated fibre.

(3a) *$f : S \to \mathbb{P}^1$ can be obtained from $\mathbb{P}^2$ by eliminating the base points of the following pencil Λ: Let L be a line on $\mathbb{P}^2$. Take a curve C_0 of degree d which has a contact of order d with L at one smooth point. Then the pencil Λ is generated by C_0 and dL.*

(3b) *$f : S \to \mathbb{P}^1$ can be obtained from $\mathbb{P}^2$, after performing a projective transformation, by eliminating the base points of the following pencil Λ: Let $(X : Y : Z)$ be homogeneous coordinates of $\mathbb{P}^2$ and L a line defined by $Y = 0$. For $t \in \mathbb{C}$, each member of Λ is defined by*

$$tY^d = X^d + YZ^{d-1} + \sum_{i=1}^{d-1} c_{i,1} X^i Y Z^{d-i-1} + \sum_{j=2}^{d} \sum_{i=0}^{d-j} c_{i,j} X^i Y^j Z^{d-i-j}, \tag{4.3}$$

where $c_{i,j}$ are complex numbers. The member of Λ corresponding to ∞ is dL.

In order to show Theorem 4.1, we prove some lemmas. As a first step, we show that the conditions (2), (3a) and (3b) are equivalent. As a second step, we deduce (2) $\Rightarrow$ (1). As a final step, we conclude (1) $\Rightarrow$ (2).

Lemma 4.2. *Let S be a smooth rational surface of $\rho(S) = d^2 + 1$ for any integer $d \geq 3$ and $f : S \to \mathbb{P}^1$ a relatively minimal fibration of plane curves of degree d. Assume that f has no multiple fibres when $d = 3$. If f has a reducible fibre F_∞ whose dual graph corresponds to the graph as in Figure* 1*, then there exists a birational morphism $\nu : S \to \mathbb{P}^2$ such that the images by ν of the fibres of f forms the pencil Λ as in* (3a) *of Theorem* 4.1.

Proof. Let Θ_k, $k = 0, 1, \cdots, d^2 - 1$ be components of the reducible fibre F_∞ that satisfy the following condition:

$$(\Theta_{i-1}.\Theta_{j-1})_{1 \leq i,j \leq d^2-1} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix},$$

$\Theta_{d^2-1}.\Theta_{d^2-d-1} = 1$, $\Theta_{d^2-1}^2 = -d + 1$ and $\Theta_{d^2-1}.\Theta_k = 0$ for $k \neq d^2 - d - 1, d^2 - 1$. We know that f has a (-1)-section E_{d^2} by the last assertion of Corollary 2.3. Since Θ_0 is a unique component whose multiplicity in F_∞ is one, E_{d^2} intersects with Θ_0. Let ν be the birational morphism contracting $E_{d^2}, \Theta_0, \Theta_1, \ldots, \Theta_{d^2-2}$ in turn. Then $(\nu_*\Theta_{d^2-1})^2 = 1$. Since $\rho(S) = d^2 + 1$, the image of S by ν is $\mathbb{P}^2$ with a line $L = \nu_*\Theta_{d^2-1}$. Furthermore, multiplicity of Θ_{d^2-1} in F_∞ implies that $\nu_* F_\infty = dL$. Let C_0 be the image by ν of $f^{-1}(0)$. By the Shioda-Tate formula (3.2) and its non-negativity, C_0 is an irreducible curve of degree d. The original fibration $f : S \to \mathbb{P}^1$ corresponds to a pencil Λ generated by C_0 and dL. From the configuration of $E_{d^2}, \Theta_0, \Theta_1, \ldots, \Theta_{d^2-1}$ and $f^{-1}(0)$, we see that the intersection point of C_0 and L is a smooth point of C_0, and we also deduce that C_0 has a contact of order d with L at the intersection point. □

LEMMA 4.3. *Let S be a smooth rational surface of $\rho(S) = d^2 + 1$ for any integer $d \geq 3$ and $f : S \to \mathbb{P}^1$ a relatively minimal fibration of plane curves of degree d. Assume that f has no multiple fibres when $d = 3$. Then the conditions* (2), (3a) *and* (3b) *of* Theorem 4.1 *are equivalent.*

PROOF. Lemma 4.2 states (2) ⇒ (3a). It suffices to show (3a) ⇒ (3b) and (3b) ⇒ (2).

(3a) ⇒ (3b): Let $(X : Y : Z)$ be homogeneous coordinates of $\mathbb{P}^2$ and

$$\sum_{j=0}^{d} \sum_{i=0}^{d-j} c_{i,j} X^i Y^j Z^{d-i-j} = 0$$

the defining equation of C_0 for some complex numbers $c_{i,j}$. We may define the line L by $Y = 0$ and assume that the unique tangent point of C_0 for L is $(0 : 0 : 1)$. Then we have $c_{0,0} = c_{1,0} = \cdots = c_{d-1,0} = 0$, $c_{d,0} \neq 0$ and $c_{0,1} \neq 0$. Furthermore, we may put $c_{d,0} = c_{0,1} = 1$ without loss of generality.

(3b) ⇒ (2): We consider a pencil Λ on $\mathbb{P}^2$ defined by (4.3), namely, each member C_t in Λ is defined by (4.3) for $t \in \mathbb{C}$ and the member C_∞ in Λ corresponding to ∞ is dL, which is defined by $Y^d = 0$. Then C_t is smooth at the point $(0 : 0 : 1)$ for all $t \in \mathbb{C}$. Furthermore, C_t has a contact of order d with L at the smooth point $(0 : 0 : 1)$. Thus any two members in Λ are disjoint on $\mathbb{P}^2 \setminus \{(0 : 0 : 1)\}$. In particular, the d^2 base points of Λ consist of the point $(0 : 0 : 1)$ and its infinitely near points. Therefore, we obtain a relatively minimal fibration $f : S \to \mathbb{P}^1$ of smooth plane curves of degree d from $\Phi_\Lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ by eliminating the base points of Λ as follows:

Let $\nu_1 : W_1 \to \mathbb{P}^2$ be the blow-up at the point $(0 : 0 : 1)$ with the exceptional curve E_1, i.e., $\nu_1(E_1) = (0 : 0 : 1)$. Let P_2 be the intersection point of E_1 and the strict transform to W_1 of L. The strict transform to W_1 of C_t has a contact of order $d-1$ with that of L at P_2 for all $t \in \mathbb{C}$. Next let $\nu_2 : W_2 \to W_1$ be the blow-up at the base point P_2 with $E_2 = \nu_2^{-1}(P_2)$. Let P_3 denote the intersection point of E_2 and the strict transform to W_2 of L. For all $t \in \mathbb{C}$ the strict transform to W_2 of C_t has a contact of order $d-2$ with that of L at P_3. In the same way, for $i = 3, 4, \ldots, d-1$, after the blow-up $\nu_i : W_i \to W_{i-1}$ at the base point P_i with $E_i = \nu_i^{-1}(P_i)$, the strict transform to W_i of C_t has a contact of order $d - i$ with that of L at P_{i+1}. Denote the pull-back of curves by the same symbols for simplicity. Then we get the irreducible decomposition

$$C_\infty - E_1 - E_2 - \cdots - E_{d-1} = d(L - E_1 - E_2 - \cdots - E_{d-1}) + \sum_{i=1}^{d-2} i(d-1)(E_i - E_{i+1}).$$

Furthermore, $C_t - E_1 - E_2 - \cdots - E_{d-1}$ has a contact of order $(d^2 - d + 1)$ with the other members at P_d for all $t \in \mathbb{C}$. Denote by $\nu_d : W_d \to W_{d-1}$ the blow-up at the base point P_d with $E_d = \nu_d^{-1}(P_d)$. Let P_{d+1} be the intersection point of E_d and the strict transform to W_d of C_d. In fact, P_{d+1} corresponds to a tangent direction of $C_t - E_1 - E_2 - \cdots - E_{d-1}$ at P_d on W_{d-1} by ν_d, and $C_t - E_1 - E_2 - \cdots - E_d$ has a contact of order $(d^2 - d)$ with the other members at P_{d+1} for all $t \in \mathbb{C}$. In the same way, for $i = d+1, d+2, \ldots, d^2 - 1$, after the blow-up $\nu_i : W_i \to W_{i-1}$ at the base point P_i with $E_i = \nu_i^{-1}(P_i)$, $C_t - E_1 - E_2 - \cdots - E_i$ has a contact of order $(d^2 - i)$ with the other members at P_{i+1}. Let $\nu_{d^2} : S \to W_{d^2-1}$ be the blow-up at the base point P_{d^2} with $E_{d^2} = \nu_{d^2}^{-1}(P_{d^2})$. Put $f = \Phi_\Lambda \circ \nu_1 \circ \nu_2 \circ \cdots \circ \nu_{d^2}$. Then $f : S \to \mathbb{P}^1$ is a relatively minimal fibration whose general fibre F is $C_t - E_1 - E_2 - \cdots - E_{d^2}$ for general $t \in \mathbb{C}$ and $f^{-1}(\infty) = C_\infty - E_1 - E_2 - \cdots - E_{d^2}$ is a reducible fibre. We remark that the irreducible components of $f^{-1}(\infty)$ consist of one $(-d+1)$-curve $L - E_1 - E_2 - \cdots - E_d$ and $(d^2 - 1)$ (-2)-curves $E_i - E_{i+1}, i = 1, 2, \ldots, d^2 - 1$. Furthermore, we see that the dual graph of the reducible fibre $f^{-1}(\infty)$ corresponds to the graph as in Figure 1. □

As a corollary of Theorem 3.1, we have the following.

Lemma 4.4. *The Mordell-Weil group of f is trivial if and only if the zero section (O) and the irreducible components of the fibres of f generate* NS(S).

(2) ⇒ (1)

Lemma 4.5. *Let S be a smooth rational surface of $\rho(S) = d^2 + 1$ for any integer $d \geq 3$ and $f : S \to \mathbb{P}^1$ a relatively minimal fibration of plane curves of degree d. Assume that f has no multiple fibres when $d = 3$. If f has the reducible fibre F_∞ whose dual graph corresponds to the graph as in Figure* 1*, then the Mordell-Weil group of f is trivial.*

Proof. We use the same notation as in Proof of Lemma 4.3. The irreducible components of F_∞ are $L - E_1 - E_2 - \cdots - E_d$ and $E_i - E_{i+1}, i = 1, 2, \ldots, d^2 - 1$. These and E_{d^2}, which is a (-1)-section of f, generate L and $E_j, j = 1, 2, \ldots, d^2$, and form a $\mathbb{Z}$-basis of NS(S). Therefore the Mordell-Weil group of f is trivial by Lemma 4.4. □

Proof of Theorem 4.1. Combining Lemmas 4.3 and 4.5, it suffices to show (1) $\Rightarrow$ (2) to prove Theorem 4.1. Let S be a smooth rational surface of $\rho(S) = d^2 + 1$ for any integer $d \geq 3$ and $f : S \to \mathbb{P}^1$ a relatively minimal fibration of plane curves of degree d. Assume that f has no multiple fibres when $d = 3$. We denote by F a general fibre of f. Let $\nu : S \to \mathbb{P}^2$ be a birational morphism as in Corollary 2.3 and E_i, $i = 1, 2, \ldots, d^2$ the pull-back to S of d^2 (-1)-curves contracted by ν. Assume that the Mordell-Weil group of f is trivial. Then a section of f is unique. We shall denote by E_{d^2} the (-1)-section of f. Furthermore, in the process of contracting by ν, we may assume that E_{i+1} corresponds to an infinitely near point of the point corresponding to E_i for $i = 1, 2, \ldots, d^2 - 1$. Since $(d^2 - 1)$ (-2)-curves $E_i - E_{i+1}$, $i = 1, 2, \ldots, d^2 - 1$ are connected, a reducible singular fibre F_∞ of f contains all of them. However, they do not generate F_∞. By the Shioda-Tate formula (3.2) and $\rho(S) = d^2 + 1$, another component of F_∞ is unique, where we denote it by Θ, and all other fibres of f are irreducible.

Let L be the pull-back by $\nu : S \to \mathbb{P}^2$ of a line. Then $\Theta = \alpha L - \sum_{i=1}^{d^2} \beta_i E_i$ for some non-negative integers α, β_i. Since $\Theta.E_{d^2}$ and $\Theta.(E_i - E_{i+1})$ are non-negative, we have $0 \leq \beta_{d^2} \leq \beta_{d^2-1} \leq \cdots \leq \beta_2 \leq \beta_1 \leq \alpha$. Lemma 4.4 implies $\alpha = 1$. These and $\Theta.F = 0$ provide $\Theta = L - E_1 - E_2 - \cdots - E_d$. Here, Θ and $(d^2 - 1)$ (-2)-curves $E_i - E_{i+1}$, $i = 1, 2, \ldots, d^2 - 1$ form a singular fibre whose dual graph corresponds to the graph as in Figure 1.

This completes the proof of Theorem 4.1. $\square$

In [1], Beauville pointed out that the minimum number of singular fibres is two over $\mathbb{P}^1$, if $f : S \to \mathbb{P}^1$ is not a trivial fibration. There are many interesting arithmetic and geometric properties in this extreme case (see [3]).

Example 4.6. Let $f : S \to \mathbb{P}^1$ be as in (3b) of Theorem 4.1. Consider the case where $c_{i,j} = 0$ for the defining equation (4.3), and recall the proof of (3b) $\Rightarrow$ (2). Let C_t be a curve on $\mathbb{P}^2$ defined by $tY^d = X^d + YZ^{d-1}$. Then C_t is smooth unless $t = 0, \infty$, namely, the number of singular fibres of f is two.

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Shinya Kitagawa,
General Education (Natural Sciences),
National Institute of Technology, Gifu College,
2236-2 Kamimakuwa, Motosu, Gifu 501-0495, Japan
e-mail: `kit058shiny@gifu-nct.ac.jp`