Splitting invariants and a π_1-equivalent Zariski-pair of conic-line arrangements

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Abstract

This article is based on the authors talk given at the Kinosaki Algebraic Geometry Symposium 2022. We give a brief overview of the subject of the embedded topology of plane curves. Furthermore, we illustrate the idea of a relatively new type of invariants called *splitting invariants* which prove effective in distinguishing the topology of plane curves. We also describe a new example of a π_1-equivalent Zariski-pair consisting of conic-line arrangements of degree 7.

1 The embedded topology of plane curves and Zariski pairs

The base field in this article is the field of complex numbers $\mathbb{C}$. First, we set up some notation and explain the subject of the embedded topology of plane algebraic curves.

Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane algebraic curve, which is possibly singular and reducible. We are interested in the *embedded topology of* $\mathcal{C}$, i.e. the homeomorphism class of the pair $(\mathbb{P}^2, \mathcal{C})$, where a homeomorphism of pairs is defined as follows:

Definition 1.1 (Homeomorphism of pairs)**.** Let X_1, X_2 be topological spaces and $Y_1 \subset X_1$, $Y_2 \subset X_2$ be subspaces. Then the pairs (X_1, Y_1) and (X_2, Y_2) are homeomorphic as pairs if and only if there exists a homeomorphism $h : X_1 \to X_2$ such that $h(Y_1) = Y_2$. When (X_1, Y_1) and (X_2, Y_2) are homeomorphic as pairs, we denote this by $(X_1, Y_1) \approx (X_2, Y_2)$.

A basic problem in this subject is to classify plane curves in terms of their embedded topology. We first make some basic observations in this area. Let $T(\mathcal{C}_1), T(\mathcal{C}_2)$ be tubular neighborhoods of $\mathcal{C}_1, \mathcal{C}_2$. The first observation is:

$$(\mathbb{P}^2, \mathcal{C}_1) \approx (\mathbb{P}^2, \mathcal{C}_2) \Rightarrow (T(\mathcal{C}_1), \mathcal{C}_1) \approx (T(\mathcal{C}_2), \mathcal{C}_2)$$
$$(T(\mathcal{C}_1), \mathcal{C}_1) \not\approx (T(\mathcal{C}_2), \mathcal{C}_2) \Rightarrow (\mathbb{P}^2, \mathcal{C}_1) \not\approx (\mathbb{P}^2, \mathcal{C}_2)$$

hence the case where $(T(\mathcal{C}_1), \mathcal{C}_1) \approx (T(\mathcal{C}_2), \mathcal{C}_2)$ becomes important. This condition is a topological condition which is relatively complicated, so we wish to translate it into more algebraic terms. In order to do this, we consider the *combinatorics* or *combinatorial type* of a plane curve, defined as follows:

Definition 1.2 (Combinatorics of $\mathcal{C}$)**.** Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane curve, $\sigma : \widehat{\mathbb{P}^2} \to \mathbb{P}^2$ the minimal good embedded resolution of $\mathcal{C}$, $\Gamma_\mathcal{C} = (\mathcal{V}_\mathcal{C}, \mathcal{E}_\mathcal{C})$ the dual graph of $\sigma^{-1}(\mathcal{C})$, $\mathrm{Str}_\mathcal{C} \subset \mathcal{V}_\mathcal{C}$ the set of vertices corresponding to the strict transforms of the irreducible components of $\mathcal{C}$, and $e_\mathcal{C} : \mathcal{V}_\mathcal{C} \to \mathbb{Z}$ the Euler map (which gives the self intersection number of each irreducible component). Then the triple $(\Gamma_\mathcal{C}, \mathrm{Str}_\mathcal{C}, e_\mathcal{C})$ is called the *combinatorics* or *combinatorial type* of $\mathcal{C}$ and is denoted by

$$\mathrm{Comb}(\mathcal{C}) := (\Gamma_\mathcal{C}, \mathrm{Str}_\mathcal{C}, e_\mathcal{C}).$$

[1]This article is based on joint work with M. Amram (Shamoon College of Engineering, Israel), T. Shirane (Tokushima U.), U. Sinichkin (Tel Aviv University, Israel) and H. Tokunaga (TMU).

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Definition 1.3 (Equivalence of combinatorics)**.** Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ be plane curves such that $\mathrm{Comb}(\mathcal{C}_i) = (\Gamma_{\mathcal{C}_i}, \mathrm{Str}_{\mathcal{C}_i}, e_{\mathcal{C}_i})$. The equivalence of $\mathrm{Comb}(\mathcal{C}_1)$ and $\mathrm{Comb}(\mathcal{C}_2)$ is defined by

$$\mathrm{Comb}(\mathcal{C}_1) = \mathrm{Comb}(\mathcal{C}_2) \quad \overset{\text{def}}{\Longleftrightarrow} \quad \left\{ \begin{array}{c} \exists \varphi : \Gamma_{\mathcal{C}_1} \to \Gamma_{\mathcal{C}_2} \text{ isomorphism of graphs s.t.} \\ \varphi(\mathrm{Str}_{\mathcal{C}_1}) = \mathrm{Str}_{\mathcal{C}_2},\ e_{\mathcal{C}_1} = e_{\mathcal{C}_2} \circ \varphi \end{array} \right. .$$

By using these concepts, we can translate topology into algebra by the following fact:

Fact 1.4. *Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ be plane curves. Then the following holds:*

$$(T(\mathcal{C}_1), \mathcal{C}_1) \approx (T(\mathcal{C}_2), \mathcal{C}_2) \Leftrightarrow \mathrm{Comb}(\mathcal{C}_1) = \mathrm{Comb}(\mathcal{C}_2)$$

Now, curves having the same combinatorial type becomes important in the study of the embedded topology, which leads to the following definition of *Zariski pairs*:

Definition 1.5 (Zariski pairs)**.** A pair of curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ is called a *Zariski pair* if the following conditions hold.

(i) $\mathrm{Comb}(\mathcal{C}_1) = \mathrm{Comb}(\mathcal{C}_2)$

(ii) $(\mathbb{P}^2, \mathcal{C}_1) \not\approx (\mathbb{P}^2, \mathcal{C}_2)$

The first example of a Zariski pair was given by O. Zariski in [19].

Example 1.6 (Zariski's Example (1929))**.** Let $\mathcal{C}_1, \mathcal{C}_2$ be sextic curves with 6 cusps. Assume that for $\mathcal{C}_1$, there exists a conic through the six cusps, where as for $\mathcal{C}_2$, no such conic exists. Then, $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_1) \not\cong \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_2)$ and $\mathcal{C}_1, \mathcal{C}_2$ is a Zariski pair.

Remark 1.7. The above example demonstrates that the embedded topology is not determined by the combinatorics, which motivates the definition of a Zariski-pair. It also illustrates that the position of singular points have an effect on the embedded topology. An interesting problem is to determine what algebraic aspects of a curve have an effect on the embedded topology.

In order to understand and classify the embedded topology of plane curves, we need to understand Zariski pairs in detail. Moreover, the following two problems become important in this direction:

- Develop a method to construct curves having prescribed combinatorics, but with subtle differences in terms of algebra.
- Find some suitable method to distinguish the embedded topology of curves based on the above subtle differences.

Concerning the first problem, there are many approaches. The author together with H. Tokunaga have utilized certain rational elliptic surfaces in order to construct curves with prescribed combinatorics. This method will be described later. As for the second problem, the basic idea is to find a suitable invariant. Some of the invariants that have been used are: fundamental groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$, Alexander polynomials $\Delta_{\mathcal{C}}(t)$, and the existence/non-existence of certain Galois covers branched within $\mathcal{C}$. The latter two are deeply related to the fundamental group. Some newer types of invariants have also been developed, such as the *linking set* ([8]) and *splitting invariants*, the latter which will be describe in the following. These newer types of invariants are independent from the fundamental group and can distinguish curves having the same fundamental group. A nice survey of Zariski pairs is given in [2].

2 Splitting Invariants and π_1-equivalent Zariski pairs

In this section, we explain the general idea of *splitting invariants* and provide some examples to illustrate the idea. The importance of splitting invariants lies in the fact that (i) they are defined and can be calculated in terms of algebraic geometry, and (ii) they can distinguish some curves that have isomorphic fundamental groups.

Our goal is to develop a method to extract data of a plane curve $\mathcal{C} \subset \mathbb{P}^2$. Instead of considering the curve directly, we consider a Galois cover $f : X \to \mathbb{P}^2$ with branch locus $\mathcal{B} := \Delta_f$. In the most general form, a splitting invariant of $\mathcal{C}$ with respect to f is any property or data extracted from $f^{-1}(\mathcal{C})$. Since the covering f is involved, the data obtained from $f^{-1}(\mathcal{C})$ should reflect the relation between $\mathcal{C}$ and $\mathcal{B}$. Moreover, it should encode how $\mathcal{C}$ and $\mathcal{B}$ are "entangled" and give information about the curve $\mathcal{C} + \mathcal{B}$. Before stating precise definitions, we give some examples that illustrate this idea.

The first example is an example found by H. Tokunaga in [17].

Example 2.1 (Tokunaga (2012))**.** Let $\mathcal{B} \subset \mathbb{P}^2$ be a smooth conic and $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the double cover branched along $\mathcal{B}$. There exist irreducible quartic curves $\mathcal{Q}_1, \mathcal{Q}_2$ such that

- $\mathcal{Q}_i$ $(i = 1, 2)$ is tangent to $\mathcal{B}$ at 4 distinct smooth points.
- $\mathrm{Comb}(\mathcal{Q}_1 + \mathcal{B}) = \mathrm{Comb}(\mathcal{Q}_2 + \mathcal{B})$

and furthermore

$$\begin{aligned} f^{-1}(\mathcal{Q}_1) &: \text{irreducible} \\ f^{-1}(\mathcal{Q}_2) &: \text{reducible} \end{aligned}$$

and moreover, the curves $\mathcal{Q}_1 + \mathcal{B}$ and $\mathcal{Q}_2 + \mathcal{B}$ form a Zariski pair.

The property of $f^{-1}(\mathcal{Q}_i)$ being irreducible or not can be regarded as a splitting invariant in this case. In fact, the term splitting invariant was inspired by this phenomenon where a curve may "split" into two irreducible components or not. Furthermore, this example leads to the following definition:

Definition 2.2 (Splitting curve)**.** Let $\mathcal{C} \subset \mathbb{P}^2$ be an irreducible curve and let $f : X \to \mathbb{P}^2$ be a double cover. Then $\mathcal{C}$ is a *splitting curve* with respect to f if $f^{-1}(\mathcal{C})$ is reducible.

The second example (although it dates earlier) is an example found by E. Artal Bartolo and H. Tokunaga in [4].

Example 2.3 (Artal-Tokunaga (2004))**.** Let f and $\mathcal{B}$ be as in Example 2.1Tokunaga (2012)thm.2.1. For any $d \geq 4$, there exists irreducible nodal rational curves $\mathcal{C}_1, \dots, \mathcal{C}_{[\frac{d}{2}]}$ such that

- $\mathcal{C}_k$ $(k = 1, \dots, [\frac{d}{2}])$ is tangent to $\mathcal{B}$ at d distinct smooth points,
- $\mathrm{Comb}(\mathcal{C}_k + \mathcal{B})$ are all equivalent,
- $\mathcal{C}_k$ $(k = 1, \dots, [\frac{d}{2}])$ is a splitting curve with respect to f,

and furthermore, if $f^{-1}(\mathcal{C}_k) = \mathcal{C}_k^+ + \mathcal{C}_k^-$ $(k = 1, \dots, [\frac{d}{2}])$, by choosing labels suitably,

$$\text{the bi-degree of } \mathcal{C}_k^+ \text{ is } (k, d-k)$$
$$\text{the bi-degree of } \mathcal{C}_k^- \text{ is } (d-k, k).$$

Moreover the curves $\mathcal{B} + \mathcal{C}_1, \dots, \mathcal{B} + \mathcal{C}_{[\frac{d}{2}]}$ are pairwise Zariski pairs (a Zariski $[\frac{d}{2}]$-tuple).

In this second example the bi-degree of the curves can be regarded as a splitting invariant. This example demonstrates that the splitting property is not enough to completely distinguish the embedded topology, and we need to consider more detailed data.

The third example is due to I. Shimada who found an equisingular family of curves with non-connected components in [11], and T. Shirane who proved that they form Zariski multiples in [14] by using a splitting invariant called the *splitting number*.

Example 2.4 (Shimada (2003), Shirane (2017))**.** Let E be a smooth cubic curve. Let $b, m, n \in \mathbb{N}$, $b \geq 4$ $n|b$, $m = \frac{b}{n}$. Let $\mathcal{B}$ be a smooth curve of degree b that intersects E at $3n$ points, each with multiplicity m. In this case, I. Shimada computed the fundamental group and found that

$$\pi_1(\mathbb{P}^2 \setminus (E + \mathcal{B})) \cong \begin{cases} \mathbb{Z} & (3 \nmid b) \\ \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & (3|b) \end{cases}.$$

He also proved that the equisingular family of such curves has $d(m)$ connected components, where $d(m)$ is the number of divisors of m. Let $\mathcal{F}$ be the equisingular family and let $\mathcal{F} = \cup_{\nu|m}\mathcal{F}_\nu$ be the decomposition of $\mathcal{F}$. Later, T. Shirane considered cyclic covers $f_{\mathcal{B}} : X \to \mathbb{P}^2$ of degree m branched along $\mathcal{B}$ and proved that for each $\mathcal{B}_\nu \in \mathcal{F}_\nu$ and $f_{\mathcal{B}_\nu} : X \to \mathbb{P}^2$,

$$f_{\mathcal{B}_\nu}^{-1}(E) = E^1 + \cdots + E^\nu$$

for a suitable labeling of the connected components $\mathcal{F}_\nu$. Namely, he computed the number of irreducible components of $f_{\mathcal{B}_\nu}^{-1}(E)$. He defined the *splitting number* of E with respect to f_ν as the number of irreducible components of $f_{\mathcal{B}_\nu}^{-1}(E)$, which can be considered as a splitting invariant. Moreover, he proved that for each b, n, m the curves

$$E + \mathcal{B}_1, \dots, E + \mathcal{B}_\nu, \dots, E + \mathcal{B}_m \quad (\nu|m)$$

form a Zariski $d(m)$-tuple.

In the above example, the fundamental group is determined solely by the degree b of $\mathcal{B}$, where as for each n and $m = \frac{b}{n}$, there are $d(m)$ curves having distinct embedded topology. Hence the embedded topology is not determined by the fundamental group. Also, the example demonstrates that splitting invariants can distinguish the embedded topology of curves that have isomorphic fundamental groups.

Definition 2.5 (π_1-equivalent Zariski pairs)**.** A pair of curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ is called a *π_1-equivalent Zariski pair* if the following conditions hold:

(i) $\mathcal{C}_1, \mathcal{C}_2$ is a Zariski pair.

(ii) $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_1) \cong \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_2)$

Many π_1-equivalent Zariski pairs are known including Example 2.4Shimada (2003), Shirane (2017)thm.2.4, for example, there are π_1-equivalent Zariski pairs of sextics with simple singularities in the list of [12]. Arrangements consisting of one smooth curve of degree $d \geq 4$ and three non-concurrent lines, called Atral-arrangements, also produce π_1-equivalent Zariski pairs (see [3] and [16]). These π_1-equivalent Zariski pairs are given by curves containing an irreducible component with either singularities or genus $g \geq 1$. In the following, we will give an example of a π_1-equivalent Zariski pair consisting of conic-line arrangements, where every component is smooth and rational. The key invariant that will be used to distinguish the embedded topology is the *splitting type* defined by the author in [5].

Definition 2.6 (Splitting Type). Let $\mathcal{B} \subset \mathbb{P}^2$ be a plane curve with $\deg \mathcal{B} = 2n$, $f : X \to \mathbb{P}^2$ be the double cover branched along $\mathcal{B}$. Let $\mathcal{C}_1, \mathcal{C}_2$ be irreducible splitting curves with respect to f, i.e.

$$f^{-1}(\mathcal{C}_1) = \mathcal{C}_1^+ + \mathcal{C}_1^-, \qquad f^{-1}(\mathcal{C}_2) = \mathcal{C}_2^+ + \mathcal{C}_2^-.$$

The splitting type of the triple $(\mathcal{C}_1, \mathcal{C}_2; \mathcal{B})$ is defined to be

$$(m_1, m_2) = (\mathcal{C}_1^+ \cdot \mathcal{C}_2^+, \mathcal{C}_1^+ \cdot \mathcal{C}_2^-)$$

where we choose suitable labels so that $m_1 \leq m_2$.

Example 2.7. Under the setting of the above definition, let $\mathcal{C}_1 = C$ be a conic and $\mathcal{C}_2 = L$ be a line. Then the possible splitting types of $(C, L; \mathcal{B})$ is $(0, 2)$ or $(1, 1)$.

Proposition 2.8. *Let $\mathcal{B}_1, \mathcal{B}_2$ be curves of even degree and $\mathcal{C}_{i1}, \mathcal{C}_{i2}$ be splitting curves with respect to $\mathcal{B}_i$ $(i = 1, 2)$. If the triples $(\mathcal{C}_{11}, \mathcal{C}_{12}; \mathcal{B}_1)$ and $(\mathcal{C}_{21}, \mathcal{C}_{22}; \mathcal{B}_2)$ have distinct splitting types, then there do not exist any homeomorphisms $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(\mathcal{B}_1) = \mathcal{B}_2$ and $h(\mathcal{C}_{11} + \mathcal{C}_{12}) = \mathcal{C}_{21} + \mathcal{C}_{22}$.*

The splitting type, together with the above proposition allows us to distinguish the embedded topology of many curves of the form $\mathcal{B} + \mathcal{C}_1 + \mathcal{C}_2$.

After the author defined the splitting type, many other splitting invariants have been developed, especially by T. Shirane, such as the *splitting number* mentioned above, the *connected number* ([15]) and the *splitting graph* ([16]).

3 Zariski pairs of conic-line arrangements of small degree

In this section, we give a list of some Zariski pairs of conic-line arrangements of small degree that can be distinguished using the splitting type defined in the previous section. **We warn the reader that we do not claim that the list is complete**, i.e. there may be other Zariski pairs that can be distinguished using alternative methods and invariants. The basic method in constructing the curves is the method developed by the author and H. Tokunaga, which uses the data of the Mordell-Weil lattices of rational elliptic surfaces. While creating the list, a π_1-equivalent Zariski pair consisting of conic-line arrangements of degree 7 was found, which we describe in detail in Example 3.2thm.3.2.

3.1 The method of construction

In this subsection, we give a rough sketch of the method to construct curves with prescribed combinatorial data.

First, we explain how to construct a rational elliptic surface associated to a quartic curve $\mathcal{B}$. Let $\mathcal{B} \subset \mathbb{P}^2$ be a quartic curve and let $z_o \in \mathcal{B}$ be a general smooth point of $\mathcal{B}$. Let Λ_{z_o} be the pencil of lines through z_o. We can construct a rational elliptic surface $S_{\mathcal{B},z_o}$ as in the following diagram

$$\begin{tikzcd} X_{\mathcal{B}} \arrow[d, "f_{\mathcal{B}}"'] & S_{\mathcal{B}} \arrow[l, "\mu"'] \arrow[d, "f'_{\mathcal{B}}"] & S_{\mathcal{B},z_o} \arrow[l, "\nu_{z_o}"'] \arrow[d, "f_{\mathcal{B},z_o}"] \\ \mathbb{P}^2 & \widehat{\mathbb{P}^2} \arrow[l, "q"] & (\widehat{\mathbb{P}^2})_{z_o}, \arrow[l, "q_{z_o}"] \end{tikzcd}$$

where $f_{\mathcal{B}}$ is the double cover branched along $\mathcal{B}$, μ is the canonical resolution of singularities of $X_{\mathcal{B}}$ and ν_{z_o} is the resolution of the pencil $\overline{\Lambda}_{z_o}$ of genus 1 curves on $S_{\mathcal{B}}$ that is induced by Λ_{z_o}. Then $S_{\mathcal{B},z_o}$ is a rational elliptic surface with a distinguished section O, which is the exceptional divisor of the second (final) blow-up in ν_{z_o}. The set of sections $\mathrm{MW}(S_{\mathcal{B},z_o})$ of $S_{\mathcal{B},z_o}$ has an abelian group structure with O being the neutral element.

Next, we explain how to obtain the curves. Given $s \in \mathrm{MW}(S_{\mathcal{B},z_o})$, $s \neq O$, the image $\mathcal{C}_s := f_{\mathcal{B}} \circ \mu \circ \nu_{z_o}(s)$ becomes a curve in $\mathbb{P}^2$. Furthermore, by the construction,

$$f_{\mathcal{B}}^{-1}(\mathcal{C}_s) = f_{\mathcal{B}} \circ \mu \circ \nu_{z_o}(s) + f_{\mathcal{B}} \circ \mu \circ \nu_{z_o}([-1]s)$$

which shows that $\mathcal{C}_s$ is a splitting curve with respect to $f_{\mathcal{B}}$. Here, $[-1]s$ is the negative of s with respect to the group structure of $\mathrm{MW}(S_{\mathcal{B},z_o})$. Hence, we can obtain splitting curves $\mathcal{C}_s$ associated to sections $s \in \mathrm{MW}(S_{\mathcal{B},z_o})$.

Finally, the height pairing $\langle \bullet, \bullet \rangle$ defined on $\mathrm{MW}(S_{\mathcal{B},z_o})$ provides additional data reflecting the geometry of the sections. This additional data allows us to calculate the splitting types through the following formula due to T. Shioda [13].

Theorem 3.1 (Shioda, 1990)**.**

$$\langle s_1, s_2 \rangle = \chi + s_1 \cdot O + s_2 \cdot O - s_1 \cdot s_2 - \sum \mathrm{contr}_v(s_1, s_2)$$

The above formula relates the height pairing $\langle s_1, s_2 \rangle$ to the intersection numbers $s_1 \cdot s_2$ of sections, which in turn give the splitting types of $(\mathcal{C}_{s_1}, \mathcal{C}_{s_2}; \mathcal{B})$. Also, the contribution term $\sum \mathrm{contr}_v(s_1, s_2)$ contains information about the components of singular fibers that intersect s_i, which in turn provides information about the singular points of B that $\mathcal{C}_{s_i}$ passes through. From this information, we can deduce the combinatorics of $\mathcal{C}_{s_i}$ and $\mathcal{B}$. We are interested in conic-line arrangements, so we focus on sections where $\mathcal{C}_s$ is a line or a conic. In each case where the resulting curve $\mathcal{C}_s$ is not a line but a (weak) contact conic, there exists a one-parameter family of curves having the same combinatorics as $\mathcal{C}_s$. We denote the family of curves corresponding to the section s by $\mathcal{F}_s$ (see [6] for details). Details of these arguments can be found in [6], [5], [7]. Here, we forgo the details and provide an example to demonstrate how we can find candidates of Zariski pairs using rational elliptic surfaces and the height pairing. This example also gives a π_1-equivalent Zariski pair.

Example 3.2. Let $\mathcal{B} = C_1 + C_2$ be the union of two smooth conics C_1, C_2 meeting transversally, and let $z_o \in C_1$ be a general point, i.e. z_o is not a nodal point of $\mathcal{B}$ and the tangent line at z_o is not a bitangent line of $\mathcal{B}$. Then the associated rational elliptic surface $S_{\mathcal{B},z_o}$ has 5 reducible singular fibers of type I_2, and by [10] the Mordell-Weil lattice $\mathrm{MW}(S_{\mathcal{B},z_o})$ is isomorphic to $(A_1^*)^{\oplus 3} \oplus \mathbb{Z}/2\mathbb{Z}$, where A_1^* is the dual lattice of the root lattice of type A_1. Let $C_1 \cap C_2 = \{P_0, P_1, P_2, P_3\}$. The strict transform of the preimages of the lines $\overline{P_0P_i}$ $(i = 1, 2, 3)$ under $f_{\mathcal{B}}$ give rise to pairs of sections $(s_1, [-1]s_1)$, $(s_2, [-1]s_2)$, $(s_3, [-1]s_3)$ which generate the $(A_1^*)^{\oplus 3}$ part of $\mathrm{MW}(S_{\mathcal{B},z_o})$. Furthermore, the strict transform of the preimage of C_2 gives rise to the torsion section t. Hence, we have a set of generators s_1, s_2, s_3, t of $\mathrm{MW}(S_{\mathcal{B},z_o})$ such that

- $\mathcal{C}_{s_i} = \overline{P_0P_i}, (i = 1, 2, 3)$,
- $\mathcal{C}_t = C_2$,
- $\langle s_i, s_i \rangle = \frac{1}{2}$, $\langle s_i, s_j \rangle = 0$ $(i \neq j)$.

We can obtain curves with prescribed combinatorial data related to $\mathcal{B}$ by using these generators. For example:

- $\mathcal{C}_{s_i \pm s_j}$ $(i \neq j)$ is a smooth conic passing through P_i, P_j and is tangent to both C_1 and C_2.
- $\mathcal{C}_{s_1 \pm s_2 \pm s_3 + t}$ is a bitangent line of $\mathcal{B}$.

The key observation is that it is possible to choose curves having the same combinatorics, but with differences in the height pairing. If we consider $\mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+s_2+s_3+t}$ and $\mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2+s_3+t}$, the two curves have the same combinatorics, but there is a difference in the height pairing:

$$\langle s_1 + s_2, s_1 + s_2 + s_3 + t \rangle = 1$$
$$\langle s_1 + s_2, s_1 + s_2 + s_3 + t \rangle = 0.$$

This difference in the height pairing leads to the difference of the splitting type, and it can be computed that the splitting type of $(\mathcal{C}_{s_1+s_1}, \mathcal{C}_{s_1+s_2+s_3+t}; \mathcal{B})$ is $(0, 2)$ where as the splitting type of $(\mathcal{C}_{s_1+s_1}, \mathcal{C}_{s_1-s_2+s_3+t}; \mathcal{B})$ is $(1, 1)$.

It is possible to compute explicit equations for the above curves.

- $C_1 : x - t^2 = 0$
- $C_2 : x^2 - 10tx + 25x - 36 = 0$
- $\mathcal{C}_{s_1+s_2} : x - (\frac{5}{4}t^2 - 2t + 3) = 0$
- $\mathcal{C}_{s_1+s_2+s_3+t} : x - (\frac{32}{5}t - \frac{256}{25}) = 0$
- $\mathcal{C}_{s_1-s_2+s_3+t} : x - (10t - 25) = 0$

The fundamental groups of the above example has been computed by the Zariski-van Kampen method using the above explicit equations, and we have our main result:

Theorem 3.3 (Amram-B-Shirane-Sinichkin-Tokunaga [1])**.** *Under the notation of Example 3.2thm.3.2, let $\mathcal{C} = C_1 + C_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+s_2+s_3+t}$ and $\mathcal{C}' = C_1 + C_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2+s_3+t}$. Then $\mathcal{C}$ and $\mathcal{C}'$ is a Zariski pair. Furthermore,*

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \cong \pi_1(\mathbb{P}^2 \setminus \mathcal{C}') \cong \mathbb{Z}^{\oplus 3}.$$

Moreover, $\mathcal{C}$ and $\mathcal{C}'$ is a π_1-equivalent Zariski pair.

3.2 The list of Zariski pairs of conic line arrangements of small degree

In this subsection, we give a list of some Zariski pairs of conic line arrangements of small degree that can be obtained by the method described in the previous subsection.

First, since we are considering conic-line arrangements and we are assuming $\deg \mathcal{B} = 4$ in order to utilize rational elliptic surfaces, so the possibilities for the components of the branch locus $\mathcal{B}$ is

- $\mathcal{B} = C_1 + C_2$ two conics,
- $\mathcal{B} = C + L_1 + C_2$ two lines,
- $\mathcal{B} = L_1 + L_2 + L_3 + L_4$ four lines.

In order to have differences in the height pairing, which leads to the difference in splitting type, the Mordell-Weil lattice should have rank greater than 2, which narrows down the possibilities of $\mathcal{B}$ to the following three cases:

- $\mathcal{B}_1 = C + L_1 + L_2$ where C is a smooth conic, L_1, L_2 are lines all of which meet transversally.
- $\mathcal{B}_2 = C_1 + C_2$ where C_1, C_2 are smooth conics meeting transversally at 4 points.
- $\mathcal{B}_3 = C_1 + C_2$ where C_1, C_2 are smooth conics tangent at 1 point with multiplicity 2.

For a general point $z_o \in \mathcal{B}_i$, we have $\mathrm{MW}(S_{\mathcal{B}_1,z_o}) = (A_1^*)^{\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, $\mathrm{MW}(S_{\mathcal{B}_2,z_o}) = (A_1^*)^{\oplus 3} \oplus (\mathbb{Z}/2\mathbb{Z})$, $\mathrm{MW}(S_{\mathcal{B}_3,z_o}) = A_1^* \oplus \langle \frac{1}{4} \rangle \oplus \mathbb{Z}/2\mathbb{Z}$.

We define some terminology to describe the curves that will appear in the list.

Definition 3.4. A *(simple) contact curve* $\mathcal{C}$ to a curve $\mathcal{B}$ is a curve that has even intersection multiplicity at all intersection points, and intersect at smooth points of $\mathcal{B}$. A *weak contact curve* is a contact curve that has intersection points at singular points if $\mathcal{B}$. When $\mathcal{B}$ has degree 4, a bitangent line is a simple contact line and a weak bitangent line is a weak contact line.

3.2.1 The case of $\mathcal{B}_1 = C + L_1 + L_2$

Let us consider the first case. We introduce the following labels to describe the curves:

$$\{P_0\} = L_1 \cap L_2, \qquad \{P_1, P_2\} = L_1 \cap C, \qquad \{P_3, P_4\} = L_2 \cap C$$

In this case, $\mathrm{MW}(S_{\mathcal{B}_1,z_o}) = (A_1^*)^{\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and we can choose generators s_1, s_2 of the $(A_1^*)^{\oplus 2}$ part and torsion sections t_1, t_2, t_3 so that

$$\mathcal{C}_{s_1} = L_{13} = \overline{P_1P_3}, \qquad \mathcal{C}_{s_2} = L_{14} = \overline{P_1P_4}, \qquad \mathcal{C}_{t_1} = L_1, \qquad \mathcal{C}_{t_2} = L_2, \qquad \mathcal{C}_{t_3} = C$$

Furthermore, the combinatorics of the lines and conics corresponding to sections are as follows:

- $\mathcal{C}_{s_1+t_1}$: weak contact conic passing through P_0, P_2, P_3
- $\mathcal{C}_{s_1+t_2}$: weak contact conic passing through P_0, P_1, P_4
- $\mathcal{C}_{s_1+t_3}$: weak bitangent line through P_2, P_4

- $\mathcal{C}_{s_2+t_1}$: weak contact conic passing through P_0, P_2, P_4
- $\mathcal{C}_{s_2+t_2}$: weak contact conic passing through P_0, P_1, P_3
- $\mathcal{C}_{s_2+t_3}$: weak bitangent line through P_2, P_3
- $\mathcal{C}_{s_1 \pm s_2}$: weak contact conic passing through P_3, P_4, tangent to L_1 and C_1
- $\mathcal{C}_{s_1 \pm s_2+t_3}$: weak contact conic passing through P_1, P_2, tangent to L_2 and C_1
- $\mathcal{C}_{s_1 \pm s_2+t_2}$: weak bitangent line passing through P_0
- $\mathcal{C}_{[2]s_i}$: simple contact conic. $(i = 1, 2)$

Now we are ready to describe the Zariski pairs $\mathcal{C}_1, \mathcal{C}_2$ obtained from these curves. First, the combinatorics of $\mathcal{C}_i$ is described and then the types of curves that give the components of $\mathcal{C}_i$ with the desired combinatorics is described in terms of sections. We note that there may be many choices of sections that give Zariski pairs with equivalent combinatorics, but we just give one example for each combinatorial type. Additional Zariski pairs can be obtained by adding further lines/conics to the listed examples, but we do not present them for simplicity.

(i) $\mathcal{C} = \mathcal{B}_1 + C_2 + L_3 = (C_1 + L_1 + L_2) + C_2 + L_3$

C_2 is a simple contact conic, L_3 is a line through P, P', where $P \in \{P_1, P_2\}$, $P' \in \{P_3, P_4\}$.

- $\mathcal{C}_1 = \mathcal{B}_1 + \mathcal{C}_{[2]s_1} + \mathcal{C}_{s_1}$
- $\mathcal{C}_2 = \mathcal{B}_1 + \mathcal{C}_{[2]s_1} + \mathcal{C}_{s_2}$

Note: This example was first found by Tokunaga ([18]).

(ii) $\mathcal{C} = \mathcal{B}_1 + C_2 + L_3 = (C_1 + L_1 + L_2) + C_2 + L_3$

C_2 is a weak contact conic through P, P' where $\{P, P'\} = \{P_1, P_2\}$ or $\{P_3, P_4\}$, L_3 is a weak bitangent line through P_0.

- $\mathcal{C}_1 = \mathcal{B}_1 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+s_2+t_2}$
- $\mathcal{C}_2 = \mathcal{B}_1 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2+t_2}$

(iii) $\mathcal{C} = \mathcal{B}_1 + C_2 + C_3 = (C_1 + L_1 + L_2) + C_2 + C_3$

C_2 is a simple contact conic and C_3 is a weak contact conic through P_0, P, P' where $\{P, P'\} = \{P_1, P_3\}$ or $\{P_2, P_4\}$.

- $\mathcal{C}_1 = \mathcal{B}_1 + \mathcal{C}_{[2]s_1} + \mathcal{C}_{s_1+t_1}$
- $\mathcal{C}_2 = \mathcal{B}_1 + \mathcal{C}_{[2]s_1} + \mathcal{C}_{s_2+t_1}$

(iv) $\mathcal{C} = \mathcal{B}_1 + C_2 + C_3 = (C_1 + L_1 + L_2) + C_2 + C_3$

C_2 and C_3 are weak contact conics both through P, P' where $\{P, P'\} = \{P_1, P_2\}$ or $\{P_3, P_4\}$.

- $\mathcal{C}_1 = \mathcal{B}_1 + \mathcal{C}_{s_1+s_2} + F_{s_1+s_2}, \quad (F_{s_1+s_2} \in \mathcal{F}_{s_1+s_2})$
- $\mathcal{C}_2 = \mathcal{B}_1 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2}$

(v) $\mathcal{C} = \mathcal{B}_1 + C_2 + C_3 = (C_1 + L_1 + L_2) + C_2 + C_3$

C_2 is a weak contact conic through P_1, P_2 and C_3 is a weak contact conic through P_3, P_4.

- $\mathcal{C}_1 = \mathcal{B}_1 + \mathcal{C}_{s_1+s_2+t_3} + \mathcal{C}_{s_1+s_2}$
- $\mathcal{C}_2 = \mathcal{B}_1 + \mathcal{C}_{s_1-s_2+t_3} + \mathcal{C}_{s_1+s_2}$

3.2.2 The case $\mathcal{B}_2 = C_1 + C_2$, transversal intersection

Let $\mathcal{B}_2 = C_1 + C_2$, where C_1, C_2 are smooth conics intersecting transversally. Let $C_1 \cap C_2 = \{P_0, P_1, P_2, P_3\}$ and $z_o \in C_1 \setminus \{P_0, P_1, P_2, P_3\}$. In this case, $\mathrm{MW}(S_{\mathcal{B}_2, z_o}) \cong (A_1^*)^{\oplus 3} \oplus (\mathbb{Z}/2\mathbb{Z})$. We can choose generators s_1, s_2, s_3 for the $(A_1^*)^{\oplus 3}$ part of the Mordell-Weil lattice so that the following hold:

$$\mathcal{C}_{s_1} = \overline{P_0 P_1}, \qquad \mathcal{C}_{s_2} = \overline{P_0 P_2}, \qquad \mathcal{C}_{s_3} = \overline{P_0 P_3}, \qquad \mathcal{C}_t = C_1$$

Furthermore the the combinatorics of the lines and conics corresponding to sections are as follows:

- $\mathcal{C}_{s_i+t}$: weak bitangent line through P_j, P_k ($\{i,j,k\} = \{1,2,3\}$)
- $\mathcal{C}_{s_i \pm s_j}$: weak contact conic through P_i, P_j ($i \neq j$)
- $\mathcal{C}_{s_i \pm s_j + t}$: weak contact conic through $\{P_0, P_1, P_2, P_3\} \setminus \{P_i, P_j\} (i \neq j)$
- $\mathcal{C}_{s_1 \pm s_2 \pm s_3 + t}$: simple bitangent line of $C_1 + C_2$
- $\mathcal{C}_{[2]s_i}$: simple contact conic. ($i = 1, 2, 3$)

The Zariski pairs obtained from these curves are as follows:

(i) $\mathcal{C} = \mathcal{B}_2 + C_3 + L_1 = (C_1 + C_2) + C_3 + L_1$

C_3 is a simple contact conic and L_1 is a line through P_i, P_j, $\{i,j\} \subset \{0,1,2,3\}$.

- $\mathcal{C}_1 = \mathcal{B}_2 + \mathcal{C}_{[2]s_1} + \mathcal{C}_{s_1}$
- $\mathcal{C}_2 = \mathcal{B}_2 + \mathcal{C}_{[2]s_1} + \mathcal{C}_{s_2}$

Note: This example was first found by Tokunaga ([18]).

(ii) $\mathcal{C} = \mathcal{B}_2 + C_3 + L_1 = (C_1 + C_2) + C_3 + L_1$

C_3 is a weak contact conic through P_i, P_j, $\{i,j\} \subset \{0,1,2,3\}$ and L_1 is a simple bitangent line.

- $\mathcal{C}_1 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+s_2+s_3+t}$
- $\mathcal{C}_2 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2+s_3+t}$

Note: This is Example 3.2thm.3.2.

(iii) $\mathcal{C} = \mathcal{B}_2 + C_3 + C_4 = (C_1 + C_2) + C_3 + C_4$

C_3 and C_4 are both simple contact conics.

- $\mathcal{C}_1 = \mathcal{B}_2 + \mathcal{C}_{[2]s_1} + F_{[2]s_1}, \quad (F_{[2]s_1} \in \mathcal{F}_{[2]s_1})$
- $\mathcal{C}_2 = \mathcal{B}_2 + \mathcal{C}_{[2]s_1} + \mathcal{C}_{[2]s_2}$

Note: This example was first found by Namba-Tsuchihashi ([9]).

(iv) $\mathcal{C} = \mathcal{B}_2 + C_3 + C_4 = (C_1 + C_2) + C_3 + C_4$
C_3 and C_4 are weak contact conics both through $\{P_i, P_j\}$, $\{i, j\} \subset \{0, 1, 2, 3\}$.

- $\mathcal{C}_1 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + F_{s_1+s_2}, \quad (F_{s_1+s_2} \in \mathcal{F}_{s_1+s_2})$
- $\mathcal{C}_2 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2}$

Note: $\mathcal{F}_{s_1+s_2}$ is the family of conics having the same combinatorics as $\mathcal{C}_{s_1+s_2}$.

(v) $\mathcal{C} = \mathcal{B}_2 + C_3 + C_4 = (C_1 + C_2) + C_3 + C_4$
C_3 is a weak contact conic through $\{P_i, P_j\}$ and C_4 is a weak contact conics both through $\{P_k, P_l\}$, $\{i, j, k, l\} \subset \{0, 1, 2, 3, \}$.

- $\mathcal{C}_1 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+s_2+t}$
- $\mathcal{C}_2 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2+t}$

(vi) $\mathcal{C} = \mathcal{B}_2 + C_3 + L_1 + L_2 = (C_1 + C_2) + C_3 + L_1 + L_2$
C_3 is a simple contact conic of $\mathcal{Q}$ and L_1, L_2 are simple bitangent lines.

- $\mathcal{C}_1 = \mathcal{B}_2 + \mathcal{C}_{2[s_1]} + \mathcal{C}_{s_1+s_2+s_3+t} + \mathcal{C}_{s_1-s_2-s_3+t}$
- $\mathcal{C}_2 = \mathcal{B}_2 + \mathcal{C}_{2[s_1]} + \mathcal{C}_{s_1+s_2+s_3+t} + \mathcal{C}_{s_1-s_2+s_3+t}$

(vii) $\mathcal{C} = \mathcal{B}_2 + C_3 + L_1 + L_2 = (C_1 + C_2) + C_3 + L_1 + L_2$
C_3 is a weak contact conic through $\{P_i, P_j\}$ and L_1, L_2 are simple bitangent lines of $\mathcal{Q}$.

- $\mathcal{C}_1 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+s_2+s_3+t} + \mathcal{C}_{s_1+s_2-s_3+t}$
- $\mathcal{C}_2 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+s_2+s_3+t} + \mathcal{C}_{s_1-s_2+s_3+t}$
- $\mathcal{C}_3 = \mathcal{B}_2 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2+s_3+t} + \mathcal{C}_{s_1-s_2-s_3+t}$

Note: We have a Zariski triple for this combinatorics. This arrangement contains Example 3.2thm.3.2 as a subarrangement.

3.2.3 The case $\mathcal{B}_3 = C_1 + C_2$, tangent at one point

Let $\mathcal{B}_3 = C_1 + C_2$, where C_1, C_2 are smooth conics tangent at one point and intersect transversally at two other points. Let P_0 be the tangent point and P_1, P_2 be the other two transversal intersection points of C_1 and C_2. Let $z_o \in C_1 \setminus \{P_0, P_1, P_2\}$. In this case, $\mathrm{MW}(S_{\mathcal{B}_3, z_o}) \cong A_1^* \oplus \langle \frac{1}{4} \rangle \oplus \mathbb{Z}/2\mathbb{Z}$. We can choose generators s_1, s_2 for the $A_1^* \oplus \langle \frac{1}{4} \rangle$ part of the Mordell-Weil lattice so that the following hold:

$$\mathcal{C}_{s_1} = \overline{P_1 P_2}, \qquad \mathcal{C}_{s_2} = \overline{P_0 P_1}, \qquad \mathcal{C}_t = C_1$$

Furthermore the conics and lines corresponding to sections are as follows:

- $\mathcal{C}_{s_2+t}$: weak bitangent line $\overline{P_0P_2}$
- $\mathcal{C}_{s_1+t}$: (weak) bitangent line of C_1, C_2 tangent to both at P_0
- $\mathcal{C}_{s_1\pm s_2}$: weak contact conic through P_0, P_2
- $\mathcal{C}_{s_1\pm s_2+t}$: weak contact conic through P_0, P_1
- $\mathcal{C}_{[2]s_2}$: simple contact conic to $C_1 + C_2$ with tangent at P_0
- $\mathcal{C}_{[2]s_2+t}$: weak contact conic through P_0, P_1, P_2
- $\mathcal{C}_{s_1\pm[2]s_2+t}$: simple bitangent line of $C_1 + C_2$
- $\mathcal{C}_{2[s_1]}$: simple contact conic

The Zariski pairs obtained from these curves are as follows:

(i) $\mathcal{C} = \mathcal{B}_3 + C_3 + L = (C_1 + C_2) + C_3 + L$

C_3 is a weak contact conic through P_0 and $P \in \{P_1, P_2\}$ and L is a simple bitangent line of $\mathcal{Q}$.

- $\mathcal{C}_1 = \mathcal{B}_3 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+[2]s_2+t}$
- $\mathcal{C}_2 = \mathcal{B}_3 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-[2]s_2+t}$

(ii) $\mathcal{C} = \mathcal{B}_3 + C_3 + C_4 = (C_1 + C_2) + C_3 + C_4$

C_3 is a weak contact conic through P_0, P_1 and C_4 is a weak contact conic through P_0, P_2.

- $\mathcal{C}_1 = \mathcal{B}_3 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1+s_2+t}$
- $\mathcal{C}_2 = \mathcal{B}_3 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2+t}$

(iii) $\mathcal{C} = \mathcal{B}_3 + C_3 + C_4 = (C_1 + C_2) + C_3 + C_4$

C_3, C_4 are weak contact conics both through P_0, P, $P \in \{P_1, P_2\}$.

- $\mathcal{C}_1 = \mathcal{B}_3 + \mathcal{C}_{s_1+s_2} + F_{s_1+s_2}, \quad (F_{s_1+s_2} \in \mathcal{F}_{s_1+s_2})$
- $\mathcal{C}_2 = \mathcal{B}_3 + \mathcal{C}_{s_1+s_2} + \mathcal{C}_{s_1-s_2}$

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