Big Cohen-Macaulay test ideals in equal characteristic zero via ultraproducts

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Let $(R, \mathfrak{m})$ be a Noetherian local ring.

Definition 1

R-algebra B is said to be a *(balanced) big Cohen-Macaulay algebra* (or simply *BCM* algebra) if every system of parameters of R is a regular sequence on B.

We explain BCM regularity introduced by [Ma, Schwede 21].

Setting 1

Let $(R, \mathfrak{m})$ be a normal local domain of dimension d.

1. We fix an embedding $R \subseteq \omega_R \subseteq \operatorname{Frac} R$. Hence we also fix an effective canonical divisor K_R.
2. $\Delta \geqslant 0$ is a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier.
3. Since $K_R + \Delta$ is effective and $\mathbb{Q}$-Cartier, there exists $n \in \mathbb{N}_{>0}$ and $f \in R$ such that $n(K_R + \Delta) = \operatorname{div}(f)$.

$R[f^{1/n}]^{\mathrm{N}} \subseteq R^+$ denotes the normalization of $R[f^{1/n}]$.

Definition 2 ([Ma, Schwede 21])

With notation as in Setting 1, if B is a BCM R-algebra and an $R[f^{1/n}]^{\mathbb{N}}$-algebra, then we define $0^{B, K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)} := \operatorname{Ker}\psi$, where ψ is the homomorphism determined by the below commutative diagram:

$$
\begin{tikzcd}
H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d] & H^d_{\mathfrak{m}}(B) \arrow[r, "\cdot f^{1/n}"] \arrow[d] & H^d_{\mathfrak{m}}(B) . \\
H^d_{\mathfrak{m}}(\omega_R) \arrow[r] \arrow[urr, bend right, "\psi"'] & H^d_{\mathfrak{m}}(B \otimes_R \omega_R) \arrow[ur] &
\end{tikzcd}
$$

Definition 3 ([Ma, Schwede 21])

Moreover, if R is complete, then we define

$$\tau_B(R, \Delta) := \operatorname{Ann}_R 0^{B, K_R + \Delta}_{H^d_{\mathfrak{m}}(\omega_R)}.$$

We call $\tau_B(R, \Delta)$ *the BCM test ideal of* (R, Δ) *w.r.t.* B. We call (R, Δ) is BCM_B-regular if $\tau_B(R, \Delta) = R$.

Proposition 1 ([Ma, Schwede 21])

Let

1. $(R, \mathfrak{m})$ *a Noetherian complete normal local domain of characteristic* $p > 0$
2. $\Delta \geqslant 0$ *a* $\mathbb{Q}$*-Weil divisor on* $\operatorname{Spec} R$ *such that* $K_R + \Delta$ *is* $\mathbb{Q}$*-Cartier*
3. B *a BCM* R^+*-algebra*

Then

$$\tau_B(R, \Delta) = \tau(R, \Delta).$$

In particular, R *is strongly* F*-regular if and only if* R *is* BCM_B*-regular.*

We fix an infinite set W. We use $P(W)$ to denote the power set of W.

Definition 4

A nonempty subset $\mathcal{F} \subseteq P(W)$ is called a *filter* if the following two conditions hold.

1. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq W$, then $B \in \mathcal{F}$.

Definition 5

Let $\mathcal{F}$ be a filter on W.

1. $\mathcal{F}$ is called an *ultrafilter* if for all $A \in P(W)$, we have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.
2. An ultrafilter $\mathcal{F}$ is called *principal* if there exists a finite subset $A \subseteq W$ such that $A \in \mathcal{F}$.

Proposition 2

Every infinite set has non-principal ultrafilters.

Definition 6

Let

1. A_w a family of set indexed by W
2. $\mathcal{F}$ a non-principal ultrafilter on W

The *ultraproduct of* A_w is defined by

$$\operatorname*{ulim}_{w} A_w = A_\infty := \prod_{w} A_w / \sim,$$

where $(a_w) \sim (b_w)$ if and only if $\{w \in W | a_w = b_w\} \in \mathcal{F}$.

Following [Schoutens 03], we explain approximations and non-standard hulls.
Let

1. R be a local ring essentially of finite type over $\mathbb{C}$.
2. $\mathcal{P}$ be the set of prime numbers
3. $\mathcal{F}$ a non-principal ultrafilter on $\mathcal{P}$.

Then we can construct an approximation R_p and the non-standard hull R_∞ of R.
They have the following properties.

1. R_p local rings essentially of finite type over $\overline{\mathbb{F}_p}$
2. $R_\infty = \mathrm{ulim}_p R_p$
3. $R \to R_\infty$ faithfully flat

Definition 7

Let

1. $\mathcal{F}$ a non-principal ultrafilter on $\mathcal{P}$
2. φ a property

Then we say $\varphi(p)$ *holds for almost all* p if $\{p \in \mathcal{P} | \varphi(p) \text{ holds}\} \in \mathcal{F}$.

Let R be a local ring essentially of finite type over $\mathbb{C}$.

Example 1

1. $\dim R_p = d$ for almost all p if and only if $\dim R = d$.
2. R_p is Cohen-Macaulay (resp. Gorenstein, regular) for almost all p if and only if R is Cohen-Macaulay (resp. Gorenstein, regular).

Let R be a local domain essentially of finite type over $\mathbb{C}$.

Definition 8 ([Schoutens 04])

We define the *canonical BCM algebra* $\mathcal{B}(R)$ by

$$\mathcal{B}(R) := \operatorname*{ulim}_{p} R_p^+,$$

where R_p is an approximation of R. (Schoutens calls this the *quasi-hull*.)

Proposition 3 ([Schoutens 04])

$\mathcal{B}(R)$ is a BCM R-algebra and an R^+-algebra.

Our main result is stated as follows:

Theorem 1 (Y 22)

Let

1. *R a normal local domain essentially of finite type over $\mathbb{C}$*
2. *$\Delta \geqslant 0$ a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier*
3. *$\mathcal{B}(R)$ is the canonical BCM algebra*
4. *$\widehat{R}$ and $\widehat{\mathcal{B}(R)}$ are the $\mathfrak{m}$-adic completions of R and $\mathcal{B}(R)$*
5. *$\widehat{\Delta}$ the flat pullback of Δ by $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$*

Then we have

$$\tau_{\widehat{\mathcal{B}(R)}}(\widehat{R}, \widehat{\Delta}) = \mathcal{J}(\widehat{R}, \widehat{\Delta}),$$

where $\mathcal{J}(\widehat{R}, \widehat{\Delta})$ is the multiplier ideal of $(\widehat{R}, \widehat{\Delta})$.

As an application of the preceding theorem, we proved the following.

Theorem 2 (Y 22)

Let

1. $R \hookrightarrow S$ *is a pure local* $\mathbb{C}$*-algebra homomorphism between normal local domains essentially of finite type over* $\mathbb{C}$
2. R *is* $\mathbb{Q}$*-Gorenstein*
3. $\Delta_S \geqslant 0$ *a* $\mathbb{Q}$*-Weil divisor such that* $K_S + \Delta_S$ *is* $\mathbb{Q}$*-Cartier*
4. $\mathfrak{a} \subseteq R$ *a nonzero ideal*
5. $t \in \mathbb{Q}_{>0}$

Then we have

$$\mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t) \cap R \subseteq \mathcal{J}(R, \mathfrak{a}^t).$$

Let R be a normal local domain essentially of finite type over $\mathbb{C}$.

Question 1

Let

1. $\Delta \geqslant 0$ *a* $\mathbb{Q}$*-Weil divisor on* $\operatorname{Spec} R$ *such that* $K_R + \Delta$ *is* $\mathbb{Q}$*-Cartier*
2. B *a BCM* R^+*-algebra*

Then does the following hold?

$$\mathcal{J}(R, \Delta) \subseteq \tau_{\widehat{B}}(\widehat{R}, \widehat{\Delta})$$

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