On the Induction Motor under Cyclical Operation.

By

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1. The present paper is devoted exclusively to the class of induction motors which work under cyclically varying loads, as for example, induction motors driving roll trains, reciprocating pumps and hoisting machines etc.. In these cases the resisting torque imposed upon the motor varies from time to time; but, with the variations which are fairly definite in their nature and after some time, the torque cycle closes itself by the last terminal conditions coinciding with the first starting point.

It is a well known fact that the capacity of an electric motor working under these conditions is limited principally by the condition that the greatest rise in temperature which the motor attains when working under a certain load, should not exceed a certain presupposed amount of rise in temperature. The rise in temperature of a motor depends on the amount of internal losses, heat dissipated from the motor body, and heat absorbed in the mass of the motor. The first varies in accordance with the variation of the load on the motor, and the last depends on the time of the supply of heat. Hence the rise in temperature of the motor, when load conditions are unstable as in our case, depends not only on the amount of the fluctuation of the load, but on the manner of variation with respect to time. Thus the greatest rise in temperature and therefore the capacity of the induction motor under consideration must be determined from the time characteristic of the load which is to be imposed upon it.

When the induction motor is applied to certain services where load conditions are very unstable, the rotor having considerable inertia becomes sometimes of great advantage. As is well known, the use of a fly-wheel attached to the rotor shaft of the motor changes, when load conditions are variable, the characteristic of the load imposed on the motor; the effect is that it takes up the peaks and equalizes the unsteady loads thrown on the motor. In this case, therefore, the capacity of the induction motor must be determined from the nature of the resisting torque and the amount of inertia of the rotating parts of the system.

In what follows the author intends, in the case of a cyclically varying load with a given characteristic, firstly to solve some dynamical problems concerning the induction motor with rotor possessing a considerable inertia and secondly to take up some problems on heating of the motor which lead to determining the capacity of the motor.

I. The Effect of Inertia on Induction Motor Operation.

2. The effect of a fly-wheel attached to the rotor shaft on the induction motor operation under cyclically varying loads has been investigated by various authors such as Ph. Ehrlich,[1] L. Kallir,[2] H. C. Sprecht[3] and F. G. Gasche,[4] in cases when the resisting torque is a sine function of time or varies intermittently from one constant value to other constant values. The purpose of this part of the present paper is to obtain more general solutions of the problem, the resisting torque being taken as any real function of time.

The characteristic of the resisting torque of a working machine depends on the nature of the work it performs and for the same class of work there

1) Elektrotech. u. Masch. XXVI Jahrg., Heft 9; S. 173. (1908).
2) Ibid., XXVI Jahrg., Heft 9; S. 465. (1908).
3) Trans. of the A. I. E. E. Vol. XXVIII, Part 2; p. 869. (1909).
4) Ibid., Vol. XXIX, Part 2; p. 1385. (1910).

is in general a well defined cycle of operations into which the machine gravitates. Thus we may obtain the resisting torque time relation of a working machine either by direct experiment on an existing machine or by calculation from physical constants. Thus in the first place we assume that by some means a definite knowlege of the resisting torque time relation for a given working machine is available.

Making further assumptions that, (1) the rotor of the induction motor and the fly-wheel are connected rigidly with the working machine, (2) the resistance of the secondary winding is constant, and (3) the impressed *E. M. F.* on the primary winding is kept constant, formulæ governing the proportion of the fly-wheel and the induction motor will be developed from the dynamics of the problem.

3. From the general assumption that under the above conditions the torque of an induction motor is proportional to the slip we have

$$M = A(\omega_s - \omega) \quad \text{..............................(1)},$$

where

M is the torque imposed on the motor at any time t,
ω is the angular velocity of the motor corresponding to the torque M,
ω_s is the synchronous angular velocity of the motor,
A is a constant.

By differentiating (1) with respect to t we get

$$\frac{dM}{dt} = -A\frac{d\omega}{dt} \quad \text{..............................(2)}.$$

The equation of motion of a motor attached with a fly-wheel is represented by

$$I\frac{d\omega}{dt} = M - m \quad \text{..............................(3)},$$

where

I is the moment of inertia of the revolving part,
m is the resisting torque on the rotor shaft at any time t.

Eliminating $\frac{d\omega}{dt}$ from (2) and (3) and putting

$$\frac{I}{A} = T_\alpha \quad \text{.............................(4)},$$

we obtain

$$\frac{dM}{dt} + \frac{1}{T_\alpha}(M - m) = 0 \quad \text{......................(5)}.$$

This is the fundamental equation representing the motion of the induction motor under consideration.

We can easily see that the constant T_α given in the above expression has an analogous meaning as the time constant of an electric circuit containing inductance. For further information on this constant the reader is referred to F. G. Gasche's valuable paper.[1]

Since, as given above, the resisting torque of the working machine m is a known function of time t, eq. (5) can be at once solved in the form

$$M = \varepsilon^{-\frac{t}{T_\alpha}}\left(\frac{1}{T_\alpha}\int \varepsilon^{\frac{t}{T_\alpha}} m\, dt + C\right),$$

where C is a constant.

If M_0 is the torque of the motor at the beginning of the cyclical operation the above equation becomes

$$M = \varepsilon^{-\frac{t}{T_\alpha}}\left(M_0 + \frac{1}{T_\alpha}\int_0^t \varepsilon^{\frac{t}{T_\alpha}} m\, dt\right) \quad \text{..............(6)}.$$

4. To determine the initial torque of the motor M_0 and to calculate the torque M at any instant t, it is necessary that the last term in the bracket of eq. (6) should be integrable. Whether it is readily integrable or not depends on the form of the function m. In certain cases it may be expressed, throughout the whole period of the cycle, as a simple function, but in many cases it is difficult unless we make use of a complex function

1) *Loc. cit.*

of t. For technical purposes, to represent m in a single complex function will be of little value, since we must obtain the solution in a form which is easily calculable. To enable the calculation in a simple form in the case of a complex resisting torque time relation, we divide the whole period into a suitable number of divisions so that during each division the resisting torque is represented or approximately represented by a simple function of time and it multiplied by $\varepsilon^{\frac{t}{T_\alpha}}$ is readily integrable.

Now assume that the complete period of the cycle is divided into n divisions and let the time of each division be

$$t_1,\ t_2,\ t_3,\ \ldots\ldots\ldots\ t_r,\ \ldots\ldots\ldots\ t_n$$

respectively, which is related by

$$t_1+t_2+t_3+\ldots\ldots\ldots+t_r+\ldots\ldots\ldots+t_n=\tau_0,$$

where τ_0 is the period of the cycle.

Let the resisting torque during each division be represented by

$$m_1,\ m_2,\ m_3,\ \ldots\ldots\ldots m_r,\ \ldots\ldots\ldots m_n$$

respectively, each being a simple function of time t.

Further let the value of the torque of the motor at the beginning of each division of time be

$$M_0,\ M_1,\ M_2,\ \ldots\ldots\ldots\ M_{r-1},\ \ldots\ldots\ldots\ M_{n-1}$$

respectively. Then the value of M at the end of each division will be

$$M_1,\ M_2,\ M_3,\ \ldots\ldots\ldots M_r,\ \ldots\ldots\ldots\ M_n$$

respectively.

Now applying eq. (6) to each division from the beginning to the end of the time we obtain

$$\left.\begin{aligned} M_1\varepsilon^{\frac{t_1}{T_\alpha}} &= M_0+\frac{1}{T_\alpha}\int_0^{t_1}\varepsilon^{\frac{t}{T_\alpha}}m_1\,dt, \\ M_2\varepsilon^{\frac{t_2}{T_\alpha}} &= M_1+\frac{1}{T_\alpha}\int_0^{t_2}\varepsilon^{\frac{t}{T_\alpha}}m_2\,dt, \end{aligned}\right|$$

$$\left.\begin{aligned} M_3\varepsilon^{\frac{t_3}{T_\alpha}} &= M_2+\frac{1}{T_\alpha}\int_0^{t_3}\varepsilon^{\frac{t}{T_\alpha}} m_3\,dt, \\ &\cdots\cdots\cdots\cdots\cdots \\ M_r\varepsilon^{\frac{t_r}{T_\alpha}} &= M_{r-1}+\frac{1}{T_\alpha}\int_0^{t_r}\varepsilon^{\frac{t}{T_\alpha}} m_r\,dt, \\ &\cdots\cdots\cdots\cdots\cdots \\ M_n\varepsilon^{\frac{t_n}{T_\alpha}} &= M_{n-1}+\frac{1}{T_\alpha}\int_0^{t_n}\varepsilon^{\frac{t}{T_\alpha}} m_n\,dt. \end{aligned}\right\} \quad \cdots\cdots\cdots(7),$$

where t is the time counted from the beginning of each devision.

Eliminating M_1, M_2, M_3, ...M_r, ...M_{n-1} from above equations and remembering that $M_n = M_0$, we get

$$M_0 = \frac{1}{T_\alpha}\,\frac{1}{\varepsilon^{\frac{\tau_0}{T_\alpha}}-1}\sum_{r=1}^{r=n}\varepsilon^{\frac{\tau_{r-1}}{T_\alpha}}\int_0^{t_r}\varepsilon^{\frac{t}{T_\alpha}} m_r\,dt \quad \cdots\cdots\cdots(8),$$

where τ_{r-1} is the time at the beginning of rth division counted from the beginning of the cycle.

The torque of the motor at the beginning of any division, rth say, is obtained from (7)

$$M_{r-1} = \varepsilon^{-\frac{\tau_{r-1}}{T_\alpha}}\left(M_0+\frac{1}{T_\alpha}\sum_{p=1}^{p=r-1}\varepsilon^{\frac{\tau_{p-1}}{T_\alpha}}\int_0^{t_p}\varepsilon^{\frac{t}{T_\alpha}} m_p dt\right) \quad \cdots\cdots(9).$$

Therefore the torque of the motor at any time t which is in rth division is given by

$$M = \varepsilon^{-\frac{t}{T_\alpha}}\left(M_{r-1}+\frac{1}{T_\alpha}\int_0^{t}\varepsilon^{\frac{t}{T_\alpha}} m_r\,dt\right) \quad \cdots\cdots\cdots\cdots\cdots(10).$$

Now for the sake of simplicity let us write

$$S_r = \frac{1}{T_\alpha}\sum_{p=1}^{p=r}\varepsilon^{\frac{\tau_{p-1}}{T_\alpha}}\int_0^{t_p}\varepsilon^{\frac{t}{T_\alpha}} m_p\,dt,$$

$$\left.\begin{aligned} S_0 &= \frac{1}{T\alpha}\sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}}{T\alpha}} \int_0^{t_p} \varepsilon^{\frac{t}{T\alpha}} m_p\, dt, \\ &\text{and in general} \\ S &= S_{r-1} + \frac{1}{T_\alpha}\varepsilon^{\frac{\tau_{r-1}}{T\alpha}} \int_0^{t} \varepsilon^{\frac{t}{T\alpha}} m_r\, dt. \end{aligned}\right\} \quad \text{.........(11).}$$

Then above equations become

$$\left.\begin{aligned} M_0 &= \frac{1}{\varepsilon^{\frac{\tau_0}{T\alpha}} - 1} S_0, \\ M_{r-1} &= \varepsilon^{-\frac{\tau_{r-1}}{T\alpha}}(M_0 + S_{r-1}), \\ M &= \varepsilon^{-\frac{\tau}{T\alpha}}(M_0 + S). \end{aligned}\right\} \quad \text{.........(I).}$$

The eq. (I) gives the means for calculating the torque of the motor at any time during the complete period.

It will be easily seen that (I) also satisfies when the resisting torque is expressed by a single function of time throughout whole cycle. In this case S_0 and S in (11) become

$$\left.\begin{aligned} S_0 &= \frac{1}{T\alpha}\int_0^{\tau_0} \varepsilon^{\frac{t}{T\alpha}} m\, dt, \\ S &= \frac{1}{T\alpha}\int_0^{\tau} \varepsilon^{\frac{t}{T\alpha}} m\, dt. \end{aligned}\right\} \quad \text{.........(12).}$$

5. The angular velocity and the slip of the induction motor at any time are given by

$$\begin{aligned} \omega &= \omega_s - \frac{M}{A} \\ &= \omega_s - \frac{1}{A}\varepsilon^{-\frac{\tau}{T\alpha}}(M_0 + S), \end{aligned}$$

and

$$\left.\begin{aligned} s &= \frac{M}{A\omega_s} \\ &= \frac{1}{A\omega_s}\varepsilon^{-\frac{\tau}{T\alpha}}(M_0+S). \end{aligned}\right\} \quad \text{.........(II)},$$

where s is the slip of the motor at any time.

The values of M's and S in the above expressions are given in (11) and (I). Thus if the resisting torque is given as a function of time the angular velocity and the slip of the motor at any time during the complete period can be determined.

Next the angular displacement θ of the induction motor is, as well known, related by the equation

$$\omega = \frac{d\theta}{dt}.$$

Hence by integration we obtain

$$\theta = \omega_s t - \frac{1}{A}\left\{\int_0^t m_r dt + T\alpha(M_{r-1}-M)\right\} \quad \text{............(13)},$$

where θ is the angular displacement of the motor from the beginning of rth division to any time t in the same division.

The angular displacement of the motor during rth division is

$$\theta_r = \omega_s t_r - \frac{1}{A}\left\{\int_0^{t_r} m_r dt + T\alpha(M_{r-1}-M_r)\right\} \quad \text{.........(14)}.$$

If we denote the angular displacement counted from the beginning of the cycle up to any time τ lying in rth division by Θ we have

$$\begin{aligned} \Theta &= \sum_{\tau=0}^{\tau}\theta \\ &= \omega_s\tau - \frac{1}{A}\left\{R + T\alpha(M_0-M)\right\} \quad \text{....................(III)}, \end{aligned}$$

where

$$R = \int_0^{\tau} m \, dt \quad \text{..............................(15).}$$

From the above equation the angular displacement during the whole period will become

$$\theta_0 = \omega_s \tau_0 - \frac{R_0}{A} \quad \text{..............................(16),}$$

where R_0 stands for

$$\int_0^{\tau_0} m \, dt \quad \text{..................................(17).}$$

It is evident that eqs. (III) and (16) also hold when the resisting torque m is expressed by a single function of time during the whole period.

The power developed by the induction motor at any instant is readily obtained by the well known relation

$$\left. \begin{aligned} P &= \omega M \\ &= \left(\omega_s - \frac{M}{A}\right) M. \end{aligned} \right\} \quad \text{........ (18),}$$

in which P is the power of the motor.

The energy supplied by the motor can be obtained from the equations

$$\left. \begin{aligned} W &= \int_{\Theta_1}^{\Theta_2} M \, d\theta \\ &= \int_{\tau_1}^{\tau_2} P \, dt \\ &= \omega_s \int_{\tau_1}^{\tau_2} M \, dt - \frac{1}{A} \int_{\tau_1}^{\tau_2} M^2 \, dt. \end{aligned} \right\} \quad \text{.........(19),}$$

where W is the energy supplied by the motor while it makes the angular displacement from θ_1 to θ_2, or the interval between τ_1 and τ_2, both limits being taken arbitrarily.

6. The formulæ deduced in the above paragraphs enable us to calculate the torque, the angular velocity, the angular displacement, the power and the energy of the induction motor, if the resisting torque time relation during the whole period of the cyclical operation is given. It is sometimes convenient, particularly when the resisting torque makes a complex function of time, to use the graphic method for calculating the above formulæ. For this purpose by usual graphic method we calculate S's in (12) and R's in (15) and (17). Then by (I), (II) and (III), M, ω, s and θ at any time will be obtained. From this (18) enables us to calculate P at any time. Hence the energy supplied by the motor during any interval of time will be found graphically from the second equation of (19).

7. Now let us apply the above general solutions to some special cases.

Case 1. When the resisting torque is a potential series of time.

Let one complete period be divided into n divisions and during each division the resisting torque be represented by the following finite potential series

$$\begin{aligned}
m_1 &= K_{10} + K_{11}t + K_{12}t^2 + \ldots\ldots + K_{1h}t^h + \ldots\ldots + K_{1m_1}t^{m_1}, \\
m_2 &= K_{20} + K_{21}t + K_{22}t^2 + \ldots\ldots + K_{2h}t^h + \ldots\ldots + K_{2m_2}t^{m_2}, \\
m_3 &= K_{30} + K_{31}t + K_{32}t^2 + \ldots\ldots + K_{3h}t^h + \ldots\ldots + K_{3m_3}t^{m_3}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
m_r &= K_{r0} + K_{r1}t + K_{r2}t^2 + \ldots\ldots + K_{rh}t^h + \ldots\ldots + K_{rm_r}t^{m_r}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
m_n &= K_{n0} + K_{n1}t + K_{n2}t^2 + \ldots\ldots + K_{nh}t^h + \ldots\ldots + K_{nm_n}t^{m_n}
\end{aligned}$$

respectively, the interval of each division being

$$t_1,\ t_2,\ t_3,\ \ldots\ldots t_r,\ \ldots\ldots t_n$$

respectively, where the K's are constants.

In this case since

$$\int_0^t m_p dt = \sum_{h=0}^{h=m_p} \frac{K_{ph}}{h+1} t^{h+1},$$

we have

$$\left.\begin{aligned} R_0 &= \sum_{p=1}^{p=n} \sum_{h=0}^{h=m_p} \frac{K_{ph}}{h+1} t_p^{h+1}, \\ R_{r-1} &= \sum_{p=1}^{p=r-1} \sum_{h=0}^{h=m_p} \frac{K_{ph}}{h+1} t_p^{h+1}, \\ R &= R_{r-1} + \sum_{h=0}^{h=m_r} \frac{K_{ph}}{h+1} t^{h+1}. \end{aligned}\right\} \quad \ldots\ldots\ldots(20),$$

and since

$$K_{ph} \int_0^t \varepsilon^{\frac{t}{T_\alpha}} t^h dt = (-1)^h K_{ph} \underline{|h} \, T_\alpha^{h+1} \left\{ \varepsilon^{\frac{t}{T_\alpha}} \sum_{g=0}^{g=h} (-1)^g \frac{1}{\underline{|g}} \left(\frac{t}{T_\alpha} \right)^g - 1 \right\},$$

$S's$ in (11) become

$$\left.\begin{aligned} S_0 &= \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}}{T_\alpha}} \sum_{h=0}^{h=m_p} (-1)^h K_{ph} \underline{|h} \, T_\alpha^h \left\{ \varepsilon^{\frac{t_p}{T_\alpha}} \sum_{g=0}^{g=h} (-1)^g \frac{1}{\underline{|g}} \left(\frac{t_p}{T_\alpha} \right)^g - 1 \right\}, \\ S_{r-1} &= \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}}{T_\alpha}} \sum_{h=0}^{h=m_p} (-1)^h K_{ph} \underline{|h} \, T_\alpha^h \left\{ \varepsilon^{\frac{t_p}{T_\alpha}} \sum_{g=0}^{g=h} (-1)^g \frac{1}{\underline{|g}} \left(\frac{t_p}{T_\alpha} \right)^g - 1 \right\}, \\ S &= S_{r-1} + \varepsilon^{\frac{\tau_{r-1}}{T_\alpha}} \sum_{h=0}^{h=m_r} (-1)^h K_{ph} \underline{|h} \, T_\alpha^h \left\{ \varepsilon^{\frac{t_p}{T_\alpha}} \sum_{g=0}^{g=h} (-1)^g \frac{1}{\underline{|g}} \left(\frac{t}{T_\alpha} \right)^g - 1 \right\}. \end{aligned}\right\} \ldots(21).$$

From the above we know the values of the R's and the S's in terms of t. Therefore by means of equations given in the preceeding paragraphs we can determine the M, ω, s, etc..

Case 2. When the resisting torque varies with constant values.

Let the resisting torques during each n division of the cycle be all constant and denoted by

$$\boldsymbol{m}_1,\ \boldsymbol{m}_2,\ \boldsymbol{m}_3,\ \ldots\ldots \boldsymbol{m}_r,\ \ldots\ldots \boldsymbol{m}_n$$

respectively.

Then we have

$$\left.\begin{aligned} R_0 &= \sum_{p=1}^{p=n} \boldsymbol{m}_p t_p, \\ R_{r-1} &= \sum_{p=1}^{p=r-1} \boldsymbol{m}_p t_p, \\ R &= R_{r-1} + \boldsymbol{m}_r t. \end{aligned}\right\} \quad \ldots\ldots\ldots(22),$$

and

$$\left.\begin{aligned} S_0 &= \sum_{p=1}^{p=n} \boldsymbol{m}_p \left(\varepsilon^{\frac{\tau_p}{T\alpha}} - \varepsilon^{\frac{\tau_{p-1}}{T\alpha}} \right), \\ S_{r-1} &= \sum_{p=1}^{p=r-1} \boldsymbol{m}_p \left(\varepsilon^{\frac{\tau_p}{T\alpha}} - \varepsilon^{\frac{\tau_{p-1}}{T\alpha}} \right), \\ S &= S_{r-1} + \boldsymbol{m}_r \left(\varepsilon^{\frac{\tau}{T\alpha}} - \varepsilon^{\frac{\tau_{r-1}}{T\alpha}} \right). \end{aligned}\right\} \quad \ldots\ldots\ldots(23).$$

Therefore from (I)

$$\left.\begin{aligned} M_0 &= \frac{1}{\varepsilon^{\frac{\tau_0}{T\alpha}} - 1} \sum_{p=1}^{p=n} \boldsymbol{m}_p \left(\varepsilon^{\frac{\tau_p}{T\alpha}} - \varepsilon^{\frac{\tau_{p-1}}{T\alpha}} \right), \\ M_{r-1} &= \varepsilon^{-\frac{\tau_{r-1}}{T\alpha}} \left\{ M_0 + \sum_{p=1}^{p=r-1} \boldsymbol{m}_p \left(\varepsilon^{\frac{\tau_p}{T\alpha}} - \varepsilon^{\frac{\tau_{p-1}}{T\alpha}} \right) \right\}, \\ M &= \varepsilon^{-\frac{\tau}{T\alpha}} \left\{ M_0 + S_{r-1} + \boldsymbol{m}_r \left(\varepsilon^{\frac{\tau}{T\alpha}} - \varepsilon^{\frac{\tau_{r-1}}{T\alpha}} \right) \right\}. \end{aligned}\right\} \quad \ldots\ldots\ldots(24),$$

or from (10)

i. e.

$$\left.\begin{aligned} M &= m_r-(m_r-M_{r-1})\varepsilon^{-\frac{t}{T\alpha}}, \\ t &= T\alpha \log\frac{m_r-M_{r-1}}{m_r-M}. \end{aligned}\right\} \quad\text{.........(25).}$$

The angular velocity and the slip at any instant are respectively obtained from (II)

$$\left.\begin{aligned} \omega &= \omega_s-\frac{1}{A}\left\{m_r-(m_r-M_{r-1})\varepsilon^{-\frac{t}{T\alpha}}\right\} \\ &= \omega_s-\frac{1}{A}\varepsilon^{-\frac{\tau}{T\alpha}}\left\{M_0+S_{r-1}+m_r\left(\varepsilon^{\frac{\tau}{T\alpha}}-\varepsilon^{\frac{\tau_{r-1}}{T\alpha}}\right)\right\}, \end{aligned}\right.$$

and

$$\left.\begin{aligned} s &= \frac{1}{A\omega_s}\left\{m_r-(m_r-M_{r-1})\varepsilon^{-\frac{t}{T\alpha}}\right\} \\ &= \frac{1}{A\omega_s}\varepsilon^{-\frac{\tau}{T\alpha}}\left\{M_0+S_{r-1}+m_r\left(\varepsilon^{\frac{\tau}{T\alpha}}-\varepsilon^{\frac{\tau_{r-1}}{T\alpha}}\right)\right\}. \end{aligned}\right\} \quad\text{.........(26).}$$

The angular displacement of the motor is from (13)

$$\theta = \left(\omega_s-\frac{m_r}{A}\right)t+\frac{T\alpha}{A}(M-M_{r-1}) \quad\text{...............(27),}$$

or eliminating t

$$\theta = T\alpha\left\{\left(\omega_s-\frac{m_r}{A}\right)\log\frac{m_r-M_{r-1}}{m_r-M}+\frac{M-M_{r-1}}{A}\right\} \quad\text{......(28).}$$

The angular displacement during the time t_r is

$$\theta_r = T\alpha\left\{\left(\omega_s-\frac{m_r}{A}\right)\log\frac{m_r-M_{r-1}}{m_r-M_r}+\frac{M_r-M_{r-1}}{A}\right\}\text{......(29).}$$

The angular displacement from the beginning of the cycle up to any time τ is given by

$$\theta = \omega_s\tau - \frac{1}{A}\left\{\sum_{p=1}^{p=r-1} m_p t_p + m_r t + T\alpha(M_0 - M)\right\} \quad \text{.........(30)},$$

and the total angular displacement during the whole period is

$$\theta_0 = \omega_s\tau_0 - \frac{1}{A}\sum_{p=1}^{p=n} m_p t_p \quad \text{.......................(31)}.$$

The power of the motor at any time is given by

$$\text{P} = \omega M,$$

where M and ω are given in (24) and (26).

Energy from the motor during $t=o$ to any time t is given by

$$W = \omega_s\int_0^t M dt - \frac{1}{A}\int_0^t M^2 dt$$

$$= m_r\theta - T\alpha(M - M_{r-1})\left\{\omega_s - \frac{1}{2A}(M + M_{r-1})\right\} \quad \text{.........(32)}.$$

The energy of the motor during the time t_r is

$$W_r = m_r\theta_r - T\alpha(M_r - M_{r-1})\left\{\omega_s - \frac{1}{2A}(M_r + M_{r-1})\right\} \quad \text{.........(33)}.$$

Therefore energy developed by the motor from the beginning of the cycle up to the end of rth division is

$$\sum_{p=1}^{p=r} W_p = \sum_{p=1}^{p=r} m_p\theta_p - T\alpha(M_r - M_0)\left\{\omega_s - \frac{1}{2A}(M_r + M_0)\right\} \quad \text{...(34)},$$

and energy during the whole period is

$$\sum_{p=1}^{p=n} W_p = \sum_{p=1}^{r=n} m_p\theta_p \quad \text{..........................(35)},$$

the result obvious from the function of the fly-wheel in the case of negligible windage loss.

In the present case, if we put the resisting torques during a series of the alternate divisions all equal, we will obtain the case treated by F. G. Gasche[1] of the induction motor driving roll trains.

Case 3. When the resisting torque is a trigonometrical series.

As an example of the resisting torque expressed by a single function of time during the complete period let us take the case which occurs when induction motors drive reciprocating pumps, compressors etc.. In this case the resisting torque can be expressed by a series of trigonometrical functions of time. Here we will take a finite trigonometrical series, because in practice a periodic function can only be analysed into a series of finite terms and such is also the case here.

Now between the limits $pt=-\pi$ and $pt=\pi$, let the resisting torque be expressed by

$$m = m_0 + m_1 \sin(pt+\varphi_1) + m_2 \sin(2pt+\varphi_2) + \ldots\ldots$$
$$+ m_r \sin(rpt+\varphi_r) + \ldots\ldots + m_n \sin(npt+\varphi_n) \quad \ldots(36),$$

where the m's are constants, p the angular velocity of the fundamental wave and the φ's the phase displacements.

In this case we have

$$\int \varepsilon^{\frac{t}{T_\alpha}} m\,dt = T_\alpha \varepsilon^{\frac{t}{T_\alpha}} \left\{ m_0 + \sum_{r=1}^{r=n} m_r \cos\psi_r \sin(rpt+\delta_r) \right\},$$

where

$$\left.\begin{aligned} \psi_r &= \tan^{-1} rpT_\alpha, \\ \delta_r &= \varphi_r - \psi_r. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

Therefore since

$$M = \varepsilon^{-\frac{t}{T_\alpha}} \left(\frac{1}{T_\alpha} \int \varepsilon^{\frac{t}{T_\alpha}} m\,dt + C \right),$$

we obtain

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1) *Loc. cit.*

$$M = m_0 + \sum_{r=1}^{r=n} m_r \cos\phi_r \sin(rpt+\delta_r) + C\varepsilon^{-\frac{t}{T_\alpha}},$$

where C is a constant.

In this case the torques of the motor when $pt=\pi$ and when $pt=-\pi$ are equal. Therefore we get

$$C = 0.$$

Hence

$$M = m_0 + \sum_{r=1}^{r=n} m_r \cos\phi_r \sin(rpt+\delta_r) \quad \text{..............(38)},$$

and the torques of the motor when $pt = 0$ and $pt = \pi$ or $-\pi$ are respectively

$$\left.\begin{aligned} M_0 &= m_0 + \sum_{r=1}^{r=n} m_r \cos\phi_r \sin\delta_r, \\ M_\pi &= M_{-\pi} \\ &= m_0 + \sum_{r=1}^{r=n} (-1)^r m_r \cos\phi_r \sin\delta_r. \end{aligned}\right\} \quad \text{.........(39).}$$

Thus we see that in the permanent state, the torque of the induction motor varies harmonically with the same frequency as that of the resisting torque.

The amplitude of each harmonic wave of the torque of the motor can be deduced from (38), i.e. its value is

$$m_r \cos\phi_r = \frac{m_r}{\sqrt{1+(rpT_\alpha)^2}} \quad \text{.................... (40).}$$

It shows that the inertia of the rotor diminishes the amplitude of each wave and the effect is more marked for the higher harmonics than for the lower. Thus we see that though the resisting torque varies much from the fundamental wave, the effect of the fly-wheel is to make the motor develope a torque of smoother form. Hence when the time constant T_α is sufficiently large, the torque of the motor, working under a resisting torque of more or less irregular form, may be taken as a simple harmonic wave of lesser amplitude.

The phase difference between each harmonic wave of the resisting torque and the torque of the motor ϕ_r is given in (37). It shows that, by virtue of the inertia effect, the phase of each harmonic wave of the torque of the motor lags behind that of the resisting torque and, the larger the time constant T_α, the more the effect becomes marked. This effect has more influence on higher harmonics than on lower.

Next the angular velocity and the slip at any time are

$$\left.\begin{aligned}
\omega &= \omega_s - \frac{M}{A} \\
&= \omega_s - \frac{1}{A}\left\{m_0 + \sum_{r=1}^{r=n} m_r \cos\phi_r \sin(rpt+\delta_r)\right\}, \\
s &= \frac{M}{A\omega_s} \\
&= \frac{1}{A\omega_s}\left\{m_0 + \sum_{r=1}^{r=n} m_r \cos\phi_r \sin(rpt+\delta_r)\right\}.
\end{aligned}\right\} \quad \ldots\ldots\ldots(41).$$

Therefore the values of ω and s when $t=0$, and $pt=\pi$ or $pt=-\pi$ become

$$\omega_0' = \omega_s - \frac{1}{A}\left(m_0 + \sum_{r=1}^{r=n} m_r \cos\phi_r \sin\delta_r\right),$$

$$s_0' = \frac{1}{A\omega_s}\left(m_0 + \sum_{r=1}^{r=n} m_r \cos\phi_r \sin\delta_r\right),$$

$$\begin{aligned}
\omega_\pi &= \omega_{-\pi} \\
&= \omega_s - \frac{1}{A}\left\{m_0 + \sum_{r=1}^{r=n} (-1)^r m_r \cos\phi_r \sin\delta_r\right\},
\end{aligned}$$

$$s_\pi = s_{-\pi}$$

$$= \frac{1}{A\omega_s}\left\{m_0 + \sum_{r=1}^{r=n}(-1)^r m_r \cos\psi_r \sin\delta_r\right\}.$$

Now since

$$R = \int_0^t m\,dt$$

$$= m_0 t + T\alpha \sum_{r=1}^{r=n} m_r \cot\psi_r \left\{\cos\varphi_r - \cos(rpt + \varphi_r)\right\}.$$

the angular displacement of the motor from $t = 0$ up to any time t is from (13)

$$\theta = \omega_s t - \frac{1}{A}\left\{R + T\alpha (M_0 - M)\right\}$$

$$= \left(\omega_s - \frac{m_0}{A}\right)t - \frac{T\alpha}{A}\sum_{r=1}^{r=n} m_r \cot\psi_r \cos\psi_r \{\cos\delta_r - \cos(rpt + \delta_r)\}$$

$$= \omega_0 t - \frac{T\alpha}{A}\sum_{r=1}^{r=n} m_r \cot\psi_r \cos\psi_r \{\cos\delta_r - \cos(rpt + \delta_r)\} \qquad \text{(42)},$$

where ω_0 is the angular velocity of the motor corresponding to the torque m_0.

From above the total angular displacement of the motor during the complete period is

$$\theta_0 = \left(\omega_s - \frac{m_0}{A}\right)\frac{2\pi}{p}$$

$$= 2\pi \frac{\omega_0}{p} \qquad \text{(43)}.$$

In the above case, if we put

$$m_2 = m_3 = \dots\dots = m_n = 0,$$

and

$$\varphi_1 = 0,$$

(36) becomes

$$m = m_0 + m_1 \sin pt \quad \text{.......................(44)},$$

i.e. the resisting torque makes a simple harmonic wave.

In this case the above expressions become

$$\left.\begin{aligned} M &= m_0 + m_1 \cos\psi \sin(pt-\psi), \\ M_0 &= m_0 - \tfrac{1}{2} m_1 \sin 2\psi, \\ M_\pi &= m_0 + \tfrac{1}{2} m_1 \sin 2\psi. \end{aligned}\right\} \text{.........(45)},$$

and

$$\left.\begin{aligned} \omega &= \omega_s - \frac{1}{A}\{m_0 + m_1 \cos\psi \sin(pt-\psi)\}, \\ s &= \frac{1}{A\omega_s}\{m_0 + m_1 \cos\psi \sin(pt-\psi)\}, \\ \theta &= \left(\omega_s - \frac{m_0}{A}\right)t - \frac{T_\alpha}{A} m_1 \cot\psi \cos\psi \{\cos\psi - \cos(pt-\psi)\}, \\ \theta_0 &= \left(\omega_s - \frac{m_0}{A}\right)\frac{2\pi}{p} \\ &= 2\pi\frac{\omega_0}{p}, \\ \text{where} & \\ \psi &= \tan^{-1} pT_\alpha. \end{aligned}\right\} \text{...(46)}.$$

Thus we see that the amplitude of the fluctuation of the torque of the induction motor becomes smaller than that of the resisting torque in the ratio

$$\cos\psi = \frac{1}{\sqrt{1+(pT_\alpha)^2}}$$

and it lags in phase by the amount ψ.

The power of the motor at any time is

$$P = \omega M$$

$$= m_0\left(\omega_s - \frac{m_0}{A}\right) + m_1 \cos\psi\left(\omega_s - \frac{2m_0}{A}\right)\sin(pt-\psi)$$

$$-\frac{1}{A}(m_1\cos\psi)^2\sin^2(pt-\psi) \quad\text{..................(47).}$$

The average power of the motor becomes

$$P_0 = m_0\left(\omega_s - \frac{m_0}{A}\right) - \frac{1}{2A}(m_1\cos\psi)^2 \quad\text{...........(48).}$$

Now the condition when the power of the motor is a maximum or a minimum is

$$\frac{dP}{dt} = 0,$$

i. e.

$$\left(\omega_s - \frac{2m_0}{A}\right)\cos(pt-\psi) - \frac{1}{A}m_1\cos\psi\sin2(pt-\psi) = 0.$$

But since in practice the greatest slip of an induction motor is generally smaller than 50 % we have

$$\omega_s - \frac{2m_0}{A} > 0,$$

and

$$\frac{A\left(\omega_s - \frac{2m_0}{A}\right)}{2m_1\cos\psi} > 1.$$

Therefore from above expression we can easily see that

$$\cos(pt-\psi) = 0$$

gives the condition of the maximum or the minimum power and that

P is the maximum when $pt-\psi = \frac{\pi}{2}$,

i. e. when M is the maximum,

and

$$P \text{ is the minimum when } pt-\psi = \frac{3\pi}{2},$$

i. e. when M is the minimum.

Hence we get

$$\left.\begin{aligned} P_{\text{max.}} &= (m_0+m_1\cos\psi)\left(\omega_s - \frac{m_0+m_1\cos\psi}{A}\right), \\ P_{\text{min.}} &= (m_0-m_1\cos\psi)\left(\omega_s - \frac{m_0-m_1\cos\psi}{A}\right). \end{aligned}\right\} \cdots\cdots\cdots(49),$$

and the fluctuation of power of the motor is

$$\frac{P_{\text{max.}}-P_{\text{min.}}}{P_0} = \frac{2m_1\cos\psi\left(\omega_s - \dfrac{2m_0}{A}\right)}{m_0\left(\omega_s-\dfrac{m_0}{A}\right)-\dfrac{1}{2A}(m_1\cos\psi)^2} \cdots\cdots(50).$$

Example. An induction motor drives a machine whose resisting torque varies in sinusoidal manner with respect to time making two cycles per revolution of the motor with

$$m_0 = 500\ MKg.,$$

$$m_1 = 400\ MKg..$$

Assuming the synchronous velocity of the induction motor 450 $r.p.m.$, it is required to calculate the necessary amount of inertia and the power of the motor to limit the maximum fluctuation of the torque of the motor to 400 $MKg.$

In this case

$$\omega_s = \frac{2\pi\,.\,450}{60} = 47{\cdot}124,$$

$$A = \frac{M_{\text{max.}}}{\omega_s - \omega_{\text{max.}}}$$

$$= \frac{M_{\text{max.}}}{s_{\text{max.}}\,\omega_s},$$

where $\omega_{\text{max.}}$ and $s_{\text{max.}}$ are the angular velocity and the slip of the motor corresponding to the maximum torque $M_{\text{max.}}$ respectively.

We take

$$s_{\text{max.}} = 10\%,$$

the value within the common practice.

Hence

$$A = \frac{700\,g}{0{\cdot}1\,.\,47{\cdot}124} = 148{\cdot}54\,g,$$

$$\omega_0 = \omega_s - \frac{m_0 g}{A}$$

$$= 47{\cdot}124 - \frac{500}{148{\cdot}54}$$

$$= 43{\cdot}758.$$

From (46)

$$\theta_0 = 2\pi\frac{\omega_0}{p} = \pi$$

$\therefore$

$$p = 2\omega_0$$

$$= 87{\cdot}516.$$

From (45)

$$M_{\text{max.}} = m_0 + \frac{m_1}{\sqrt{1+(pT_\alpha)^2}},$$

$\therefore$

$$700 = 500 + \frac{400}{\sqrt{1+(pT_\alpha)^2}}.$$

Hence

$$pT_\alpha = \sqrt{3}.$$

$\therefore$

$$T_\alpha = \frac{\sqrt{3}}{87{\cdot}516} = 0{\cdot}0198 \text{ Sec}$$

But

$$T_\alpha = \frac{I}{A}.$$

$\therefore$

$$I = 0{\cdot}0198 \,.\, 148{\cdot}54 \,.\, 9{\cdot}8$$

$$= 28{\cdot}8 \quad M.\,Kg.\,S.$$

Now

$$\tan\psi = pT_\alpha = \sqrt{3}\,.$$

$\therefore$

$$\psi = \frac{\pi}{3}.$$

Therefore (45) gives

$$M = 500 + 200 \sin\left(87{\cdot}516\, t - \frac{\pi}{3}\right).$$

The maximum and the minimum power of the motor are from (49)

$$P_{\text{max.}} = (500+200)\left(47{\cdot}124 - \frac{500+200}{148{\cdot}54}\right)$$

$$= 29688 \;\; Kg\,M.\text{ per Sec.}$$

$$= 395{\cdot}8 \text{ H.P.},$$

$$P_{\text{min.}} = (500-200)\left(47{\cdot}124 - \frac{500-200}{148{\cdot}54}\right)$$

$$= 13530 \;\; Kg\,M.\text{ per Sec.}$$

$$= 180{\cdot}4 \text{ H.P.}$$

II. The Rating of the Induction Motor.

8. In what follows the problem of the heating of induction motors running under cyclically varying loads will be considered. In the usual method of treating the problem of heating an electric motor, the assump-

tion is made that the motor is a homogeneous body, and that a certain amount of heat per unit time is generated in it uniformly throughout the body. This assumption is of course not correct except in the case of totally enclosed motors, in which the assumption approximately holds. The present part of the paper also starts from this approximate hypothesis.

With the above assumption let

W be the amount of heat developed per unit time in the motor,

u be the temperature of the motor, assumed uniform, at any time t, measured above that of the room,

a be the heat stored in the body per degree rise in temperature of the motor, it being assumed constant,

b be the amount of heat dissipated from the body per unit time per degree rise in temperature of the motor.

Then we have

$$a\frac{du}{dt}+bu=W \quad \text{..............................(1).}$$

This is the well known equation employed by various authors for treating the problem of heating an electric motor. But it must be noticed that W in eq. (1) is not in our case, as in the case of previous investigators, a constant value, but it is a known function of time characterized from the nature of the service of the motor. Even under the above general assumption, b in (1) is not a constant unless the motor runs with a constant speed or we use forced ventilation. It varies with the change of the speed of the motor which in turn depends on the variation of the load of the motor. Therefore it is a function of time. In this general case, where W and b are any function of time, the above equation can be solved, provided they are known functions of time. But in practice, even when the motor works under very unstable resisting torque, the variation of the speed is limited within a certain not large amount so that its effect on the value of b is not greatly appreciable. From the above consideration, together with the ambiguity involved in the fundamental equation (1), we may take, when the variation of load on the motor is moderate, b as a constant.

Assuming now a and b constants and W a given function of time t the equation (1) is at once solved in the form

$$u = \varepsilon^{-\frac{t}{T_\beta}}\left(u_0 + \frac{1}{a}\int_0^t \varepsilon^{\frac{t}{T_\beta}} W dt\right) \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where u_0 is the temperature of the motor at the beginning of the cycle and T_β stands for $\frac{a}{b}$.

T_β is a constant known as the time constant of the motor. It is a constant so far as the speed of the motor and therefore the load on it is constant or is ventilated with constant air velocity.

To determine the initial temperature u_0, let the period of the cycle be τ_0. Then putting

$$u = u_0 \quad \text{and} \quad t = \tau_0$$

in the above equation we have

$$u_0 = \frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1} \cdot \frac{1}{a}\int_0^{\tau_0} \varepsilon^{\frac{t}{T_\beta}} W\, dt \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

9. When the integrals of the above expressions are difficult or the speed of the motor varies much, so that T_β can not be taken as constant, we divide the period of the cycle into a suitable number of divisions and treat as in the preceeding chapter, during each division T_β being assumed constant.

Assume, as in the preceeding chapter, that the complete period of the cycle is divided into n divisions and let the time of each division be

$$t_1,\ t_2,\ t_3,\ \ldots\ldots t_r,\ \ldots\ldots t_n$$

respectively, which is related by

$$t_1 + t_2 + t_3 + \ldots\ldots + t_r + \ldots\ldots + t_n = \tau_0.$$

Let the values of W and T_β during each division be represented by

$$W_1,\ W_2,\ W_3,\ \ldots\ldots W_r,\ \ldots\ldots W_n$$

and

$$T_{\beta^1},\ T_{\beta^2},\ T_{\beta^3},\ \ldots\ldots T_{\beta^r},\ \ldots\ldots T_{\beta^n}$$

respectively, where the W's are known functions of time and the T_β's are constants.

Further, let the values of u at the beginning and the end of each division be

$$u_0,\ u_1,\ u_2,\ \ldots\ldots u_{r-1},\ \ldots\ldots u_{n-1}$$

and

$$u_1,\ u_2,\ u_3,\ \ldots\ldots u_r,\ \ldots\ldots u_n$$

respectively.

Then applying (2) to each division from the beginning to the end of the time we get

$$u_1\,\varepsilon^{\frac{t_1}{T_{\beta^1}}} = u_0 + \frac{1}{a}\int_0^{t_1} \varepsilon^{\frac{t}{T_{\beta^1}}}\, W_1\, dt,$$

$$u_2\,\varepsilon^{\frac{t_2}{T_{\beta^2}}} = u_1 + \frac{1}{a}\int_0^{t_2} \varepsilon^{\frac{t}{T_{\beta^2}}}\, W_2\, dt,$$

$$u_3\,\varepsilon^{\frac{t_3}{T_{\beta^3}}} = u_2 + \frac{1}{a}\int_0^{t_3} \varepsilon^{\frac{t}{T_{\beta^3}}}\, W_3\, dt,$$

....

$$u_r\,\varepsilon^{\frac{t_r}{T_{\beta^r}}} = u_{r-1} + \frac{1}{a}\int_0^{t_r} \varepsilon^{\frac{t}{T_{\beta^r}}}\, W_r\, dt,$$

....

$$u_n\,\varepsilon^{\frac{t_n}{T_{\beta^n}}} = u_{n-1} + \frac{1}{a}\int_0^{t_n} \varepsilon^{\frac{t}{T_{\beta^n}}}\, W_n\, dt,$$

where the time t is counted from the beginning of each division.

Now eliminating u_1, u_2, u_3, etc., u_{n-1} from the above equations we get

$$u_n \varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} = u_0 + \frac{1}{a} \sum_{r=1}^{r=n} \varepsilon^{\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}} \int_0^{t_r} \varepsilon^{\frac{t}{T_{\beta r}}} W_r \, dt.$$

But since for cyclical operation

$$u_n = u_0,$$

we have

$$u_0 = \frac{1}{\varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} - 1} \; \frac{1}{a} \sum_{r=1}^{r=n} \varepsilon^{\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}} \int_0^{t_r} \varepsilon^{\frac{t}{T_{\beta r}}} W_r \, dt \text{......(4)}.$$

In a similar way the temperature at the beginning of rth division is obtained by

$$u_{r-1} = \varepsilon^{-\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}} \left(u_0 + \frac{1}{a} \sum_{p=1}^{p=r-1} \varepsilon^{\sum_{q=1}^{q=p-1} \frac{t_q}{T_{\beta q}}} \int_0^{t_p} \varepsilon^{\frac{t}{T_{\beta p}}} W_p dt \right) \text{.........(5)}.$$

Hence from (2) the temperature of the motor at any time t in any division rth, say, will be given by

$$u = \varepsilon^{-\frac{t}{T_{\beta r}}} \left(u_{r-1} + \frac{1}{a} \int_0^t \varepsilon^{\frac{t}{T_{\beta r}}} W_r \, dt \right) \text{..................(6)}.$$

Now, for the sake of simplicity let us write

$$\left. \begin{aligned} N_0 &= \frac{1}{a} \sum_{p=1}^{p=n} \varepsilon^{\sum_{q=1}^{q=p-1} \frac{t_q}{T_{\beta q}}} \int_0^{t_p} \varepsilon^{\frac{t}{T_{\beta p}}} W_p \, dt, \\ N_{r-1} &= \frac{1}{a} \sum_{p=1}^{p=r-1} \varepsilon^{\sum_{q=1}^{q=p-1} \frac{t_q}{T_{\beta q}}} \int_0^{t_p} \varepsilon^{\frac{t}{T_{\beta p}}} W_p \, dt, \end{aligned} \right\} \text{.........(7)}.$$

$$N = N_{r-1} + \frac{1}{a} \varepsilon^{\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}} \int_0^t \varepsilon^{\frac{t}{T_{\beta r}}} W_r \, dt.$$

Then the above expressions become

$$\left.\begin{aligned} u_0 &= \frac{1}{\varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} - 1} N_0, \\ u_{r-1} &= \varepsilon^{-\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}} (u_0 + N_{r-1}), \\ u &= \varepsilon^{-\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}} - \frac{t}{T_{\beta r}}} (u_0 + N). \end{aligned}\right\} \quad \ldots\ldots\ldots(8).$$

If the time constant T_β's may be assumed equal during the complete period, the above expressions become

$$\left.\begin{aligned} N_0 &= \frac{1}{a} \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}} \int_0^{t_p} \varepsilon^{\frac{t}{T_\beta}} W_p \, dt, \\ N_{r-1} &= \frac{1}{a} \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}} \int_0^{t_p} \varepsilon^{\frac{t}{T_\beta}} W_p \, dt, \\ N &= N_{r-1} + \frac{1}{a} \varepsilon^{\frac{\tau_{r-1}}{T_\beta}} \int_0^t \varepsilon^{\frac{t}{T_\beta}} W_r \, dt. \end{aligned}\right\} \quad \ldots\ldots\ldots(9),$$

and

$$\left.\begin{aligned} u_0 &= \frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1} N_0, \\ u_{r-1} &= \varepsilon^{-\frac{\tau_{r-1}}{T_\beta}} (u_0 + N_{r-1}), \\ u &= \varepsilon^{-\frac{\tau}{T_\beta}} (u_0 + N). \end{aligned}\right\} \quad \ldots\ldots\ldots(10).$$

10. When the motor runs continuously with a constant load, the

internal losses are constant. Therefore in this case putting W constant in (2) we get

$$u = \frac{W}{b} - \left(\frac{W}{b} - u_0\right) \varepsilon^{-\frac{t}{T_\beta}}.$$

Let now $u = U$ when $t = \infty$,
then we have

$$\left. \begin{aligned} U &= \frac{W}{b}, \\ u &= U - (U - u_0)\, \varepsilon^{-\frac{t}{T_\beta}}. \end{aligned} \right\} \quad \text{.........(11).}$$

If $u_0 = 0$, i.e. the temperature of the motor be initially equal to that of the room

$$u = U(1 - \varepsilon^{-\frac{t}{T_\beta}}) \quad \text{................................(12).}$$

This is the well known formula used for treating the problem of heating a motor. Here U is the maximum temperature or the temperature of the continuous running when the internal losses are equal to W.

11. All formulæ deduced in the preceeding paragraphs hold true for any electric motor. Now let us apply the above propositions to the case of induction motors running under cyclical loads.

Heat generated in the motor per unit time W is due to its internal losses. When the induction motor with constant secondary resistance runs under constant voltage and frequency, W can be very approximately expressed by

$$W = W_0 + BM^2 \quad \text{.............................(13),}$$

where W_0 is the no load losses of the motor excluding friction loss and B is a constant standing for

$$\frac{1}{A}\,\frac{r_1 + r_2}{r_2},$$

in which

r_1 is the resistance of the primary winding per phase,

r_2 is the resistance of the secondary winding per phase, reduced to the primary.

The above equation shows that the internal losses of an induction motor with secondary winding of constant resistance working under constant impressed E. M. F. and frequency varies with the second power of the torque.

12. Now let us take the case where the induction motor works under cyclically varying loads, during the complete period the motor being never stopped.

Substituting (13) into (3) and (2) we get

$$\left.\begin{aligned} u_0 &= \frac{W_0}{b} + \frac{B}{a}\,\frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1}\int_0^{\tau_0} \varepsilon^{\frac{t}{T_\beta}} M^2\,dt, \\ u &= \frac{W_0}{b} + \frac{B}{a}\,\varepsilon^{-\frac{t}{T_\beta}}\left\{\frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1}\int_0^{\tau_0} \varepsilon^{\frac{t}{T_\beta}} M^2\,dt + \int_0^{t} \varepsilon^{\frac{t}{T_\beta}} M^2\,dt\right\}. \end{aligned}\right\} \quad (14).$$

Thus we see that in this case the variation of temperature of the motor during the cycle depends upon the variation of the torque of the motor and the greatest rise in temperature of the motor occurs when

$$\varepsilon^{-\frac{t}{T_\beta}}\left\{\frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1}\int_0^{\tau_0} \varepsilon^{\frac{t}{T_\beta}} M^2\,dt + \int_0^{t} \varepsilon^{\frac{t}{T_\beta}} M^2\,dt\right\}$$

is a maximum. The above expression does not depend upon the construction of the motor, except the time constant, and therefore the condition of the greatest rise in temperature is found from the manner of variation of M, provided the time constant T_β is known. But since M is a known function of time depending upon the nature of service the condition can be met. Let is occur when $t = t_{max.}$. Then the greatest rise in temperature becomes

$$u_{max.} = \frac{W_0}{b} + \frac{B}{b} \varepsilon^{-\frac{t_{max.}}{T_\beta}} \left\{ \frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1} \int_0^{\tau_0} \varepsilon^{\frac{t}{T_\beta}} M^2 \, dt + \int_0^{t_{max.}} \varepsilon^{\frac{t}{T_\beta}} M^2 \, dt \right\} \text{...(15).}$$

Now from (11) the temperature of a motor running continuously under a constant load is

$$U = \frac{W}{b},$$

or substituting (13), in the case of an induction motor we have

$$U = \frac{W_0}{b} + \frac{B}{b} M^2 \quad \text{.........................(16).}$$

Let

$$U = u_{max.} \quad \text{when} \quad M = M_{mean},$$

then we get

$$u_{max.} = \frac{W_0}{b} + \frac{B}{b} M_{mean}^2 \quad \text{......................(17).}$$

Now equating (15) and (17) we get

$$M_{mean}^2 = \frac{1}{T_\beta} \varepsilon^{-\frac{t_{max.}}{T_\beta}} \left\{ \frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1} \int_0^{\tau_0} \varepsilon^{\frac{t}{T_\beta}} M^2 \, dt + \int_0^{t_{max.}} \varepsilon^{\frac{t}{T_\beta}} M^2 \, dt \right\},$$

or

$$M_{mean} = \sqrt{\frac{1}{T_\beta} \varepsilon^{-\frac{t_{max.}}{T_\beta}} \left\{ \frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1} \int_0^{\tau_0} \varepsilon^{\frac{t}{T_\beta}} M^2 \, dt + \int_0^{t_{max.}} \varepsilon^{\frac{t}{T_\beta}} M^2 \, dt \right\}}. \quad \text{(I).}$$

M_{mean} in the above equation means the torque which, if applied continuously on the same induction motor, would give the same greatest rise in temperature as it does when running under the given cyclically varying load. Hence for this class of service an induction motor is to be selected which would develope M_{mean} continuously with the rise in temperature not exceeding a certain presupposed amount.

For calculating formulæ given in this paragraph the graphic method similar to that given in § 6 will be convenient.

In the above if the period of the cycle τ_0 is very small compared to the time constant T_β, (I) reduces to

$$M_{\text{mean}} = \sqrt{\frac{1}{\tau_0}\int_0^{\tau_0} M^2\, dt} \quad \text{........................(18).}$$

In this case the M_{mean} can be determined independent of T_β.

13. When the torque cycle is divided into n divisions, in the case of an induction motor if we put

$$\left.\begin{aligned}
K_0 &= \sum_{p=1}^{p=n} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1} \frac{t_q}{T_{\beta q}}} \left(\varepsilon^{\frac{t_p}{T_{\beta p}}} - 1\right), \\
K_{r-1} &= \sum_{p=1}^{p=r-1} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1} \frac{t_q}{T_{\beta q}}} \left(\varepsilon^{\frac{t_p}{T_{\beta p}}} - 1\right), \\
K &= K_{r-1} + T_{\beta r}\, \varepsilon^{\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}} \left(\varepsilon^{\frac{t}{T_{\beta r}}} - 1\right), \\
&\text{and} \\
L_0 &= \sum_{p=1}^{p=n} \varepsilon^{\sum_{q=1}^{q=p-1} \frac{t_q}{T_{\beta q}}} \int_0^{t_p} \varepsilon^{\frac{t}{T_{\beta p}}} M^2\, dt, \\
L_{r-1} &= \sum_{p=1}^{p=r-1} \varepsilon^{\sum_{q=1}^{q=p-1} \frac{t_q}{T_{\beta q}}} \int_0^{t_p} \varepsilon^{\frac{t}{T_{\beta p}}} M^2\, dt, \\
L &= L_{r-1} + \varepsilon^{\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}} \int_0^{t} \varepsilon^{\frac{t}{T_{\beta r}}} M^2\, dt.
\end{aligned}\right\} \text{.........(19),}$$

(7) gives

$$N_0 = \frac{W_0}{a} K_0 + \frac{B}{a} L_0,$$

$$N_{r-1} = \frac{W_0}{a} K_{r-1} + \frac{B}{a} L_{r-1},$$

$$N = \frac{W_0}{a} K + \frac{B}{a} L.$$

Therefore from (8)

$$\left.\begin{aligned}
u_0 &= \frac{W_0}{a} \frac{K_0}{\varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} - 1} + \frac{B}{a} \frac{L_0}{\varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} - 1}, \\
u_{r-1} &= \frac{W_0}{a}\left(\frac{K_0}{\varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} - 1} + K_{r-1}\right) \varepsilon^{-\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}} \\
&\quad + \frac{B}{a}\left(\frac{L_0}{\varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} - 1} + L_{r-1}\right) \varepsilon^{-\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}}}, \\
u &= \frac{W_0}{a}\left(\frac{K_0}{\varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} - 1} + K\right) \varepsilon^{-\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}} - \frac{t}{T_{\beta r}}} \\
&\quad + \frac{B}{a}\left(\frac{L_0}{\varepsilon^{\sum_{r=1}^{r=n} \frac{t_r}{T_{\beta r}}} - 1} + L\right) \varepsilon^{-\sum_{p=1}^{p=r-1} \frac{t_p}{T_{\beta p}} - \frac{t}{T_{\beta r}}}.
\end{aligned}\right\} \quad \ldots\ldots(20).$$

Now in (16) let

$$U = U_x \quad \text{when} \quad M = M_x.$$

Then

$$U_x = \frac{W_0}{b_x} + \frac{B}{b_x} M_x^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

where b_x is the value of b corresponding to the torque M_x.

Equating u in (20) and U_x in (21) we get

$$M_x^2 = \frac{W_0}{B}\left\{\frac{1}{T_{\beta x}}\left(\frac{K_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1}+K\right)\varepsilon^{-\sum_{p=1}^{p=r-1}{}'\frac{t_p}{T_{\beta p}}-\frac{t}{T_{\beta r}}}-1\right\}$$

$$+\frac{1}{T_{\beta x}}\left(\frac{L_0}{\varepsilon^{\sum_{r=1}^{r=n}{}'\frac{t_r}{T_{\beta r}}}-1}+L\right)\varepsilon^{-\sum_{p=1}^{p=r-1}{}'\frac{t_p}{T_{\beta p}}-\frac{t}{T_{\beta r}}},$$

where

$$T_{\beta x} = \frac{a}{b_x}.$$

Now put

$$W_0 = \lambda B M_x^2,$$

where λ is a constant.

Then the above expression becomes

$$M_x^2 = \frac{\left(\dfrac{L_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1}+L\right)\varepsilon^{-\sum_{p=1}^{p=r-1}{}'\frac{t_p}{T_{\beta p}}-\frac{t}{T_{\beta r}}}}{(1+\lambda)T_{\beta x}-\lambda\left(\dfrac{K_0}{\varepsilon^{\sum_{r=1}^{r=n}{}'\frac{t_r}{T_{\beta r}}}-1}+K\right)\varepsilon^{-\sum_{p=1}^{p=r-1}{}'\frac{t_p}{T_{\beta p}}-\frac{t}{T_{\beta r}}}}.$$

λ in the above expression is a constant depending upon the design of the motor. It shows that the no load losses of the motor are λ times the variable loss corresponding to M_x.

Now assume the value of λ corresponding to the torque M_{mean} and obtain the greatest value of the above expression. Then it will give the M_{mean}^2 looked for. Let $t_{max.}$ be the time in that case, then we get

$$M^2_{\text{mean}} = \frac{\left(\dfrac{L_0}{\varepsilon^{\sum\limits_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1}+L_{\text{max.}}\right)\varepsilon^{-\sum\limits_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\text{max.}}}{T_{\beta r}}}}{(1+\lambda)T_{\beta_{\text{mean}}}-\lambda\left(\dfrac{K_0}{\varepsilon^{\sum\limits_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1}+K_{\text{max.}}\right)\varepsilon^{-\sum\limits_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\text{max.}}}{T_{\beta r}}}} \quad \text{(II)},$$

where $K_{\text{max.}}$ and $L_{\text{max.}}$ are respectively the values of K and L corresponding to $t = t_{\text{max.}}$, and $T_{\beta_{\text{mean}}}$ is the time constant corresponding to M_{mean}.

In the above expression if we put $\lambda = 1$, i. e. if the motor is designed so that the no load losses are equal to the variable loss when the torque is M_{mean} we have

$$M^2_{\text{mean}} = \frac{\left(\dfrac{L_0}{\varepsilon^{\sum\limits_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1}+L_{\text{max.}}\right)\varepsilon^{-\sum\limits_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\text{max.}}}{T_{\beta r}}}}{2T_{\beta_{\text{mean}}}-\left(\dfrac{K_0}{\varepsilon^{\sum\limits_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1}+K_{\text{max.}}\right)\varepsilon^{-\sum\limits_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\text{max.}}}{T_{\beta r}}}} \quad \text{(III)}.$$

If it is permissible to put all the T_β's equal we get

$$\left.\begin{aligned} K_0 &= T_\beta\left(\varepsilon^{\frac{\tau_0}{T_\beta}}-1\right), \\ K_{r-1} &= T_\beta\left(\varepsilon^{\frac{\tau_{r-1}}{T_\beta}}-1\right), \\ K &= T_\beta\left(\varepsilon^{\frac{\tau}{T_\beta}}-1\right), \end{aligned}\right|$$

and

$$\left.\begin{aligned} L_0 &= \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}} \int_0^{t_p} \varepsilon^{\frac{t}{T_\beta}} M^2\, dt, \\ L_{r-1} &= \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}} \int_0^{t_p} \varepsilon^{\frac{t}{T_\beta}} M^2\, dt, \\ L &= L_{r-1} + \varepsilon^{\frac{\tau_{r-1}}{T_\beta}} \int_0^{t} \varepsilon^{\frac{t}{T_\beta}} M^2\, dt. \end{aligned}\right\} \quad \text{.........(22)},$$

and

$$\left.\begin{aligned} u_0 &= \frac{W_0}{b} + \frac{B}{a}\,\frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1} L_0, \\ u_{r-1} &= \frac{W_0}{b} + \frac{B}{a}\left(\frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1} L_0 + L_{r-1}\right)\varepsilon^{-\frac{\tau_{r-1}}{T_\beta}}, \\ u &= \frac{W_0}{b} + \frac{B}{a}\left(\frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1} L_0 + L\right)\varepsilon^{-\frac{\tau}{T_\beta}}. \end{aligned}\right\} \quad \text{.........(23)}.$$

In this case $u_{\text{max.}}$ can be readily found. Assume that it occurs when $\tau = \tau_{\text{max.}}$, then

$$u_{\text{max.}} = \frac{W_0}{b} + \frac{B}{a}\left(\frac{1}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1} L_0 + L_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}} \quad \text{.........(24)},$$

where $L_{\text{max.}}$ is the value of L when $\tau = \tau_{\text{max.}}$.

Again equating (17) and (24) we obtain

$$M^2_{\text{mean}} = \frac{1}{T_\beta}\left(\frac{L_0}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1} + L_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}}$$

i. e.

$$M_{\text{mean}} = \sqrt{\frac{1}{T_\beta}\left(\frac{L_0}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1}+L_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}}} \quad \ldots\ldots\ldots(\text{IV}).$$

The induction motor working under the given cyclical load should be rated for continuous load by the formula given above.

14. We will now apply the above solutions to some special cases.

Case 1. When the induction motor having a rotor attached with a fly-wheel works under a cyclically varying load in which the resisting torque varies with constant values successively and without stopping as given in Case 2 of § 7. In this case, using the same notation as in the preceeding, the torque of the motor at any instant is given by

$$M = m_p-(m_p-M_{p-1})\,\varepsilon^{-\frac{t}{T_\alpha}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

Since in this case we may without much error assume that the T_β's are all equal, substituting (25) into (22) we have

$$\left.\begin{aligned}
L_0 = T_\beta\sum_{p=1}^{p=n}\Bigg[\varepsilon^{\frac{\tau_p}{T_\beta}}\left\{m_p^2+\frac{2}{\frac{T_\beta}{T_\alpha}-1}m_p(m_p-M_p)-\frac{1}{2\frac{T_\beta}{T_\alpha}-1}(m_p-M_p)^2\right\}\\
-\varepsilon^{\frac{\tau_{p-1}}{T_\beta}}\left\{m_p^2+\frac{2}{\frac{T_\beta}{T_\alpha}-1}m_p(m_p-M_{p-1})-\frac{1}{2\frac{T_\beta}{T_\alpha}-1}(m_p-M_{p-1})\right\}\Bigg],\\
L_{r-1}= T_\beta\sum_{p=1}^{p=r-1}\Bigg[\varepsilon^{\frac{\tau_p}{T_\beta}}\left\{m_p^2+\frac{2}{\frac{T_\beta}{T_\alpha}-1}m_p(m_p-M_p)-\frac{1}{2\frac{T_\beta}{T_\alpha}-1}(m_p-M_p)^2\right\}\\
-\varepsilon^{\frac{\tau_{p-1}}{T_\beta}}\left\{m_p^2+\frac{2}{\frac{T_\beta}{T_\alpha}-1}m_p(m_p-M_{p-1})-\frac{1}{2\frac{T_\beta}{T_\alpha}-1}(m_p-M_{p-1})\right\}\Bigg],
\end{aligned}\right\}\ (26).$$

$$L = L_{r-1} + T_\beta\left[\varepsilon^{\frac{\tau}{T_\beta}}\left\{m_r^2 + \frac{2}{\frac{T_\beta}{T_\alpha}-1}m_r(m_r - M) - \frac{1}{2\frac{T_\beta}{T_\alpha}-1}(m_r - M)^2\right\}\right.$$
$$\left. - \varepsilon^{\frac{\tau_{r-1}}{T_\beta}}\left\{m_r^2 + \frac{2}{\frac{T_\beta}{T_\alpha}-1}m_r(m_r - M_{r-1}) - \frac{1}{2\frac{T_\beta}{T_\alpha}-1}(m_r - M_{r-1})^2\right\}\right].$$

Let $u_{max.}$ occur when $\tau = \tau_{max.}$ and designate

$$\left.\begin{aligned}
L_0 &= T_\beta L_0', \\
L_{r-1} &= T_\beta L'_{r-1}, \\
L'_{max.} &= L'_{r-1} + \varepsilon^{\frac{\tau_{max.}}{T_\beta}}\left\{m_r^2 + \frac{2}{\frac{T_\beta}{T_\alpha}-1}m_r(m_r - M_{max.}) - \frac{1}{2\frac{T_\beta}{T_\alpha}-1}(m_r - M_{max.})^2\right\} \\
&\quad - \varepsilon^{\frac{\tau_{r-1}}{T_\beta}}\left\{m_r^2 + \frac{2}{\frac{T_\beta}{T_\alpha}-1}m_r(m_r - M_{r-1}) - \frac{1}{2\frac{T_\beta}{T_\alpha}-1}(m_r - M_{r-1})^2\right\}.
\end{aligned}\right\} \quad \ldots\ldots(27),$$

where $M_{max.}$ stands for M when $\tau = \tau_{max.}$.

Then from (IV) we obtain

$$M_{mean} = \sqrt{\left(\frac{L'_0}{\varepsilon^{\frac{\tau_0}{T_\beta}} - 1} + L'_{max.}\right)\varepsilon^{-\frac{\tau_{max.}}{T_\beta}}} \quad \ldots\ldots\ldots\ldots(28).$$

In the above case if we put the resisting torques in a series of alternate divisions all equal we will obtain from (28) the continuous rating of an induction motor with a rotor having considerable inertia which drives roll trains.

Case 2. In Case 1 when the motor has no fly-wheel.

In this case since the M's are all constants (18) becomes

$$\left.\begin{aligned}
K_0 &= \sum_{p=1}^{p=n} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1}\frac{t_q}{T_{\beta q}}} \left(\varepsilon^{\frac{t_p}{T_{\beta p}}}-1\right),\\
K_{r-1} &= \sum_{p=1}^{p=r-1} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1}\frac{t_q}{T_{\beta q}}} \left(\varepsilon^{\frac{t_p}{T_{\beta p}}}-1\right),\\
K &= K_{r-1} + T_{\beta r}\, \varepsilon^{\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}} \left(\varepsilon^{\frac{t}{T_{\beta r}}}-1\right),\\
\text{and}\qquad &\\
L_0 &= \sum_{p=1}^{p=n} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1}\frac{t_q}{T_{\beta q}}} \left(\varepsilon^{\frac{t_p}{T_{\beta p}}}-1\right) M_p^2,\\
L_{r-1} &= \sum_{p=1}^{p=r-1} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1}\frac{t_q}{T_{\beta q}}} \left(\varepsilon^{\frac{t_p}{T_{\beta p}}}-1\right) M_p^2,\\
L &= L_{r-1} + T_{\beta r}\, \varepsilon^{\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}} \left(\varepsilon^{\frac{t}{T_{\beta r}}}-1\right) M_r^2.
\end{aligned}\right\}\quad\ldots\ldots\ldots(29).$$

Hence from (II) and (III)

$$M_{\text{mean}}^2 = \frac{\left(\dfrac{L_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1} + L_{\text{max.}}\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\text{max.}}}{T_{\beta r}}}}{(1+\lambda)T_{\beta\,\text{mean}} - \lambda\left(\dfrac{K_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1} + K_{\text{max.}}\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\text{max.}}}{T_{\beta r}}}},$$

when $\lambda = 1$,

$$M^2_{\text{mean}} = \frac{\left(\dfrac{L_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1}+L_{\max.}\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\max.}}{T_{\beta r}}}}{2T_{\beta\text{mean}}-\left(\dfrac{K_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}}-1}+K_{\max.}\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\max}}{T_{\beta r}}}} \quad \text{...(30)},$$

where $K_{\max.}$ and $L_{\max.}$ are respectively the values of K and L when $t=t_{\max.}$, which are determined as in (II) and (III).

When we may put all the T_β's equal, from (29)

$$\left.\begin{aligned}
L_0 &= T_\beta \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}}\left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right)M_p^2, \\
L_{r-1} &= T_\beta \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}}\left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right)M_p^2, \\
L &= L_{r-1} + T_\beta\, \varepsilon^{\frac{\tau_{r-1}}{T_\beta}}\left(\varepsilon^{\frac{t}{T_\beta}}-1\right)M_r^2 .
\end{aligned}\right\} \quad \text{.........(31)}.$$

From (24) the greatest rise in temperature is

$$u_{\max.} = \frac{W_0}{b} + \frac{B}{a}\left(\frac{L_0}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1}+L_{\max.}\right)\varepsilon^{-\frac{\tau_{\max.}}{T_\beta}},$$

where $L_{\max.}$ is the value of L when u is greatest.

Finally the mean torque of the motor is obtained from (IV)

$$M_{\text{mean}} = \sqrt{\left(\frac{L_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1}+L'_{\max.}\right)\varepsilon^{-\frac{\tau_{\max}}{T_\beta}}} \quad \text{............(32)},$$

where

$$\left.\begin{aligned} L_0' &= \frac{1}{T_\beta} L_0, \\ L'_{\max.} &= \frac{1}{T_\beta} L_{\max.}. \end{aligned}\right\} \quad \text{.........(33).}$$

If the period of the cycle τ_0 is very small compared to the time constant T_β, the above equation reduces to

$$M_{\text{mean}} = \sqrt{\sum_{p=1}^{p=n} M_p^2 \frac{t_p}{\tau_0}} \quad \text{..............................(34).}$$

In this case M_{mean} becomes independent of the time constant of the motor.

As a more special case of the above, let the cycle be completed with two constant torques M_1 and M_2 working during the time t_1 and t_2.

In this case since $n = 2$, from (29)

$$\left.\begin{aligned} K_0 &= T_{\beta^1}(\varepsilon^{\frac{t_1}{T_{\beta^1}}} - 1) + T_{\beta^2}(\varepsilon^{\frac{t_1}{T_{\beta^1}} + \frac{t_2}{T_{\beta^2}}} - \varepsilon^{\frac{t_1}{T_{\beta^1}}}), \\ K_1 &= T_{\beta^1}(\varepsilon^{\frac{t_1}{T_{\beta^1}}} - 1), \\ L_0 &= T_{\beta^1}(\varepsilon^{\frac{t_1}{T_{\beta^1}}} - 1)M_1^2 + T_{\beta^2}(\varepsilon^{\frac{t_1}{T_{\beta^1}} + \frac{t_2}{T_{\beta^2}}} - \varepsilon^{\frac{t_1}{T_{\beta^1}}})M_2^2, \\ L_1 &= T_{\beta^1}(\varepsilon^{\frac{t_1}{T_{\beta^1}}} - 1)M_1^2. \end{aligned}\right\} \quad \text{......(35).}$$

Hence if we put

$$\left.\begin{aligned} \frac{\varepsilon^{\frac{t_1}{T_{\beta^1}} + \frac{t_2}{T_{\beta^2}}} - \varepsilon^{\frac{t_2}{T_{\beta^2}}}}{\varepsilon^{\frac{t_1}{T_{\beta^1}} + \frac{t_2}{T_{\beta^2}}} - 1} &= k_1, \end{aligned}\right\} \quad \text{.........(36),}$$

$$\frac{\varepsilon^{\frac{t_2}{T_{\beta^2}}}-1}{\varepsilon^{\frac{t_1}{T_{\beta^1}}+\frac{t_2}{T_{\beta^2}}}-1}=k_2.$$

we get

$$u_1=(k_1T_{\beta^1}+k_2T_{\beta^2})\frac{W_0}{a}+(k_1T_{\beta^1}M_1^2+k_2T_{\beta^2}M_2^2)\frac{B}{a}.$$

Let us assume that M_1 is greater than M_2 and that the maximum temperature occurs when $t=t_1$, i.e. u_1 is the maximum. Then (30) becomes

$$M^2_{\text{mean}}=\frac{k_1T_{\beta^1}M_1^2+k_2T_{\beta^2}M_2^2}{(1+\lambda)T_{\beta_{\text{mean}}}-\lambda(k_1T_{\beta^1}+k_2T_{\beta^2})},$$

or if $\lambda=1$, ...(37).

$$M^2_{\text{mean}}=\frac{k_1T_{\beta^1}M_1^2+k_2T_{\beta^2}M_2^2}{2T_{\beta_{\text{mean}}}-(k_1T_{\beta^1}+k_2T_{\beta^2})}.$$

If we may put

$$T_{\beta^1}=T_{\beta^2}=T_{\beta\text{mean}}=T_\beta,$$

we have

$$k_1=\frac{\varepsilon^{\frac{t_1+t_2}{T_\beta}}-\varepsilon^{\frac{t_2}{T_\beta}}}{\varepsilon^{\frac{t_1+t_2}{T_\beta}}-1},$$

$$k_2=\frac{\varepsilon^{\frac{t_2}{T_\beta}}-1}{\varepsilon^{\frac{t_1+t_2}{T_\beta}}-1}. \qquad \text{.........(38),}$$

and

$$M_{\text{mean}}=\sqrt{k_1M_1^2+k_2M_2^2} \qquad \text{..................(39).}$$

Again if t_1+t_2 is very small compared to T_β the above formulæ become approximately

$$\left.\begin{aligned} k_1 &= \frac{t_1}{t_1+t_2}, \\ k_2 &= \frac{t_2}{t_1+t_2}. \end{aligned}\right\} \quad \text{.........(40)},$$

so that

$$M_{\text{mean}} = \sqrt{\frac{t_1 M_1^2 + t_2 M_2^2}{t_1+t_2}} \quad \text{..........................(41)}.$$

Numerical example.

In the above case let $T_\beta = 1$ hour, and the torque and the time during each division be respectively

$M_1 = 500$ *MKg.*	$t_1 = 5$ minutes
$M_2 = 20$,,	$t_2 = 3$,,
$M_3 = 400$,,	$t_3 = 10$,,
$M_4 = 20$,,	$t_4 = 5$,,
$M_5 = 600$,,	$t_5 = 5$,,
$M_6 = 20$,,	$t_6 = 7$,,
$M_7 = 400$,,	$t_7 = 15$,,
$M_8 = 20$,,	$t_8 = 10$,,

the cycle being completed with 8 divisions.

Now since

$$\varepsilon^{\frac{\tau_1}{T_\beta}} = 1{\cdot}08690, \qquad \varepsilon^{\frac{\tau_2}{T_\beta}} = 1{\cdot}14263,$$

$$\varepsilon^{\frac{\tau_3}{T_\beta}} = 1{\cdot}34986, \qquad \varepsilon^{\frac{\tau_4}{T_\beta}} = 1{\cdot}46717,$$

$$\varepsilon^{\frac{\tau_5}{T_\beta}} = 1{\cdot}59467, \qquad \varepsilon^{\frac{\tau_6}{T_\beta}} = 1{\cdot}79200,$$

$$\varepsilon^{\frac{\tau_7}{T_\beta}} = 2{\cdot}30098, \qquad \varepsilon^{\frac{\tau_0}{T_\beta}} = 2{\cdot}71828,$$

from (31)

$$L_1' = \overline{500}^2\,(1{\cdot}08690-1) = 21726,$$
$$L_2' = L_1'+\overline{20}^2\,(1{\cdot}14263-1{\cdot}08690) = 21748,$$
$$L_3' = L_2'+\overline{400}^2\,(1{\cdot}34986-1{\cdot}14263) = 54905,$$
$$L_4' = L_3'+\overline{20}^2\,(1{\cdot}46717-1{\cdot}34986) = 54952,$$
$$L_5' = L_4'+\overline{600}^2\,(1{\cdot}59467-1{\cdot}46717) = 100852,$$
$$L_6' = L_5'+\overline{20}^2\,(1{\cdot}79200-1{\cdot}59467) = 100930,$$
$$L_7' = L_6'+\overline{400}^2\,(2{\cdot}30098-1{\cdot}79200) = 182370,$$
$$L_0' = L_7'+\overline{20}^2\,(2{\cdot}71828-2{\cdot}30098) = 182537.$$

$\therefore$

$$\frac{L_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1} = \frac{182537}{1{\cdot}71828} = 106233.$$

Denote

$$X_r = \left(\frac{L_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}}-1} + L_r'\right)\varepsilon^{-\frac{\tau_r}{T_\beta}},$$

then we have

$$X_1 = 117728, \qquad X_2 = 112006,$$
$$X_3 = 119374, \qquad X_4 = 109861,$$
$$X_5 = 129861, \qquad X_6 = 115604,$$
$$X_7 = 125426, \qquad X_0 = 106233.$$

Since X_5 is the largest we may consider that the greatest rise in temperature occurs at the end of the fifth division.

Hence from (32)

$$M_{\text{mean}} = \sqrt{X_5}$$
$$= \sqrt{129861}$$
$$= 360\ MKg.$$

If the induction motor should develope the maximum torque at 3% slip of the synchronous velocity, 720 r. p. m. say, we have

$$\omega_s = 2\pi \frac{720}{60} = 75{\cdot}40$$

$\therefore$

$$A = \frac{600}{0{\cdot}03 \times 75{\cdot}40} = 265{\cdot}26.$$

The angular velocity corresponding to M_{mean} is

$$\omega_{\text{mean}} = \omega_s - \frac{M_{\text{mean}}}{A}$$
$$= 75{\cdot}40 - \frac{360}{265{\cdot}26}$$
$$= 74{\cdot}0$$

Hence the slip is

$$s_{\text{mean}} = \frac{\omega_s - \omega_{\text{mean}}}{\omega_s} \times 100$$
$$= 1{\cdot}82\%$$

The power of the motor is

$$P_{\text{mean}} = M_{\text{mean}}\ \omega_{\text{mean}}$$
$$= \frac{360 \times 74{\cdot}0}{75}$$
$$= 355 \text{ H.P.}$$

Case 3. When the induction motor works intermittently with constant loads making a cycle.

Let the constant torques on the motor be respectively

$$M_1,\ M_2,\ M_3,\ \ldots\ldots M_r,\ \ldots\ldots M_n,$$

each working during the time

$$t_1,\ t_2,\ t_3,\ \ldots\ldots t_r,\ \ldots\ldots t_n,$$

with the values of the T_β's

$$T_{\beta 1},\ T_{\beta 2},\ T_{\beta 3},\ \ldots\ldots T_{\beta r},\ \ldots\ldots T_{\beta n}$$

respectively, and let the interval of pause after each load be

$$t_1',\ t_2',\ t_3',\ \ldots\ldots t_r',\ \ldots\ldots t_n'$$

respectively, during which the value of the time constant being $T_{\beta 0}$.

In this case (19) becomes

$$\left.\begin{aligned}
K_0 &= \sum_{p=1}^{p=n} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1}\frac{t_q}{T_{\beta q}}+\frac{\tau'_{p-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t_p}{T_{\beta p}}}-1\right),\\
K_{r-1} &= \sum_{p=1}^{p=r-1} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1}\frac{t_q}{T_{\beta q}}+\frac{\tau'_{p-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t_p}{T_{\beta p}}}-1\right),\\
K &= K_{r-1}+T_{\beta r}\, \varepsilon^{\sum_{q=1}^{q=r-1}\frac{t_q}{T_{\beta q}}+\frac{\tau'_{r-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t}{T_{\beta r}}}-1\right),\\
&\text{and}\\
L_0 &= \sum_{p=1}^{p=n} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1}\frac{t_q}{T_{\beta q}}+\frac{\tau'_{p-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t_p}{T_{\beta p}}}-1\right) M_p^2,\\
L_{r-1} &= \sum_{p=1}^{p=r-1} T_{\beta p}\, \varepsilon^{\sum_{q=1}^{q=p-1}\frac{t_q}{T_{\beta q}}+\frac{\tau'_{p-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t_p}{T_{\beta p}}}-1\right) M_p^2,\\
L &= L_{r-1}+T_{\beta r}\, \varepsilon^{\sum_{q=1}^{q=r-1}\frac{t_q}{T_{\beta q}}+\frac{\tau'_{r-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t}{T_{\beta r}}}-1\right) M_r^2.
\end{aligned}\right\}\ \ldots(42),$$

where

$$\tau_q' = t_1' + t_2' + t_3' + \ldots\ldots + t_q'.$$

The temperature of the motor at any working time is from (20)

$$u = \frac{W_0}{a}\left(\frac{K_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t}{T_{\beta r}}-\frac{\tau'_{r-1}}{T_{\beta 0}}}$$

$$+\frac{B}{a}\left(\frac{L_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}+\frac{\tau_0'}{T_{\beta 0}}}-1} + L\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t}{T_{\beta r}}-\frac{\tau'_{r-1}}{T_{\beta 0}}},$$

where

$$\tau_0' = t_1' + t_2' + t_3' + \ldots\ldots + t_n'.$$

Hence (II) becomes

$$M^2_{\text{mean}} = \frac{\left(\frac{L_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}+\frac{\tau_0'}{T_{\beta 0}}}-1} + L_{\max.}\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\max.}}{T_{\beta r}}-\frac{\tau'_{r-1}}{T_{\beta 0}}}}{(1+\lambda)T_{\beta\text{mean}} - \lambda\left(\frac{K_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K_{\max.}\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\max.}}{T_{\beta r}}-\frac{\tau'_{r-1}}{T_{\beta 0}}}} \quad (43),$$

where $L_{\max.}$ and $K_{\max.}$ are respectively the values of L and K corresponding to $t = t_{\max.}$.

When $\lambda = 1$, (43) becomes

$$M^2_{\text{mean}} = \frac{\left(\frac{L_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}+\frac{\tau_0'}{T_{\beta 0}}}-1} + L_{\max.}\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\max.}}{T_{\beta r}}-\frac{\tau'_{r-1}}{T_{\beta 0}}}}{2T_{\beta\text{mean}} - \left(\frac{K_0}{\varepsilon^{\sum_{r=1}^{r=n}\frac{t_r}{T_{\beta r}}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K_{\max.}\right)\varepsilon^{-\sum_{p=1}^{p=r-1}\frac{t_p}{T_{\beta p}}-\frac{t_{\max.}}{T_{\beta r}}-\frac{\tau'_{r-1}}{T_{\beta 0}}}} \quad (44).$$

If we may consider all the T_β's during the working time equal we have

$$\left.\begin{aligned}
K_0 &= T_\beta \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}+\frac{\tau'_{p-1}}{T_{\beta 0}}} \left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right),\\
K_{r-1} &= T_\beta \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}+\frac{\tau'_{p-1}}{T_{\beta 0}}} \left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right),\\
K &= K_{r-1} + T_\beta\, \varepsilon^{\frac{\tau_{r-1}}{T_\beta}+\frac{\tau'_{r-1}}{T_{\beta 0}}} \left(\varepsilon^{\frac{t}{T_\beta}}-1\right),\\
&\text{and}\\
L_0 &= T_\beta \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}+\frac{\tau'_{p-1}}{T_{\beta 0}}} \left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right) M_p^2,\\
L_{r-1} &= T_\beta \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}+\frac{\tau'_{p-1}}{T_{\beta 0}}} \left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right) M_p^2,\\
L &= L_{r-1} + T_\beta\, \varepsilon^{\frac{\tau_{r-1}}{T_\beta}+\frac{\tau'_{r-1}}{T_{\beta 0}}} \left(\varepsilon^{\frac{t}{T_\beta}}-1\right) M_r^2 .
\end{aligned}\right\} \quad \ldots\ldots(45),$$

where T_β is the value of the time constant during the working time and

$$\tau_r = t_1 + t_2 + \ldots\ldots + t_r,$$
$$\tau_0 = t_1 + t_2 + \ldots\ldots + t_n.$$

Hence (43) and (44) become

$$M^2_{\text{mean}} = \frac{\left(\dfrac{L'_0}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + L'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}}{(1+\lambda)\dfrac{T_{\beta\text{mean}}}{T_\beta} - \lambda\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}},$$

when $\lambda = 1$,

$$M^2_{\text{mean}} = \frac{\left(\dfrac{L'_0}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + L'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}}{2\dfrac{T_{\beta\text{mean}}}{T_\beta} - \left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}} \quad \text{(46)},$$

where

$$K' = \frac{1}{T_\beta}K, \qquad L' = \frac{1}{T_\beta}L. \quad \text{.........(47).}$$

Further if we may consider $T_{\beta_{\text{mean}}} = T_\beta$,

$$M^2_{\text{mean}} = \frac{\left(\dfrac{L'_0}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + L'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}}{(1+\lambda)-\lambda\left(\dfrac{K'_0}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau'_0}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}},$$

when $\lambda = 1$, ...(48).

$$M^2_{\text{mean}} = \frac{\left(\dfrac{L_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau'_0}{T_{\beta 0}}}-1} + L'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}}{2-\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}}.$$

Moreover if we may put $T_\beta = T_{\beta 0}$,

$$
\left.
\begin{aligned}
K'_0 &= \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}+\tau'_{p-1}}{T_\beta}}\left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right),\\
K'_{r-1} &= \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}+\tau'_{p-1}}{T_\beta}}\left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right),\\
K' &= K'_{r-1}+\varepsilon^{\frac{\tau_{r-1}+\tau'_{r-1}}{T_\beta}}\left(\varepsilon^{\frac{t}{T_\beta}}-1\right),\\
\text{and}\qquad &\\
L'_0 &= \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}+\tau'_{p-1}}{T_\beta}}\left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right)M_p^2,\\
L'_{r-1} &= \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}+\tau'_{p-1}}{T_\beta}}\left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right)M_p^2,\\
L &= L_{r-1}+\varepsilon^{\frac{\tau_{r-1}+\tau'_{r-1}}{T_\beta}}\left(\varepsilon^{\frac{t}{T_\beta}}-1\right)M_r^2.
\end{aligned}
\right\}\ldots(49),
$$

and

$$
\left.
\begin{aligned}
M^2_{\text{mean}} &= \frac{\left(\dfrac{L'_0}{\varepsilon^{\frac{\tau_0+\tau_0'}{T_\beta}}-1}+L'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}+\tau'_{r-1}}{T_\beta}}}{(1+\lambda)-\lambda\left(\dfrac{K'_0}{\varepsilon^{\frac{\tau_0+\tau_0'}{T_\beta}}-1}+K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}+\tau'_{r-1}}{T_\beta}}},\\
\text{when } \lambda = 1,\qquad &\\
M^2_{\text{mean}} &= \frac{\left(\dfrac{L_0'}{\varepsilon^{\frac{\tau_0+\tau_0'}{T_\beta}}-1}+L'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}+\tau'_{r-1}}{T_\beta}}}{2-\left(\dfrac{K'_0}{\varepsilon^{\frac{\tau_0+\tau_0'}{T_\beta}}-1}+K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}+\tau'_{r-1}}{T_\beta}}}.
\end{aligned}
\right\}\ldots(50).
$$

In the above if the M's are all equal from (45) we have

$$\left.\begin{aligned} K_0 &= T_\beta \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}+\frac{\tau'_{p-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right), \\ K_{r-1} &= T_\beta \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}}{T_\beta}+\frac{\tau'_{p-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t_p}{T_\beta}}-1\right), \\ K &= K_{r-1}+T_\beta\, \varepsilon^{\frac{\tau_{r-1}}{T_\beta}+\frac{\tau'_{r-1}}{T_{\beta 0}}}\left(\varepsilon^{\frac{t}{T_\beta}}-1\right), \\ &\text{and} \\ L_0 &= K_0 M^2, \\ L_{r-1} &= K_{r-1} M^2, \\ L &= K M^2. \end{aligned}\right\} \quad \ldots(51).$$

Therefore the temperature of the motor at any time becomes

$$u = \left(\frac{W_0}{a}+\frac{B}{a}M^2\right)\left(\frac{K_0}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1}+K\right)\varepsilon^{-\frac{\tau}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}.$$

In this case the time when u is greatest can be readily found. Let it occur when $\tau=\tau_{\max.}$ and $K=K_{\max.}$.

Then the M^2_{mean} becomes

$$M^2_{\text{mean}} = \frac{\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1}+K'_{\max.}\right)\varepsilon^{-\frac{\tau_{\max.}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}M^2}{(1+\lambda)\dfrac{T_{\beta\text{mean}}}{T_\beta}-\lambda\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1}+K'_{\max.}\right)\varepsilon^{-\frac{\tau_{\max.}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}},$$

when $\lambda=1$,

$$M^2_{\text{mean}} = \frac{\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}} M^2}{2\,\dfrac{T_{\beta_{\text{mean}}}}{T_\beta} - \left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}},$$

where

$$K' = \frac{1}{T_\beta} K. \qquad \ldots(52).$$

When we may put

$$T_{\beta_{\text{mean}}} = T_\beta,$$

$$M^2_{\text{mean}} = \frac{\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}} M^2}{(1+\lambda) - \lambda\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}},$$

when $\lambda = 1$, $\ldots(53).$

$$M^2_{\text{mean}} = \frac{\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'}{T_{\beta 0}}} M^2}{2 - \left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0}{T_\beta}+\frac{\tau_0'}{T_{\beta 0}}}-1} + K'_{\text{max.}}\right)\varepsilon^{-\frac{\tau_{\text{max.}}}{T_\beta}-\frac{\tau'_{r-1}}{T_{\beta 0}}}}.$$

If we may put $T_\beta = T_{\beta 0}$,

$$K_0' = \sum_{p=1}^{p=n} \varepsilon^{\frac{\tau_{p-1}+\tau'_{p-1}}{T_\beta}} \left(\varepsilon^{\frac{t_p}{T_\beta}} - 1\right),$$

$$K'_{r-1} = \sum_{p=1}^{p=r-1} \varepsilon^{\frac{\tau_{p-1}+\tau'_{p-1}}{T_\beta}} \left(\varepsilon^{\frac{t_p}{T_\beta}} - 1\right),$$

$$K' = K'_{r-1} + \varepsilon^{\frac{\tau_{r-1}+\tau'_{r-1}}{T_\beta}}\left(\varepsilon^{\frac{t}{T_\beta}} - 1\right), \qquad \ldots(54),$$

and

$$L'_0 = K'_0 M^2,$$

$$L'_{r-1} = K'_{r-1} M^2,$$

$$L' = K' M^2.$$

and

$$M^2_{\text{mean}} = \frac{\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0+\tau_0'}{T_\beta}} - 1} + K'_{\text{max.}}\right) \varepsilon^{-\frac{\tau_{\text{max.}}+\tau'_{r-1}}{T_\beta}} M^2}{(1+\lambda) - \lambda\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0+\tau_0'}{T_\beta}} - 1} + K'_{\text{max.}}\right) \varepsilon^{-\frac{\tau_{\text{max.}}+\tau'_{r-1}}{T_\beta}}},$$

when $\lambda = 1$, $\ldots(55)$.

$$M^2_{\text{mean}} = \frac{\left(\dfrac{K_0'}{\varepsilon^{\frac{\tau_0+\tau'_0}{T_\beta}} - 1} + K'_{\text{max.}}\right) \varepsilon^{-\frac{\tau_{\text{max.}}+\tau'_{r-1}}{T_\beta}} M^2}{2 - \left(\dfrac{K'_0}{\varepsilon^{\frac{\tau_0+\tau_0'}{T_\beta}} - 1} + K'_{\text{max.}}\right) \varepsilon^{-\frac{\tau_{\text{max.}}+\tau'_{r-1}}{T_\beta}}}.$$

When the period of the cycle is very small compared to the time constant, the above formulæ (50) and (55) become respectively

$$M_{\text{mean}} = \sqrt{\frac{1}{\tau_0 + (1+\lambda)\tau_0'} \sum_{p=1}^{p=n} t_p M_p^2},$$

when $\lambda = 1$, $\ldots(50)'$,

$$M_{\text{mean}} = \sqrt{\frac{1}{\tau_0 + 2\tau_0'} \sum_{p=1}^{p=n} t_p M_p^2}.$$

and

$$\left.\begin{aligned} &M_{\text{mean}} = M\sqrt{\frac{\tau_0}{\tau_0+(1+\lambda)\tau_0'}}\,, \\ &\text{when } \lambda = 1, \\ &M_{\text{mean}} = M\sqrt{\frac{\tau_0}{\tau_0+2\,\tau_0'}}\,. \end{aligned}\right\} \quad (55)'.$$

Thus in these cases M_{mean} can be approximately determined independent of the time constant of the motor.

SUMMARY.

In Chapter 1 the interaction of the fly-wheel and the induction motor under cyclically varying load of any given form was considered. The motion of the induction motor during the whole period has been determined, and formulæ governing the proportion of the fly-wheel and the motor have been developed.

The general formulæ have been applied to some special cases of practical importance.

Since when the resisting torque of an induction motor under cyclical operation is given, we can in general find the manner of variation of the torque imposed upon the motor, whether the rotor has or has not a fly-wheel, in Chapter II formulæ for determining the capacity of the induction motor under cyclical operation of any form of load rated for continuous running have been developed.

The general formulæ have been applied to some special cases.

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