On the Envelope Gearing

By

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Spur wheel teeth are at present invariably of cycloidal or of involute from. In this paper I propose a form of teeth, which may for the present be called "envelope," as possessing certain advantages over the ordinary cycloidal or involute form.

The pair of curves for wheel teeth here referred to is a straight line and its envelope.

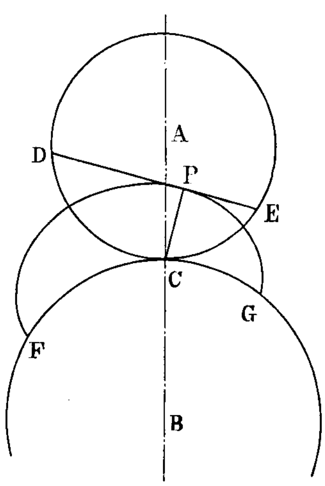

Fig. 1.

Let, in Fig. 1, A and B be pitch circles, DE a chord fixed in the circle A. When the latter rolls on the circle B, DE occupies different positions of which an envelope FPG is formed, called the envelope of a carried right line.[1] Since the moving chord touches its envelope in each of its positions, the path of its point of contact at any instant must be tangential to the envelope; hence the normal at their common point must pass through C, the point of contact of the pitch circles. The condition of a constant velocity ratio in toothed gearing is that the common normal of two curves in contact must cut the line of centres at a fixed point. This condition is satisfied by our pair,

1) Williamson's Differential Calculus, Art. 292.

the chord and its envelope. When the envelope FPG is fixed to the circle B and the two circles are set in motion by rolling contact round A and B as their respective centres, the two lines touch each other continuously, the point of contact moving along a certain curve, but for all positions of the point the common normal passes through C, a fixed point as required. The curve above referred to or the locus of the perpendicular foot let fall from C on DE is called limaçon and is of the form shown by the dotted

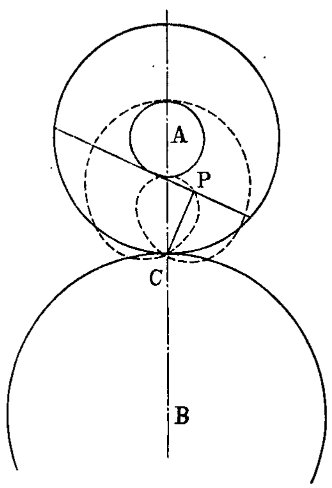

Fig. 2.

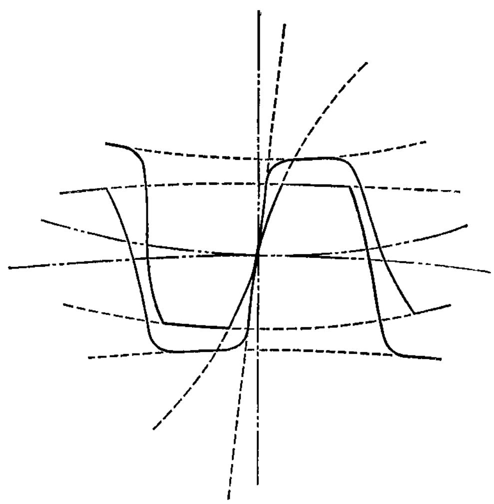

Fig. 3.

line in Fig. 2. Two pairs of the chord and the envelope form the working edges of the teeth in contact as shown in Fig. 3.

Let us now proceed to compare the new form with the cycloidal and involute forms in regard to the abrasion of teeth.

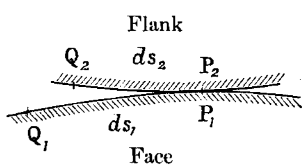

Fig. 4.

In the pair of face and flank, Fig. 4, if P_1 and P_2 are in contact at the beginning of the time dt and Q_1 and Q_2 at the end of it, then the amount of sliding during dt is

$$P_1Q_1 - P_2Q_2 \qquad \text{or} \qquad ds_1 - ds_2$$

which, divided by ds_2 is the measure of abrasion[1] of the flank at P_2 and may be called the specific sliding at that point. Or if we denote by σ_2 and σ_1 the specific slidings for flank and face respectively

$$\left.\begin{aligned}\sigma_2 &= \frac{ds_1 - ds_2}{ds_2},\\ \sigma_1 &= \frac{ds_1 - ds_2}{ds_1}.\end{aligned}\right\} \qquad \text{.............................(1)}$$

The expression for the specific sliding in cycloidal, involute and envelope teeth will be found in the following:

1. Cycloidal teeth:

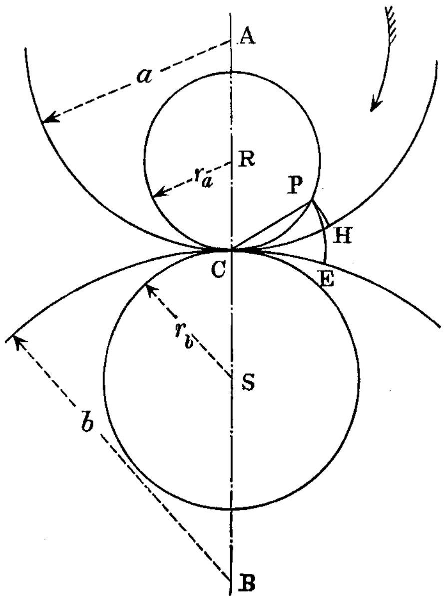

Fig. 5.

Let s_2 = hypocycloidal arc HP,

s_1 = epicycloidal arc EP,

y = chord CP,

x = arc CP,

α = angle of obliquity of action.

If by the rolling contact the circumferences of the circles R and A move through dx, the corresponding rotations are $\frac{dx}{r_a}$ and $\frac{dx}{a}$ in the same direction. Hence the relative rotation of R against A is

$$dx\left(\frac{1}{r_a} - \frac{1}{a}\right).$$

This multiplied by y is the length of the elementary arc described by P i.e. ds_2 or

$$ds_2 = y\,dx\left(\frac{1}{r_a} - \frac{1}{a}\right).$$

1) The effect of variation of the normal pressure due to the change of obliquity of action being neglected.

Similarly the rotations of R and B corresponding to their rolling motion through dx are $\dfrac{dx}{r_a}$ and $\dfrac{dx}{b}$, now in the opposite directions. Hence the relative rotation of one against the other is

$$dx\left(\frac{1}{r_a}+\frac{1}{b}\right)$$

and the length of the elementary arc ds_1 is

$$ds_1 = y\,dx\left(\frac{1}{r_a}+\frac{1}{b}\right).$$

Then by (1) we get

$$\left.\begin{aligned} \sigma_2 &= \frac{\left(\dfrac{1}{r_a}+\dfrac{1}{b}\right)-\left(\dfrac{1}{r_a}-\dfrac{1}{a}\right)}{\dfrac{1}{r_a}-\dfrac{1}{a}} = \frac{\dfrac{1}{b}+\dfrac{1}{a}}{\dfrac{1}{r_a}-\dfrac{1}{a}}, \\ \sigma_2 &= \frac{\left(\dfrac{1}{r_a}+\dfrac{1}{b}\right)-\left(\dfrac{1}{r_a}-\dfrac{1}{a}\right)}{\dfrac{1}{r_a}+\dfrac{1}{b}} = \frac{\dfrac{1}{b}+\dfrac{1}{a}}{\dfrac{1}{r_a}+\dfrac{1}{b}}. \end{aligned}\right\} \quad \ldots\ldots(2)$$

The specific sliding in cycloidal gearing is a constant for different points of the face and another constant for different points of the flank, while in involute and envelope gearings it is a function of y or of the radial distance of the point of contact from the pitch circle as will be seen later.

For the expression for the maximum angle of obliquity α_m in cycloidal gearing we have, referring to Fig. 6

$$y = -\,b\sin\alpha_m + \sqrt{b_1^2 - b^2\cos^2\alpha_m}$$

and $$y = 2r_a\sin\alpha_m\,.$$

Equating $$(2r_a + b)\sin\alpha_m = \sqrt{b_1^2 - b^2\cos^2\alpha_m}\,,$$

whence $$\sin\alpha_m = \sqrt{\frac{b_1^2 - b^2}{4r_a(r_a+b)}}\,.$$

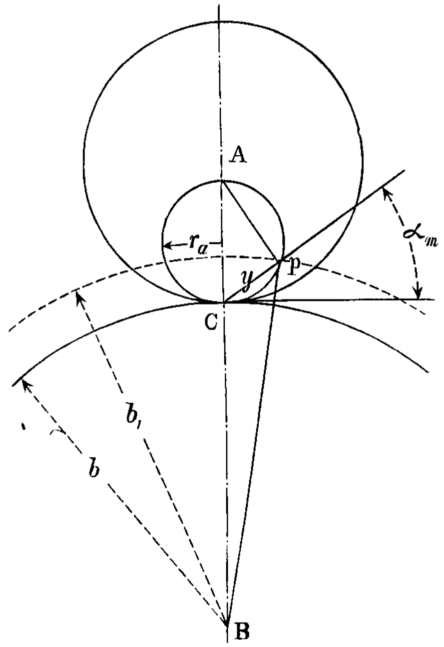

Fig. 6.

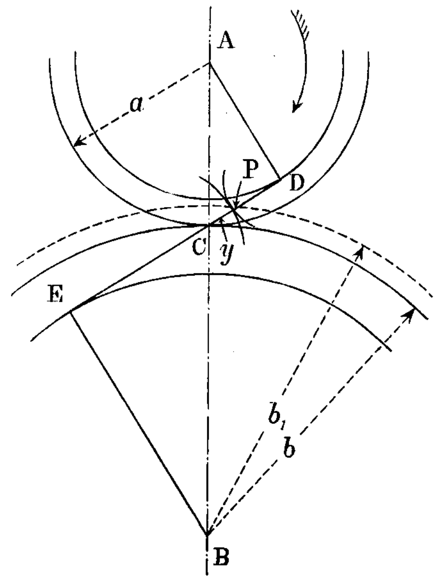

Fig. 7.

2. Involute teeth:

As the circumferences of the pitch circles A and B, Fig. 7, move through dx by rolling contact, the moving point P describes on the material planes of A and B elements of involute, whose lengths are respectively

$$ds_2 = (a \sin \alpha - y)\,\frac{dx}{a}$$

and

$$ds_1 = (b \sin \alpha + y)\,\frac{dx}{b}$$

Therefore by (1) we have

$$\left.\begin{aligned} \sigma_2 &= \frac{y\left(\dfrac{1}{b} + \dfrac{1}{a}\right)a}{a \sin \alpha - y} = \frac{y\left(1 + \dfrac{a}{b}\right)}{a \sin \alpha - y}, \\ \sigma_1 &= \frac{y\left(\dfrac{1}{b} + \dfrac{1}{a}\right)b}{b \sin \alpha + y} = \frac{y\left(1 + \dfrac{b}{a}\right)}{b \sin \alpha + y}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4)$$

where $$y = -b\sin\alpha + \sqrt{b_1^2 - b^2\cos^2\alpha} \quad\text{.....................(5)}$$

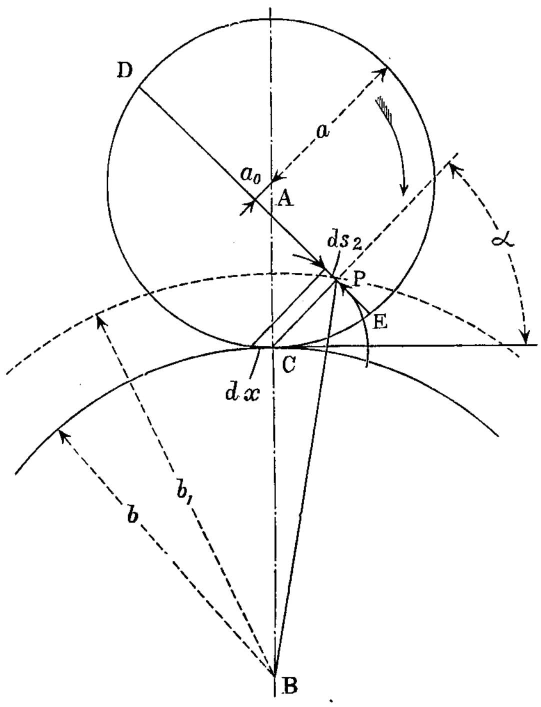

Fig. 8.

3. Envelope teeth:

Referring to Fig. 8, if the circumferences of the pitch circles A and B move through dx by rolling contact, the relative rotation of one against the other is

$$dx\left(\frac{1}{a} + \frac{1}{b}\right)$$

which, multiplied by y, is the amount of sliding of the flank over the face or

$$ydx\left(\frac{1}{a} + \frac{1}{b}\right) = ds_1 - ds_2$$

and

$$ds_2 = dx\sin\alpha\,.$$

Therefore $$ds_1 = dx\sin\alpha + ydx\left(\frac{1}{a} + \frac{1}{b}\right).$$

Then by (1) we have

$$\left.\begin{aligned} \sigma_2 &= \frac{y}{\sin\alpha}\left(\frac{1}{a} + \frac{1}{b}\right), \\ \sigma_1 &= \frac{y\left(\frac{1}{a} + \frac{1}{b}\right)}{\sin\alpha + y\left(\frac{1}{a} + \frac{1}{b}\right)} = \frac{\sigma_2}{1+\sigma_2}, \end{aligned}\right\} \quad\text{..............(6)}$$

in which $\dfrac{y}{\sin\alpha}$ is still unknown. From Fig. 8

$$y = -b \sin \alpha + \sqrt{b_1^2 - b^2 \cos^2 \alpha}$$

or $$\frac{y}{\sin \alpha} = b + \frac{1}{\sin \alpha} \sqrt{b_1^2 - b^2 \cos^2 \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

and $$a_0 = a \sin \alpha - y ,$$

a_0 being the perpendicular distance from A to DE.

Putting the above value of y in the last equation

$$a_0 = (a + b) \sin \alpha - \sqrt{b_1^2 - b^2 \cos^2 \alpha} ,$$

whence

$$\sin \alpha = \frac{2a_0(a+b) + \sqrt{4a_0^2(a+b)^2 + 4a(a+2b)(b_1^2 - b^2 - a_0^2)}}{2a(a+2b)} . \qquad (8)$$

This value being put in (6) $\dfrac{y}{\sin \alpha}$ is obtained.

All the forgoeing results are for the path of approach; for the path of recess letters a and b occuring therein are to the interchanged.

The values of the specific sliding in different forms of teeth will be compared by examples.

Example 1. Let the number of teeth ef one wheel be 20 and that of the other wheel 80.

1. Cycloidal teeth:

Take the diameter of rolling circles in the following ratio:

$$a : b : r_a : r_b = 1 : 4 : 0{\cdot}35 : 1 .$$

The height of teeth h_1 above pitch circle is usually taken at 0·3 pitch. Hence

$$b_1 = b + h_1 = b + 0{\cdot}3 \frac{2\pi b}{80}$$

or $$b : b_1 = 4 : 4{\cdot}094 ,$$

also $$a : a_1 = 1 : 1{\cdot}094 .$$

The maximum angle of obliquity and the values of σ_2 and σ_1 calculated by (3) and (2) are

	α_m	σ_2 (flank)	σ_1 (face)
Path of approach	20°, 42′	0·67	0·40
Path of recess	9°, 1′	1·67	0·63

2. Involute teeth:

Taking the angle of obliquity at 20° the values of y, σ_2 and σ_1 for different points of paths of approach and recess calculated by (5) and (4) are

		y	σ_2 (flank)	σ_1 (face)
Path of approach	$b_1 = b + h_1$	0·251	3·45	0·76
	$b_1 = b + \frac{2}{3} h_1$	0·171	1·25	0·56
	$b_1 = b + \frac{1}{3} h_1$	0·085	0·41	0·29
	$b_1 = b$	0	0	0
Path of recess	$b_1 = b + h_1$	0·218	0·95	0·49
	$b_1 = b + \frac{2}{3} h_1$	0·154	0·63	0·39
	$b_1 = b + \frac{1}{3} h_1$	0·082	0·32	0·24
	$b_1 = b$	0	0	0

3. Envelope teeth:

Taking $a_0 = 0{\cdot}1\,a$ and $b_0 = 0{\cdot}1\,b$ the values of α, $\frac{y}{\sin \alpha}$, σ_2 and σ_1 for different points of paths of approach and of recess calculated by (8), (7) and (6) are

		α	$\frac{y}{\sin \alpha}$	σ_2 (flank)	σ_1 (face)
Path of approach	$b_1 = b + h_1$	20°, 28′	0·714	0·89	0·47
	$b_1 = b + \frac{2}{3} h_1$	17°, 17′	0·667	0·83	0·45
	$b_1 = b + \frac{1}{3} h_1$	13°, 18′	0·550	0·69	0·41
	$b_1 = b$	5°, 44′	0	0	0
Path of recess	$b_1 = b + h_1$	10°, 6′	1·720	2·15	0·68
	$b_1 = b + \frac{2}{3} h_1$	9°, 8′	1·480	1·85	0·65
	$b_1 = b + \frac{1}{3} h_1$	7°, 55′	1·092	1·37	0·58
	$b_1 = b$	5°, 44′	0	0	0

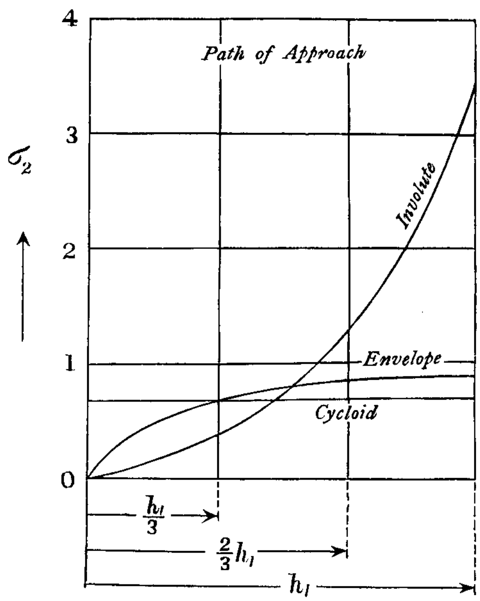

Fig. 9.

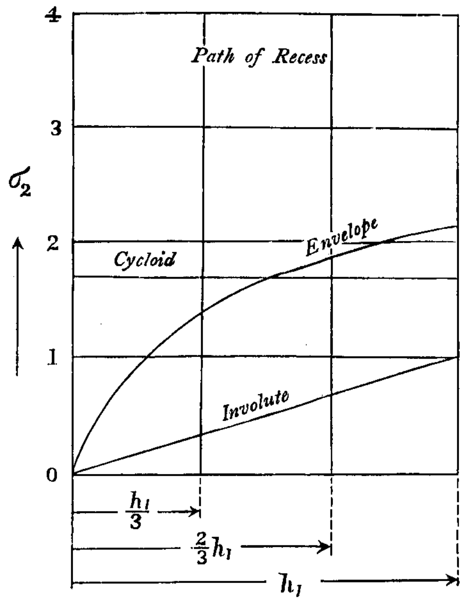

Fig. 10.

As seen from these results for the three forms of teeth σ_2 is always greater than the corresponding value of σ_1. We therefore compare σ_2 alone.

Fig. 9 and 10 are graphical comparisons of σ_2. In these figures abscissa represents the radial distance of the point of contact on the face from the pitch circle and ordinate the specific sliding of the corresponding point of contact on the flank forming the pair with the face.

Example 2. Let the two wheels have each 32 teeth. In this case σ_2, σ_1 etc are the same for the paths of approach and recess.

1. Cycloidal teeth:

Take the diameter of the rolling circles equal to 0·32 of that of the pitch circles. Then $\quad a : b : r_a : r_b = 1 : 1 : 0{\cdot}32 : 0{\cdot}32$

$$b_1 = b + h_1 = b + 0{\cdot}3\,\frac{2\pi b}{32}$$

or $\quad b : b_1 = 1 : 1{\cdot}059\,.$

The results calculated by (3) and (2) are

a_m	σ_2 (flank)	σ_1 (face)
15°, 33′	0·94	0·48

2. Involute teeth:

Taking the obliquity at 16° we get by (5) and (4)

	y	σ_2 (flank)	σ_1 (face)
$b_1 = b + h_1$	0·169	3·15	0·76
$b_1 = b + \frac{2}{3} h_1$	0·119	1·51	0·60
$b_1 = b + \frac{1}{3} h_1$	0·065	0·62	0·38
$b_1 = b$	0	0	0

3. Envelope teeth:

Taking $a_0 = 0{\cdot}1\, a$ as before we get by (8), (7) and (6)

	α	$\frac{y}{\sin \alpha}$	σ_2 (flank)	σ_1 (face)
$b_1 = b + h_1$	15°, 42′	0·630	1·26	0·56
$b_1 = b + \frac{2}{3} h_1$	13°, 28′	0·571	1·14	0·53
$b_1 = b + \frac{1}{3} h_1$	10°, 48′	0·466	0·93	0·48
$b_1 = b$	5°, 44′	0	0	0

σ_2 is compared graphically in Fig. 11.

Examining Figs 9, 10 and 11 we see that involute teeth have the greatest value of σ_2, which would come out still greater if the obliquity α be taken smaller.[1] The abrasion is thus very great at the root of the teeth of the wheel A. As the wheel A in Ex. 1 is the smaller of the two, the defect is greatly magnified because a tooth of the smaller wheel partakes in the transmission several times as often as that of the larger wheel in a given time. Moreover as σ_2 fluctuates greatly along the flank, the quick destruction of the involute form is easily conceivable.

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1) A calculation with $\alpha = 18°$ in Ex. 1 gave the greatest value of 10·82.

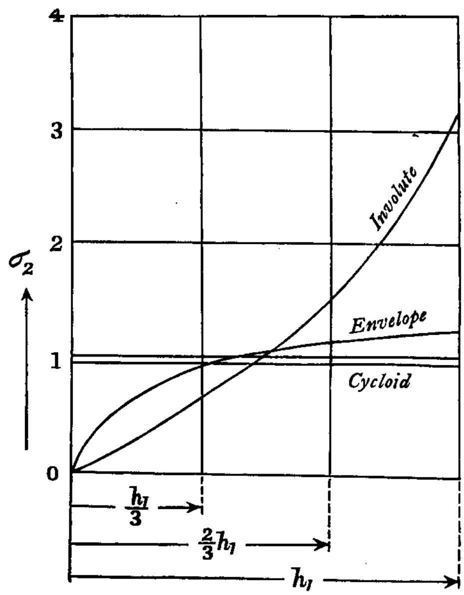

Fig. 11.

In cycloidal teeth σ_2 is constant and is not equal to zero at the pitch point. This fact may be considered as a defect. It was stated that the specific sliding is the measure of abrasion, but examining the matter more accurately there appears another factor. If the arc of contact is less than twice the pitch, as is usually the case, two pairs of teeth come in contact simultaneously in the first and last parts of the arc, while in the middle part only a single pair is in action. Hence if in Fig. 12 MN be the path of contact MM' and $N'N$ the parts of the path corresponding to the pitch of teeth, the pressure between a pair of teeth is one-half of the total force transmitted

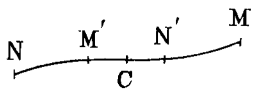

Fig. 12.

while the point of contact is moving through MN' and $M'N$, but the pair has to bear the whole amount while the point is moving through $N'M'$. Thus during this motion through $N'M'$ the wearing effect is doubled and consequently it is desirable that σ_2 be of small value in the vicinity of the pitch point. This condition is satisfied by our envelope teeth.

Fig. 13 is a graphical compari-

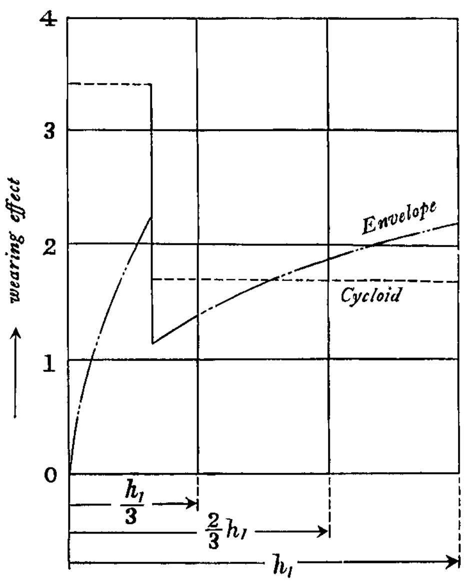

Fig. 13.

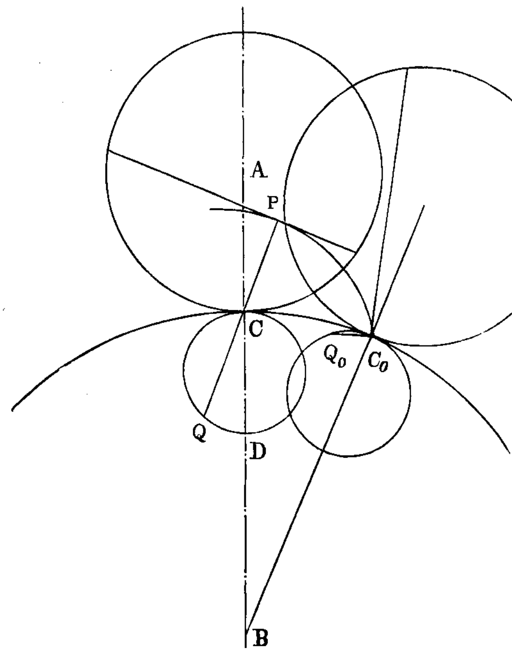

Fig. 14.

son of actual wearing effect in the cycloidal and envelope teeth in Ex. 1 for the path of recess.

The envelope of a carried right line is treated in Williamson's Differential Calculus, Art. 292, referring to which, if CD in Fig. 14 is the circle of inflexion or

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{CD},$$

the centre of curvature for P is situated at Q the intersection of PC produced with the circle of inflexion. For the initial point C_0 the radius of curvature is

$$C_0Q_0 = \frac{ab}{a+b}\,\frac{a_0}{a} = \frac{b}{a+b}\,a_0 \quad \text{.....................(9)}$$

In epicycloid and hypocycloid the radius of curvature for the initial point is zero, while in the envelope it is finite. From this fact it follows that the envelope can be more closely approximated than cycloids by a circular arc. Moreover the simplicity of the form of envelope teeth facilitates pattern making and the manufacture of milling cutters, an advantage at least when compared with cycloidal teeth.