Flattening-Out of Aeroplanes after Steep Glides.

By

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I Introduction.

A wide region of unexplored ground of study is opened up to us by the development of the aeroplane. Among the large number of interesting problems concerning the aeroplane, the one to determine the motion at flattening-out after a steep glide is worthy of close investigation.

This subject naturally involves two important problems: one the determination of time required to restore the machine from a steep glide, the other the determination of "wing-loading" set up during this motion. The urgency of the first both for the practical aviator and for the designer is obvious. The latter is of utmost importance to the designer of the aeroplane. It is necessary to know the distributed loading due to the air forces, generated by the movement through the air, before proper consideration can be given to the stresses in structure.

The working stresses in the wing structure and in the other parts of the machine are generally calculated for the case of a horizontal flight in still air, the normal loading on the wings in this case being equal to the weight of the whole machine exclusive of the wings. This loading may be increased by different conditions of flight to many times its normal value. Thus, the effects of banking, sudden recovery from a steep glide, sharp or irregular gusts, landing, etc., all impose higher loadings upon the machine. A. W. Judge* gives the following figures, as the ratios of the loadings under the stated conditions to the normal loading upon the planes:

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* A.W. Judge, "Design of Aeroplane," P. 56.

for banking	1·5
for Wind gusts	4 to 5
for flattening-out after steep glide	5 to 7.

He adds that the figures given for flattening-out after a steep dive represent an extreme case, which would not be realized by a careful pilot; and that the maximum abnormal loadings in practice could hardly exceed about five times the normal. G. C. Leoning* says :—" careful observation indicates that the forces of sharp puffs, or sudden changes in wind direction, may easily give stresses three to four times the weight of the machine." Prof. E.B. Wilson† developed the theory of the longitudinal or symmetric motion of the aeroplane under gusts, and investigated the motions of the free and the constrained aeroplanes encountering gusts of different kinds, and with various degrees of sharpness. He found that the upgusts, which operate chiefly to lift the machine and accordingly impose the greatest loading on the wings in comparison with others, do not seriously stress the machine which is designed to stand upward accelerations of 6g to 8g in manoeuvering. The stresses introduced by landing shocks are in a separate class, requiring careful consideration. These stresses are minimized in good designs of landing gear by employing the proper shock absorber. The greatest source of danger in flying, due to imposing great stress on the wings, arises, without question, from flattening-out sharply after a long steep dive.

The aeroplane should be so designed that the strength of its weakest structural part will at least be great enough to withstand a possible maximum stress. The usual method hitherto employed to allow for these extra stresses has been to choose a " factor of safety," with reference to the normal flying load, determined by the weight of the machine. The excess stress induced by conditions other than ordinary horizontal flight, is taken account of in the " factor of safety " itself.

At present there is a tendency to apply a high factor of safety in the

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* G.C. Leoning, "Military Aeroplane," P. 108.

† First Annual Report of the National Advisory Committee of U.S.A. for Aeronautics, No. 1, 1915.

design of aeroplanes. The British Government before the war accepted aeroplanes with overall factors of safety not lower than 6, but stating that the future machine accepted will require factors of safety of double this figure. The memorandum on the "Military Aeroplane," prepared in the office of the Aviation Section of the Signal Corps, U.S.A., specifies a factor of safety of 7.5 for military training aeroplanes, 6 for land pursuit machine, and 7 for land gun-carrying machines. Judge advises a load factor of 5 to allow for the maximum abnormal loading and an actual factor of safety of between 2 and 3 to allow for material strength; the overall factor of safety lying between 10 and 15.

It must be remembered, however, that there are many features opposing the employment of a high factor of safety; for instance, there will naturally result an increased weight of the machine with consequent difficulty in landing.

If the stress caused by a sharp flattening-out is to determine the load factor of an aeroplane, the designer must of necessity estimate the stresses for possible extreme cases by calculations or determine them exactly by experiments.

In the present work we start from the general equations of rigid dynamics, and the discussion will be confined to the symmetric motion of the aeroplane. The first scheme is to solve the simultaneous differential equations representing motions in a vertical plane. There are two possible methods of attack:—one to confine the motion to small oscillations, to simplify the mathematical analysis; the other to apply an approximate calculation, which shall include all the complicated conditions which enter in applying it to a definite movement.

The former method was applied by Bryan* to the dynamical investigation of stability of motion of aeroplanes and extended by Bairstow.† Bairtrow says,—"from preliminary calculations it appears that much of the analysis can be applied to problems in which the mathematical assumption that the oscillations are small is not made."

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* G.H. Bryan, "Stability in Aviation."

† Technical Report of the Committee for Aeronautics for the year 1912–1913, No. 77.

This theory is applicable with sufficient accuracy over a fairly great range of movement of the aeroplane; particularly it applies when flying at a high velocity. This does not hold good, however, in the case when the pilot attempts to recover his machine sharply from a steep dive at high speed. This problem must be solved by use of the second method, i.e.. approximate calculation. The theory of small oscillation is, however, suitably applicable to the motion when the aeroplane approaches its final steady attitude of flight from the violently disturbed condition caused by sharp manoeuvering.

As numerical examples, the method of approximate calculation will be applied to various cases of extremely sharp flattening-out of a typical military biplane tractor known as Curtiss JN2. A model with span of 18 inches representing this machine was tested in the wind tunnel of the Massachusetts Institute of Technology; and the lift, drift, and pitching moment were measured for a series of angles of incidence. The damping of the pitching oscillations was also determined experimentally. For the purpose of comparison with one case of the Curtiss plane a parallel calculation will be made of an equivalent case of another biplane tractor designed by Captain Clark. A model of the latter aeroplane, $\frac{1}{26}$ size, was tested in the same wind tunnel of the Massachusetts Institute of Technology. The investigation of the inherent longitudinal stability of Curtiss JN 2 was carried out by Dr. Hunsaker,* as a preliminary calculation to the "Discussion of the Effect of Wind Gusts" by Prof. Wilson. The longitudinal and lateral stability of the Clark plane were fully discussed by Hunsaker† and its aeronautical properties were compared to those of Curtiss JN2.

II Derivation of the Dynamical Equation of Motion.

We take the centre of gravity of the aeroplane as origin and choose three axes mutually at right angles, fixed to the aeroplane and moving

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* J.C. Hunsaker, "Experimental Analysis of Inherent Stability for a Typical Biplane," First Annual Report, U.S.A. No. 1, Part 1.

† J.C. Hunsaker "Dynamical Stability of Aeroplane," Smithsonian Miscellaneous Collection, vol. 62, No. 5.

with it in space. We use the same notation as that of Bairstow, viz:

$$ox,\ oy,\ oz, \ldots\ldots$$

moving axes directed, respectively, backward, to the left, and upward relative to the pilot;

$$u, v, w, \text{ and } p, q, r, \ldots\ldots$$

linear and angular velocitiés, resolved along these axes;

$$X, Y, Z \text{ and } L, M, N, \ldots\ldots$$

forces and moments of forces measured per unit mass of the aeroplane;

$$A, B, C, \text{ and } D, E, F, \ldots\ldots$$

moments and products of inertia relative to the moving axes fixed in the moving body;

$$h_1, h_2, h_3, \ldots\ldots$$

Components of angular momentum about moving axes;

$$m \ldots.,$$

the mass of the aeroplane. Then the dynamical equations of motion are:

$$\left.\begin{aligned} \frac{du}{dt} + wq - vr &= X \\ \frac{dv}{dt} + ur - wp &= Y \\ \frac{dw}{dt} + vp - uq &= Z \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} \frac{dh_1}{dt} - rh_2 + qh_3 &= mL \\ \frac{dh_2}{dt} - ph_3 + rh_1 &= mM \\ \frac{dh_3}{dt} - qh_1 + ph_2 &= mN \end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned} h_1 &= pA - qF - rE \\ h_2 &= qB - rD - pF \\ h_3 &= rC - pE - qD \end{aligned}\right\} \qquad (3)$$

In the case of a symmetrical aeroplane, such as commonly exists in practice, the xoz plane will be the plane of symmetry, so that

$$D = o \text{ and } F = o.$$

If the motion is supposed to be in the vertical plane of symmetry, so that the "line of flight" is confined in the vertical plane, then

$$Y=L=N=0$$

and

$$v=p=r=0.$$

The axes of reference ox, oy, and oz at any instant are supposed to coincide instantaneously with a second set fixed in the air, and u, v, w are the velocities of the centre of gravity of the machine along the axes of x, y, z relative to the latter. If we choose the direction of x-axis always tangential to the line of flight, then w would be zero.

Putting these values in the equations of motion, we have:

$$\left.\begin{aligned} \frac{du}{dt}&=X \\ -uq&=Z \\ B\frac{dq}{dt}&=mM. \end{aligned}\right\} \qquad (4)$$

In the above defined system of axes, the axes of reference in a steady horizontal flight would always remain in the same direction with the axes fixed in spase, the z-axis always being vertical. For investigation of a finite movement of an aeroplane along any line in a vertical plane in space, it is convenient to make some slight modifications of the usual system for defining the angular position of the aeroplane.

Let PoQ be a line of flight, *i.e.*, the path of the centre of gravity 0 of an aeroplane (Fig 1); ox and oz the instantaneous axes, remaining always along and normal to the path; ox' the x-axis fixed to the aeroplane, directing toward the chord of the wing; ox_0 and oz_0 the horizontal and the vertical axes fixed in space; y-axis of all set lying perpendicular to the plane of figure. Measure the inclination, τ, of ox, counter clockwise from the

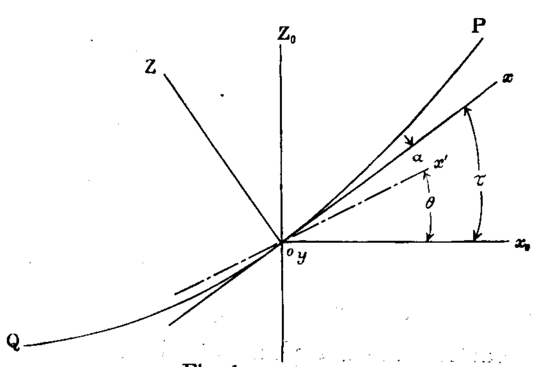


Fig. 1.

horizontal; and also measure the inclination, θ, of ox' in the same sense, so that the angle of attack, α, to the wing is:

$$\alpha = \tau - \theta.$$

In *eq.* (4) u has always a negative value and may be put

$$u = -U,$$

if U is the velocity of flight.

q represents the angular velocity about oy-axis relative to the axes fixed in space; and as the angles τ and θ are measured in the opposite sense to that which was stipulated for measurement of q, *i.e.*, left hand direction, it is put:

$$q = -\frac{d\tau}{dt}$$

$\frac{dq}{dt}$ in the last equation in (4) is, however, the rate of change of angular velocity of the aeroplane about its own transverse axis. Hence it is expressed:

$$\frac{dq}{dt} = -\frac{d^2\theta}{dt^2}$$

In the new system of axes *eq.* (4) takes the following form:

$$\left.\begin{aligned} -\frac{dU}{dt} &= X \\ -U\frac{d\tau}{dt} &= Z \\ -B\frac{d^2\theta}{dt^2} &= mM. \end{aligned}\right\} \qquad (4')$$

In the right hand side of *eq* (4′) the external forces X and Z along the instantaneous axes ox and oz and the moment of force about the oy-axis are all functions of U, τ, and θ. X and Z consist of components of the gravity force, components of the propeller thrust and components of the air resistance, along and normal, respectively, to the wind direction.

Thus:

$$\left.\begin{aligned} X &= \frac{1}{m}\Big[D - W\sin\tau - T\cos(\alpha+\beta)\Big] \\ Z &= \frac{1}{m}\Big[L - W\cos\tau + T\sin(\alpha+\beta)\Big] \end{aligned}\right\} \qquad (5)$$

where,

L = the total lift in pounds,

D = the total drift in pounds,

W = the weight of the machine in pounds,

T = the propeller thrust in pounds,

β = the angle between the axis of propeller and the chord of wing.

As is well known, the air resistance on many parts of the machine does not follow the "law of square" relative to the speed, on account of the skin friction. But, if the change of velocity be very small, we may put with sufficient accuracy,

$$\left.\begin{aligned} D&=k_x U^2 \\ L&=k_y U^2 \end{aligned}\right\} \tag{6}$$

where k_x and k_y are the drift and lift coefficients in pounds per unit speed; that is, components of the air resistance upon the whole machine, respectively, along and normal to the direction of the wind. As experiments show, these coefficients are not constant, but are certain functions of the angle of incidence α. Within certain variations of α they may, in brevity, be expressed:

$$\left.\begin{aligned} k_x&=G_0+G\,\alpha^2 \\ k_y&=H_0+H\alpha. \end{aligned}\right\} \tag{7}$$

These expressions give values close enough to the result of experiments to serve in our mathematical treatment of the problem, if the values of the constants G_0, G, H_0, and H are chosen for a small variation of α.

If we assume no change of the propeller thrust with small change in forward speed and the axis of propeller to be parallel to the wind direction, as has been done in many discussions of stability, then terms containing the thrust vanish out in equations of small oscillations, due to conditions of equilibrium. In our case here, the propeller thrust is negligibly small in comparison with the head resistance. For simplicity we assume that the motor is shut off in gliding and the machine has no thrust exerted by the propeller, as is often actually the case.

Substituting the values of (6) and (7) in *eq* (5), we have:

$$\left.\begin{aligned} X &= \frac{G_0 + G\alpha^2}{m} U^2 - g \sin \tau \\ Z &= \frac{H_0 + H\alpha}{\mathrm{m}} U^2 - g \cos \tau. \end{aligned}\right\} \tag{8}$$

For mM in *eq* (4′) we may put:

$$mM = M_e - M_q\left(-\frac{d\theta}{dt}\right) \tag{9}$$

in which M_e is the pitching moment of the aeroplane with its elevator set at a certain angle; M_q is the damping moment against pitching oscillations of the whole machine per unit pitching velocity.

We now assume a relation,

$$M_e = (M_0 + M\alpha)\, U^2, \tag{10}$$

in which M_0 and M are certain constants to be determined from experimental data.

Experimental data is scanty concerning the pitching moment of the aeroplane with the elevator set at various angles. To find the lift, drift, and pitching moment of the complete machine experiments on models with stabilizer and elevator made in one have been performed in the wind tunnel of the National Physical Laboratory, Teddington, England, and in that of the Massachusetts Institute of Technology, U. S. A. Values resulting from these experiments were plotted taking as abscissa the angle of incidence.

In a small change of the angle of incidence we can follow the curves by properly choosing the constants M_0 and M in the above expression.

In all these experiments the elevators occupied the neutral position. There may be certain washing effects of the stabilizer, when the elevator is placed at an angle relative to the former. In "Dynamical Stability of Aeroplanes" Dr. Hunsaker gives the curves of pitching moments of a complete model with the horizontal tail surface making angles of $-2.^\circ 75$, -5°, and -7° with the wing-chord. These curves show that the pitching moments vary about proportionately to the change of angle of incidence within values smaller than twelve degrees, while at higher angles the usual characteristics of pitching moments are considerably changed. This subject will be considered later in numerical calculations.

$M_q \frac{d\theta}{dt}$ is the damping moment due to wings, horizontal tail, body, and all other parts forward and aft of the centre of gravity. The damping of a surface should depend on the area of the surface, the linear velocity with which it swings through the air, and the velocity of advance; thus:

$$M_q \frac{d\theta}{dt} \propto \left[l^2 \times l \frac{d\theta}{dt} \times f(U) \right] l$$

for similar aeroplanes, l being a linear dimension. Therefore, for a particular aeroplane, we may put:

$$M_q \frac{d\theta}{dt} = N f(U) \frac{d\theta}{dt},$$

where N is a certain constant.

By tests on models it is found that the damping is proportional to U, i.e.,

$$f(U) = U$$

in the case of aeroplanes; and that N is not a real constant. The value of N for the model of the Clark plane decreases as α increases, while that of the Curtiss *JN* 2 increases with α. For this strange discrepancy we cannot get any satisfactory explanation; but in both cases the variations of N are relatively slight.

Assuming that the value of N varies proportionately to the change of α, we may put:

$$M_q \frac{d\theta}{dt} = (N_0 + N\alpha) U \frac{d\theta}{dt} \tag{11}$$

where values of constants N_0 ahd N should be determined, so as to follow the experimental results.

Substituting the valnes of (8), (9), (10), and (11) in eq (4′), we have:

$$\left.\begin{aligned} \frac{dU}{dt} &= -\frac{G_0 + G\alpha^2}{m} U^2 + g \sin\tau \\ U \frac{d\tau}{dt} &= -\frac{H_0 + H\alpha}{m} U^2 + g \cos\tau \\ \frac{d^2\theta}{dt^2} &= -\frac{M_0 + M\alpha}{B} U^2 - \frac{N_0 + N\alpha}{B} U \frac{d\theta}{dt}; \end{aligned}\right\} \tag{12}$$

or

$$\frac{1}{2} \frac{dU^2}{ds} = -\frac{G_0 + G\alpha^2}{m} U^2 + g \sin\tau$$

$$\left.\begin{aligned} U^2\frac{d\tau}{ds}&=-\frac{H_0+Ha}{m}U^2+g\cos\tau \\ U^2\frac{d^2\theta}{ds}+\frac{1}{2}\cdot\frac{dU^2}{ds}\cdot\frac{d\theta}{ds}&=-\frac{M_0+Ma}{B}U^2-\frac{N_0+Na}{B}U^2\frac{d\theta}{ds}, \end{aligned}\right\}\quad(12')$$

where s is the length of path in feet.

Putting, $U^2=V,$

eq (12′) is rewritten as follows:

$$\left.\begin{aligned} \frac{dV}{ds}&=-2\,\frac{G_0+Ga^2}{m}V+2g\sin\tau \\ \frac{d\tau}{ds}&=-\frac{H_0+Ha}{m}+g\,\frac{\cos\tau}{V} \\ \frac{d^2\theta}{ds^2}+\frac{1}{2V}\frac{dV}{ds}\cdot\frac{d\theta}{ds}&=-\frac{M_0+Ma}{B}-\frac{N_0+Na}{B}\cdot\frac{d\theta}{ds}. \end{aligned}\right\}\quad(14)$$

These are the general differential equations of symmetric motion of the aeroplane containing $V, \tau, \theta,$ and s as variables. If the general solution of these equations is obtained, the symmetric motion of an aeroplane in a vertical plane would be traced within the limits between which the assumptions made are held correct. Accordingly we can get by calculation the maximum loading on the wing at sudden flattening-out after a steep glide, and also obtain the time required.

III The First Method of Solution.

(Equations of Small Oscillations.)

If δ denotes the small variation of V, τ, and θ; then, by neglecting the small quantities of higher order, we have the following equations from *eq*(14).

$$\left.\begin{aligned} &\frac{d}{ds}(\delta V)=-2\,\frac{G_0+Ga^2}{m}(\delta V)-4\,\frac{Ga}{m}V(\delta\tau-\delta\theta)+2g\cos\tau.(\delta\tau) \\ &\frac{d}{ds}(\delta\tau)=-\frac{H}{m}(\delta\tau-\delta\theta)-g\,\frac{\cos\tau}{V^2}(\delta V)-g\frac{\sin\tau}{V}(\delta\tau) \\ &\frac{d^2}{ds^2}(\delta\theta)+\frac{1}{2V}\cdot\frac{dV}{ds}\cdot\frac{d}{ds}(\delta\theta)+\frac{1}{2V}\cdot\frac{d\theta}{ds}\cdot\frac{d}{ds}(\delta V) \\ &\qquad-\frac{1}{2V^2}\cdot\frac{d\theta}{ds}\cdot\frac{dV}{ds}\cdot(\delta V) \end{aligned}\right\}\quad(15)$$

$$= -\frac{M}{B}(\delta\tau-\delta\theta)-\frac{N_0+Na}{B}\cdot\frac{d}{ds}(\delta\theta)-\frac{N}{B}\cdot\frac{d\theta}{ds}(\delta\tau-\delta\theta)$$

These are the differential equations for small symmetric occillations, in which three variables δV, $\delta\tau$, and $\delta\theta$ are each a function of the path ds; and the three equations should be satisfied at any point of the path by a concordant set of values of δV, $\delta\tau$, and $\delta\theta$. The equations are, therefore, simultaneous and are linear differential equations without constant terms; $\frac{dV}{ds}$, $\frac{d\tau}{ds}$ and $\frac{d\theta}{ds}$ being all zero.

Writing the operator D to indicate differentiation with regard to the path or $\frac{d}{ds}$, we have:

$$\left.\begin{aligned}&\left[D+2\frac{G_0+Ga^2}{m}\right]\delta V+\left[4\frac{Ga}{m}V-2g\cos\tau\right]\delta\tau-\left[4\frac{Ga}{m}V\right]\delta\theta=0\\&\left[g\frac{\cos\tau}{V^2}\right]\delta V+\left[D+\frac{H}{m}+g\frac{\sin\tau}{V}\right]\delta\tau-\left[\frac{H}{m}\right]\delta\theta=0.\\&\left[\frac{M}{B}\right]\delta\tau+\left[D^2-\frac{M}{B}+\frac{N_0+Na}{B}D\right]\delta\theta=0.\end{aligned}\right\}\quad(16)$$

The physical condition that the three equations shall be simultaneous is expressed mathematically by equating to zero the determinant formed by the coefficient of the variables δV, $\delta\tau$, and $\delta\theta$. Thus:

$$\begin{vmatrix} D+2\frac{G_0+Ga^2}{m}, & 4\frac{Ga}{m}V-2g\cos\tau, & -4\frac{Ga}{m}V \\ g\frac{\cos\tau}{V^2}, & D+\frac{H}{m}+g\frac{\sin\tau}{V}, & -\frac{H}{m} \\ 0, & \frac{M}{B}, & D^2-\frac{M}{B}+\frac{N_0+Na}{B}D \end{vmatrix}=0 \quad (17)$$

Expanding the determinant we obtatn:

$$D^4+A_1D^3+B_1D^2+C_1D+E_1=0, \quad (18)$$

where
$$\begin{aligned}A_1&=2\frac{G_0+Ga^2}{m}+\frac{H}{m}+g\frac{\sin\tau}{V}+\frac{N_0+Na}{B}\\B_1&=2\frac{G_0+Ga^2}{m}\left[\frac{H}{m}+g\frac{\sin\tau}{V}\right]+\left[2\frac{G_0+Ga^2}{m}+\frac{H}{m}\right.\\&\left.+g\frac{\sin\tau}{V}\right]\frac{N_0+Na}{B}-g\frac{\cos\tau}{V^2}\left[g\frac{Ga}{m}V-2g\cos\tau\right]\end{aligned}$$

$$\left.\begin{aligned} C_1 &= -\left[2\,\frac{G_0+Ga^2}{m}+g\frac{\sin\tau}{V}\right]\frac{M}{B}+2\frac{G_0+Ga^2}{m}\left[\frac{H}{m}\right. \\ &\quad \left.+g\frac{\sin\tau}{V}\right]\frac{N_0+Na}{B}-g\frac{\cos\tau}{V^2}\left[4\frac{Ga}{m}V-2g\cos\tau\right]\frac{N_0+Na}{B} \\ E_1 &= -2\,\frac{G_0+Ga^2}{m}\cdot g\,\frac{\sin\tau}{V}\cdot\frac{M}{B}-g\,\frac{\cos\tau}{V^2}\cdot\frac{M}{B}\cdot 4\,\frac{Ga}{m}V \\ &\quad +g\,\frac{\cos\tau}{V^2}\left[4\,\frac{Ga}{m}V-2g\cos\tau\right]\frac{M}{B} \end{aligned}\right\} \tag{19}$$

If $\lambda_1, \lambda_2, \lambda_3$, and λ_4 be the roots of the biquadratic equation (18), the solution of the simultaneous linear differential equations (15) is of the type:

$$\left.\begin{aligned} \delta V &= C_{V1}e^{\lambda_1 s}+C_{V2}e^{\lambda_2 s}+C_{V3}e^{\lambda_3 s}+C_{V4}e^{\lambda_4 s} \\ \delta\tau &= C_{\tau1}e^{\lambda_1 s}+C_{\tau2}e^{\lambda_2 s}+C_{\tau3}e^{\lambda_3 s}+C_{\tau4}e^{\lambda_4 s} \\ \delta\tau &= C_{\theta1}e^{\lambda_1 s}+C_{\theta2}e^{\lambda_2 s}+C_{\theta3}e^{\lambda_3 s}+C_{\theta4}e^{\lambda_4 s} \end{aligned}\right\} \tag{20}$$

where C_{V1}, $C_{\tau1}$, $C_{\theta1}$, etc., are integration constants and determined by initial conditions.

The condition of stability of motion is that $\delta V, \delta\tau$; and $\delta\theta$ shall vanish as time goes on. Hence, each of the roots of the biquadratic equation must be negative if real, or, if imaginary, must have its real part negative. Bryan has shown that by use of Routh's discriminant the biquadratic need not be solved. The necessary and sufficient condition that a biquadratic equation have negative roots or imaginary roots with real parts negative, is that A_1, B_1, C_1, E_1, and $A_1B_1C_1-C_1^2-A_1^2E_1$ are each positive. Solving the biquadratic for λ_1, λ_2, λ_3, and λ_4, however, we can calculate the period and the decreament of small oscillations expressed by *eq* (20). Furthermore, to get the complete solution we have to find the values of constants C_{V1}, $C_{\tau1}$. $C_{\theta1}$, etc., by applying the initial conditions.

Substituting δV. $\delta\tau$, and $\delta\theta$ in *eq* (16), we obtain:

$$\begin{aligned} &\left[\left(\lambda_1+2\,\frac{G_0+Ga^2}{m}\right)C_{V1}+\left(4\,\frac{Ga}{m}V-2g\cos\tau\right)C_{\tau1}-\left(4\,\frac{Ga}{m}V\right)C_{\theta1}\right]e^{\lambda_1 s} \\ &\quad+\left[\left(\lambda_2+2\,\frac{G_0+Ga^2}{m}\right)C_{V2}+\left(4\,\frac{Ga}{m}V-2g\cos\tau\right)C_{\tau2}-\left(4\,\frac{Ga}{m}V\right)C_{\theta2}\right]e^{\lambda_2 s} \\ &\quad+\cdots\cdots\cdots \qquad = 0. \end{aligned}$$

$$\left[\left(g\frac{\cos\tau}{V^2}\right)C_{V1}+\left(\lambda_1+\frac{H}{m}+g\frac{\sin\tau}{V}\right)C_{\tau1}-\left(\frac{H}{m}\right)C_{\theta1}\right]e^{\lambda_1 s}$$
$$+\left[\left(g\frac{\cos\tau}{V^2}\right)C_{V2}+\left(\lambda_2+\frac{H}{m}+g\frac{\sin\tau}{V}\right)C_{\tau2}-\left(\frac{H}{m}\right)C_{\theta2}\right]e^{\lambda_2 s}$$
$$+\cdots\cdots\cdots\cdots \qquad =0.$$

$$\left[\left(\frac{M}{B}\right)C_{\tau1}+\left(\lambda_1^2-\frac{M}{B}+\frac{N_0+Na}{B}\lambda_1\right)C_{\theta1}\right]e^{\lambda_1 s}$$
$$+\left[\left(\frac{M}{B}\right)C_{\tau2}+\left(\lambda_2^2-\frac{M}{B}+\frac{N_0+Na}{B}\lambda_2\right)C_{\theta2}\right]e^{\lambda_2 s}$$
$$+\cdots\cdots\cdots\cdots \qquad =0.$$

These relations should hold identically for any value of s, and therefore the coefficient of $e^{\lambda_1 s}, e^{\lambda_2 s}$, etc., in each must vanish. Accordingly we obtain the three homogeneous equations for each three of unknowns, C_{V1}, $C_{\tau1}$, and $C_{\theta1}$, etc. Thus.

$$\left(\lambda_1+2\frac{G_0+Ga^2}{m}\right)C_{V1}+\left(4\frac{Ga}{m}V-2g\cos\tau\right)C_{\tau1}-\left(4\frac{Ga}{m}V\right)C_{\theta1}=0$$
$$\left(g\frac{\cos\tau}{V^2}\right)C_{V1}+\left(\lambda_1+\frac{H}{m}+g\frac{\sin\tau}{V}\right)C_{\tau1}-\left(\frac{H}{m}\right)C_{\theta1}=0$$
$$\left(\frac{M}{B}\right)C_{\tau1}+\left(\lambda_1^2-\frac{M}{B}+\frac{N_0+Na}{B}\lambda_1\right)C_{\theta1}=0,$$
$$\text{etc., etc.,} \tag{21}$$

The relation,

$$\begin{vmatrix} \lambda_1+2\frac{G_0+Ga^2}{m}, & 4\frac{Ga}{m}V-2g\cos\tau, & -4\frac{Ga}{m}V \\ g\frac{\cos\tau}{V^2}, & \lambda_1+\frac{H}{m}+g\frac{\sin\tau}{V}, & -\frac{H}{m} \\ 0, & \frac{M}{B}, & \lambda_1^2-\frac{M}{B}+\frac{N_0+Na}{B}\lambda_1 \end{vmatrix}=0,$$
$$\text{etc., etc..} \tag{22}$$

are consistent, because λ_1, λ_2, etc., are the roots of these determinants.

Hence the solutions of C_{V1}, C_{V2}, etc., are:

$C_{V1}: C_{\tau1}: C_{\theta1}=$

$$\begin{vmatrix} 4\frac{Ga}{m}V-2g\cos\tau, & 4\frac{Ga}{m}V \\ \lambda_1+\frac{H}{m}+g\frac{\sin\tau}{V}, & \frac{H}{m} \end{vmatrix} : \begin{vmatrix} 4\frac{Ga}{m}V, & \lambda_1+2\frac{G_0+Ga^2}{m} \\ \frac{H}{m}, & g\frac{\cos\tau}{V^2} \end{vmatrix} :$$

$$\begin{vmatrix} \lambda_1+2\dfrac{G_0+Ga^2}{m}, & 4\dfrac{Ga}{m}V-2g\cos\tau \\ g\dfrac{\cos\tau}{V^2}, & \lambda_1+\dfrac{H}{m}+g\dfrac{\sin\tau}{V} \end{vmatrix}$$

etc., etc. (23)

The general solutions of *eq* (15) are:

$$\left.\begin{aligned} \delta V &= C_{V1}e^{\lambda_1 s}+C_{V1}e^{\lambda_2 s}+C_{V3}e^{\lambda_3 s}+C_{V4}e^{\lambda_4 s} \\ \delta\tau &= P_1C_{V1}e^{\lambda_1 s}+P_2C_{V2}e^{\lambda_2 s}+P_3C_{V1}e^{\lambda_3 s}+P_4C_{V4}e^{\lambda_4 s} \\ \delta\theta &= Q_1C_{V1}e^{\lambda_1 s}+Q_2C_{V2}e^{\lambda_2 s}+Q_3C_{V3}e^{\lambda_3 s}+Q_4C_{V4}e^{\lambda_4 s}; \end{aligned}\right\} \quad (24)$$

where

$$P_1=\frac{\begin{vmatrix} 4\dfrac{Ga}{m}V, & \lambda_1+2\dfrac{G_0+Ga^2}{m} \\ \dfrac{H}{m}, & g\dfrac{\cos\tau}{V^2} \end{vmatrix}}{\begin{vmatrix} 4\dfrac{Ga}{m}V-2g\cos\tau, & 4\dfrac{Ga}{m}V \\ \lambda_1+\dfrac{H}{m}+g\dfrac{\sin\tau}{V}, & \dfrac{H}{m} \end{vmatrix}}, \quad \text{etc.};$$

$$Q_1=\frac{\begin{vmatrix} \lambda_1+2\dfrac{G_0+Ga^2}{m}, & 4\dfrac{Ga}{m}V-2g\cos\tau \\ g\dfrac{\cos\tau}{V^2}, & \lambda_1+\dfrac{H}{m}+g\dfrac{\sin\tau}{V} \end{vmatrix}}{\begin{vmatrix} 4\dfrac{Ga}{m}V-2g\cos\tau, & 4\dfrac{Ga}{m}V \\ \lambda_1+\dfrac{H}{m}+g\dfrac{\sin\tau}{V}, & \dfrac{H}{m} \end{vmatrix}}, \quad \text{etc.} \quad (25)$$

It now remains to determine the values of constants C_{V1}, C_{V2}, etc. from the initial conditions of steep glide. Let initial values of variables be:

$$(\delta V)_{s=0}=(\delta V)_0,$$
$$(\delta\tau)_{s=0}=(\delta\tau)_0,$$
$$(\delta\theta)_{s=0}=(\delta\theta)_0,$$

and $\left(\dfrac{d}{ds}d\theta\right)_{s=0}=q_0$; then

$$(\delta V)_0=C_{V1}+C_{V2}+C_{V3}+C_{V4}$$

$$
\begin{aligned}
(\delta\tau)_0 &= P_1 C_{V1} + P_2 C_{V2} + P_3 C_{V3} + P_4 C_{V4} \\
(\delta\theta)_0 &= Q_1 C_{V1} + Q_2 C_{V2} + Q_3 C_{V3} + Q_4 C_{V4} \\
q_0 &= Q_1 \lambda_1 C_{V1} + Q_2 \lambda_2 C_{V2} + Q_3 \lambda_3 C_{V3} + Q_4 \lambda_4 C_{V4}
\end{aligned}
\tag{26}
$$

The values of C_{V1}, C_{V2}, etc., are obtained by solving *eq* (26).

Thus:

$$
C_{V1} = \begin{vmatrix} (\delta V)_0, & 1, & 1, & 1 \\ (\delta\tau)_0, & P_2, & P_3, & P_4 \\ (\delta\theta)_0, & Q_2, & Q_3, & Q_4 \\ q_0, & Q_2\lambda_2, & Q_3\lambda_3, & Q_4\lambda_4 \end{vmatrix} \div \Delta
$$

$$
C_{V2} = \begin{vmatrix} 1, & (\delta V)_0, & 1, & 1 \\ P_1, & (\delta\tau)_0, & P_3, & P_4 \\ Q_1, & (\delta\theta)_0, & Q_3, & Q_4 \\ Q_1\lambda_1, & q_0, & Q_3\lambda_3, & Q_4\lambda_4 \end{vmatrix} \div \Delta
$$

$$
\text{etc.;} \tag{27}
$$

where

$$
\Delta = \begin{vmatrix} 1, & 1, & 1, & 1 \\ P_1, & P_2, & P_3, & P_4 \\ Q_1, & Q_2, & Q_3, & P_4 \\ Q_1\lambda_1, & Q_2\lambda_2, & Q_3\lambda_3, & Q_4\lambda_4 \end{vmatrix}.
$$

In general the roots of the biquadratic are of two couples of the conjugate imaginaries, thus:

$$
\begin{matrix} \lambda_1 = p + iq \\ \lambda_2 = p - iq \end{matrix} \quad \text{and} \quad \left. \begin{matrix} \lambda_3 = m + in \\ \lambda_4 = m - in \end{matrix} \right\} \tag{28}
$$

Putting these in *eq* (24), we have:

$$
\begin{aligned}
\delta V &= e^{ps}(E_V \cos qs + F_V \sin qs) + e^{ms}(J_V \cos ns + K_V \sin ns) \\
\delta\tau &= e^{ps}(E_\tau \cos qs + F_\tau \sin qs) + e^{ms}(J_\tau \cos ns + K_\tau \sin ns) \\
\delta\theta &= e^{ps}(E_\theta \cos qs + F_\theta \sin qs) + e^{ms}(J_\theta \cos ns + K_\theta \sin ns)
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
&E_V = C_{V1} + C_{V2}, && E_\tau = P_1 C_{V1} + P_2 C_{V2}, && E_\theta = Q_1 C_{V1} + Q_2 C_{V2}, \\
&F_V = i(C_{V1} - C_{V2}), && F_\tau = i(P_1 C_{V1} - P_2 C_{V2}), && F_\theta = i(Q_1 C_{V1} - Q_2 C_{V2}), \\
&J_V = C_{V3} + C_{V4}, && J_\tau = P_3 C_{V3} + P_4 C_{V4}, && J_\theta = Q_3 C_{V3} + Q_4 C_{V4}, \\
&K_V = i(C_{V3} - C_{V4}); && K_\tau = i(P_3 C_{V3} - P_4 C_{V4}); && K_\theta = i(Q_3 C_{V3} - Q_4 C_{V4}).
\end{aligned}
\tag{30}
$$

In the above analysis we built up the equations of small oscillations (15) upon the assumptions, sin $\delta\tau = \delta\tau$ and cos $\delta\tau = 1$, and also by neglecting squares and products of $\delta\tau$, $\delta\theta$, and δV. These are approximations commonly employed in treatises on theoretical mechanics, when small oscillations are concerned, in order to get linear differential equations.

Finding the roots, $\lambda_1, \lambda_2, \lambda_3$, and λ_4 of the biquadratic (18) which are deduced from the simultaneous linear differential equations (15), and then calculating the unmerical values of integration constants, C_{V1}, C_{V2}, etc., from initial conditians hy means of *eq* (27), we can obtain the general solution, *eq* (29). In these calculations the roots $\lambda_1, \lambda_2, \lambda_3$, and λ_4 must be found with considerable accuracy by trial, as was done by Wilson. The rough approximations, which were indicated by Bairstow and which will be employed in the preliminary investigation of the gliding stability later, would be insufficient for present use; because an intolerable inaccuracy would be introduced into the results of calculation of eight numerical coefficients, $C_{\tau1}, \ldots\ldots, C_{\theta1}, \ldots\ldots$, by *eq* (23), by such rough values of λ_1, etc.

As may be easily seen, in approximations made for small oscillations the assumptions, sin $\delta\tau = \delta\tau$ and cos $\delta\tau = 1$, lead to the greatest inaccuracy in results of calculation of oscillations in gliding; and consequently this limits the application of *eq* (29) to the oscillatory motion set up by a disturbance from a steady gliding. Suppose the case of an aeroplane gliding steadily at the angle τ, whose elevator is suddenly turned by such an angle, that it disturbes the equilibrium condition of motion and brings finally the machine to a new equilibrium condition gliding at angle, $\tau + 8°$. To apply *eq* (29) to the disturbed motion in this case, would cause an inaccuracy of about one per cent in assuming, cos $\delta\tau = 1$; the initial variation of τ being, $(\delta\tau)_0 = 0.1396$ radian. The variations, of V and θ are, however, so slight in such a case that the higher terms may be safely neglected.

After all, the equations of small oscillations developed above are applicable merely to the disturbed motion set up by a minute change of position of the elevator; and we come to the necessity of finding another way of approach for the solution of our problem.

IV The Second Method of Solution.

(Approximate Equations for Successive Calculations.)

By putting $U^2=V$ as before, *eq* (12′) may be written:

$$\frac{dV}{ds}=-\frac{2G_0}{m}V-2\frac{G}{m}(\tau-\theta)^2V+2g\sin\tau \tag{31a}$$

$$V\frac{d\tau}{ds}=-\frac{H_0}{m}V-\frac{H}{m}(\tau-\theta)V+g\cos\tau \tag{31b}$$

$$V\frac{d^2\theta}{ds^2}+\frac{1}{2}\frac{dV}{ds}\cdot\frac{d\theta}{ds}=-\frac{M_0}{B}V-\frac{M}{B}(\tau-\theta)V-\frac{N_0}{B}\cdot\frac{d\theta}{ds}V-\frac{N}{B}(\tau-\theta)\frac{d\theta}{ds}V \tag{31c}$$

Assuming $\tau, \theta,$ and V to be expressed as exponential series of s, thus:

$$\left.\begin{aligned}\tau&=a_0+a_1s+a_2s^2+a_3s^3+a_4s^4+\ldots\ldots\\ \theta&=b_0+b_1s+b_2s^2+b_3s^3+b_4s^4+\ldots\ldots\\ V&=c_0+c_1s+c_2s^2+c_3s^3+c_4s^4+\ldots\ldots\end{aligned}\right\} \tag{32}$$

where $a_0, a_1,\ldots\ldots$ $b_0, b_1,\ldots\ldots$ $c_0. c_1,\ldots\ldots$ are certain constants. Accordingly we have:

$$\left.\begin{aligned}\frac{d\tau}{ds}&=a_1+2a_2s+3a_3s^2+4a_4s^3+\ldots\ldots\\ \frac{d\theta}{ds}&=b_1+2b_2s+3b_3s^2+4b_4s^3+\ldots\ldots\\ \frac{d^2\theta}{ds^2}&=2b_2+6b_3s+12b_4s^2+\ldots\ldots\\ \frac{dV}{ds}&=c_1+2c_2s+3c_3s^2+4c_4s^3+\ldots\ldots\end{aligned}\right\} \tag{33}$$

and
$$\left.\begin{aligned}\sin\tau&=\sin(a_0+\delta\tau)=\sin a_0\cos\delta\tau+\cos a_0\sin\delta\tau\\ \cos\tau&=\cos(a_0+\delta\tau)=\cos a_0\cos\delta\tau-\sin a_0\sin\delta\tau,\end{aligned}\right\} \tag{34}$$

in which
$$\delta\tau=a_1s+a_2s^2+a_3s^3+\ldots\ldots;$$

so that
$$\left.\begin{aligned}\sin\tau&=\delta\tau-\frac{\delta\tau^3}{6}+\frac{\delta\tau^5}{120}\ldots\ldots\\ &=a_1s+a_2s^2+\left(a_3-\frac{1}{6}a_1^3\right)s^3+\ldots\ldots\\ \cos\delta\tau&=1-\frac{\delta\tau^2}{2}+\frac{\delta\tau^4}{24}\ldots\ldots\end{aligned}\right\} \tag{35}$$

$$=1-\frac{1}{2}a_1^2s^2-a_1a_2s^3+\cdots\cdots$$

Substituting these values in *eq* (31*a*), we have:

$$c_1+2c_2s+3c\,s^2+4c_3s^3+\cdots\cdots$$

$$=-\frac{2G_0}{m}\Big[c_0+c_1s+c_2s^2+c_3s^3+\cdots\cdots\Big]$$

$$-\frac{2G}{m}\Big[c_0(a_0-b_0)^2+\Big\{2c_0(a_0-b_0)(a_1-b_1)+c_1(a_0-b_0)^2\Big\}s$$

$$+\Big\{2c_0(a_0-b_0)(a_2-b_2)+c_0(a_1-b_1)^2+2c_1(a_0-b_0)(a_1-b_1)+c_2(a_0-b_0)^2\Big\}s^2$$

$$+\Big\{2c_0(a_0-b_0)(a_3-b_3)+2c_0(a_1-b_1)(a_2-b_2)+2c_1(a_0-b_0)(a_2-b_2)$$

$$+c_1(a_1-b_1)^2+2c_0(a_0-b_0)(a_1-b_1)+c_3(a_0-b_0)^2\Big\}s_3+\cdots\cdots\Big]$$

$$+2g\sin a_0\Big[1-\frac{1}{2}a_1^2s^2-a_1a_2s^3+\cdots\cdots\Big]$$

$$+2g\cos a_0\Big[a_1s+a_2s^2+\Big(a_3-\frac{1}{6}a_1^3\Big)s^3+\cdots\cdots\Big]$$

Equating the coefficients of the same power of s on both sides of the above equation, we get the following relations:

$$c_1=-\frac{2G_0}{m}c_0-\frac{2G}{m}c_0(a_0-b_0)^2+2g\sin a_0$$

$$2c_2=-\frac{2G_0}{m}c_1-\frac{2G}{m}\Big\{2c_0(a_0-b_0)(a_1-b_1)+c_1(a_0-b_0)^2\Big\}+2g\,a_1\cos a_0$$

$$3c_3=-\frac{2G_0}{m}c_2-\frac{2G}{m}\Big\{2c_0(a_0-b_0)(a_2-b_2)+c_0(a_1-b_1)^2+2c_1(a_0-b_0)(a_1-b_1)$$
$$+c_2(a_0-b_0)^2\Big\}-ga_1^2\sin a_0+2ga_2\cos a_0$$

$$4c_4=-\frac{2G_0}{m}c_3-\frac{2G}{m}\Big\{2c_0(a_0-b_0)(a_3-b_3)+2c_0(a_1-b_1)(a_2-b_2)$$
$$+2c_1(a_0-b_0)(a_2-b_2)+c_1(a_1-b_1)^2+2c_2(a_0-b_0)(a_1-b_1)$$
$$+c_3(a_0-b_0)^2\Big\}$$
$$-3ga_1a_2\sin a_0+2g\Big(a_3-\frac{1}{6}a_1^3\Big)\cos a_0$$

....

.... (36)

Similarly from *eq* (31b)

$$c_0a_1=-\frac{H_0}{m}c_0-\frac{H}{m}c_0(a_0-b_0)+g\cos a_0$$

$$2c_0a_2+c_1a_1$$
$$=-\frac{H_0}{m}c_1-\frac{H}{m}\Big\{(a_0-b_0)c_1+(a_1-b_1)c_0\Big\}-ga_1\sin a_0$$
$$3c_0a_3+2c_1a_2+c_2a_1$$
$$=-\frac{H_0}{m}c_2-\frac{H}{m}\Big\{(a_0-b_0)c_2+(a_1-b_1)c_1+(a_2-b_2)c_0\Big\}$$
$$-\frac{1}{2}ga_1^2\cos a_0-ga_2\sin a_0$$
$$4c_0a_4+3c_1a_3+2c_2a_2+c_3a_1$$
$$=-\frac{H_0}{m}c_3-\frac{H}{m}\Big\{(a_0-b_0)c_3+(a_1-b_1)c_2+(a_2-b_2)c_1+(a_3-b_3)c_0\Big\}$$
$$-ga_1a_2\cos a_0-g\left(a_3-\frac{1}{6}a_1^3\right)\sin a_0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots; \tag{37}$$

and from eq (31c)

$$2c_0b_2+\frac{1}{2}b_1c_1=-\frac{M_0}{B}c_0-\frac{M}{B}c_0(a_0-b_0)-\frac{N_0}{B}c_0b_1-\frac{N}{B}c_0b_1(a_0-b_0)$$
$$6c_0b_3+3c_1b_2+c_2b_1$$
$$=-\frac{M_0}{B}c_1-\frac{M}{B}\Big\{c_0(a_1-b_1)+c_1(a_0-b_0)\Big\}-\frac{N_0}{B}(2c_0b_2+c_1b_1)$$
$$-\frac{N}{B}\Big\{2c_0b_2(a_0-b_0)+c_0b_1(a_1-b_1)+b_1c_1(a_0-b_0)\Big\}$$
$$12c_0b_4+\frac{15}{2}c_1b_3+4c_2b_2+\frac{3}{2}b_1c_3$$
$$=-\frac{M_0}{B}c_2-\frac{M}{B}\Big\{c_0(a_2-b_2)+c_1(a_1-b_1)+c_2(a_0-b_0)\Big\}$$
$$-\frac{N_0}{B}\Big\{3c_0b_3+2c_1b_2+c_2b_1\Big\}$$
$$-\frac{N}{B}\Big\{3c_0b_3(a_0-b_0)+c_0b_1(a_2-b_2)+b_1c_2(a_2-b_2)+2c_1b_2(a_0-b_0)$$
$$+2c_0b_2(a_1-b_1)+b_1c_1(a_1-b_1)\Big\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{38}$$

The constants in eq (32) are, therefore, solved by means of eq (36), eq (37), and eq (38). For that purpose the initial conditions of the motion have to be determined.

Taking the starting point at the beginning of the turn from the gliding:

Let τ_0, θ_0, and V_0 be the respective values of τ, θ, and V, when $s=o$, so that

$$a_0=\tau_0\,,\quad b_0=\theta_0\,,\quad c_0=V_0$$

and

$$a_0-b_0=\alpha_0. \tag{39}$$

When the pilot turns up the elevator attempting to flatten out his machine from a steep glide, there is a sudden change of the pitching moment, and the motion is governed by the new conditions. There will be, therefore, the discontinuity of motion between the paths before and after the control.

Substituting the inital conditions (39) in *eq* (36), *eq* (37), and *eq* (38) we can get the values of constants by successive solutions. Thus:

$$c_1=-2\left[\frac{G_0+G\alpha_0^2}{m}-g\,\frac{\sin\tau_0}{V_0}\right]V_0$$

$$a_1=-\left[\frac{H_0+H\alpha_0}{m}-g\,\frac{\cos\tau_0}{V_0}\right]$$

$$c_2=-\left[\frac{G_0+G\alpha_0^2}{m}\,\frac{c_1}{V_0}+g\,\frac{G\alpha_0}{m}\,(a_1-b_1)-g\,\frac{\cos\tau_0}{V_0}\,a_1\right]V_0$$

$$a_2=-\frac{1}{2}\left[\frac{H_0+H\alpha_0}{m}.\frac{c_1}{V_0}+\frac{H}{m}\,(a_1-b_1)+g\,\frac{\sin\tau_0}{V_0}\,a_1+\frac{c_1}{V_0}\,a_1\right]$$

$$b_2=-\frac{1}{2}\left[\frac{M_0+M\alpha_0}{B}+\frac{N_0+N\alpha_0}{B}\,b_1+\frac{1}{2}\,\frac{c_1}{V_0}\,b_1\right]$$

$$c_3=-\frac{2}{3}\left[\frac{G_0+G\alpha_0^2}{m}.\frac{c_2}{V_0}+2\frac{G\alpha_0}{m}(a_2-b_2)+\frac{G}{m}(a_1-b_1)^2+2\,\frac{G\alpha_0}{m}.\frac{c_1}{V_0}(a_1-b_1)\right.$$
$$\left.+\frac{1}{2}\,g\,\frac{\sin\tau_0}{V_0}\,a_1^2-g\,\frac{\cos\tau_0}{V_0}.a_2\right]V_0$$

$$a_3=-\frac{1}{3}\left[\frac{H_0+H\alpha_0}{m}.\frac{c_2}{V_0}+\frac{H}{m}\left\{\frac{c_1}{V_0}\,(a_1-b_1)+(a_2-b_2)\right\}+\frac{1}{2}\,g\,\frac{\cos\tau_0}{V_0}.a_1^2\right.$$
$$\left.+g\,\frac{\sin\tau_0}{V_0}\,a_2+2\,\frac{c_1}{V_0}\,a_2+\frac{c_2}{V_0}\,a_1\right]$$

$$b_3=-\frac{1}{6}\left[\frac{M_0+M\alpha_0}{B}.\frac{c_1}{V_0}+\frac{M}{B}\,(a_1-b_1)+\frac{N_0+N\alpha_0}{B}\left(2b_2+\frac{c_1}{V_0}\,b_1\right)\right.$$
$$\left.+\frac{N}{B}\,b_1(a_1-b_1)+3\,\frac{c_1}{V_0}\,b_2+\frac{c_2}{V_0}\,b_1\right]$$

$$c_4=-\frac{1}{2}\Big[\frac{G_0+Ga_0}{m}\cdot\frac{c_3}{V_0}+2\frac{Ga_0}{m}\Big\{(a_3-b_3)+\frac{c_1}{V_0}(a_2-b_2)+\frac{c_2}{V_0}(a_1-b_1)\Big\}$$

$$+\frac{G}{m}\Big\{2(a_1-b_1)(a_2-b_2)+\frac{c_1}{V_0}(a_1-b_1)^2\Big\}$$

$$+g\frac{\sin\tau_0}{V_0}a_1a_2-g\frac{\cos\tau_0}{V_0}\Big(a_3-\frac{1}{6}a_1^3\Big)\Big]V_0 \quad .$$

$$a_4=-\frac{1}{4}\Big[\frac{H_0+Ha_0}{m}\cdot\frac{c_3}{V_0}+\frac{H}{m}\Big\{\frac{c_2}{V_0}(a_1-b_1)+\frac{c_1}{V_0}(a_2-b_2)+(a_3-b_3)\Big\}$$

$$+3\frac{c_1}{V_0}a_3+2\frac{c_2}{V_0}a_2+\frac{c_3}{V_0}a_1+g\frac{\cos\tau_0}{V_0}a_1a_2+g\frac{\sin\tau_0}{V_0}\Big(a_3-\frac{1}{6}a_1^3\Big)\Big]$$

$$b_4=-\frac{1}{12}\Big[\frac{M_0+Ma_0}{B}\cdot\frac{c_2}{V_0}+\frac{M}{B}\Big\{(a_2-b_2)+\frac{c_1}{V_0}(a_1-b_1)\Big\}$$

$$+\frac{N_0+Na_0}{B}\Big\{3b_3+2\frac{c_1}{V_0}b_2+\frac{c_2}{V_0}b_1\Big\}+\frac{N}{B}\Big\{b_1(a_2-b_2)$$

$$+2b_2(a_1-b_1)+\frac{c_1}{V_0}(a_1-b_1)b_1\Big\}+\frac{15}{2}\cdot\frac{c_1}{V_0}b_3+4\frac{c_2}{V_0}b_2+\frac{3}{2}\frac{c_3}{V_0}b_1\Big]$$

....

.... (40)

If the aeroplane reaches its maximum velocity during the glide, then it will advance along a straight path at that maximum velocity; and there will be no change in velocity nor in direction of glide. When this is the condition previous to the flattening-out, we have:

$$\left(\frac{d\tau}{ds}\right)_{s=0}=a_1=0,$$

$$\left(\frac{dV}{ds}\right)_{s=0}=c_1=0,$$

and

$$\left(\frac{d\theta}{ds}\right)_{s=0}=b_1=0,$$

since

$$\left(\frac{d\theta}{dt}\right)_{s=0}=\left(\frac{d\theta}{ds}\right)_{s=0}\cdot\left(\frac{ds}{dt}\right)_{s=0}$$

$$=\left(\frac{d\theta}{ds}\right)_{s=0}\cdot V_0=0.$$

In this case *cq* (40) is simplified as follows:

$a_2=0$

$c_2=0$

$$b_2=-\frac{1}{2}\,\frac{M_0+Ma_0}{B}$$

$$c_3=-\frac{2}{3}\,\frac{Ga_0}{m}.V_0\cdot\frac{M_0+Ma_0}{B}$$

$$a_3=-\frac{1}{6}\,\frac{H}{m}\cdot\frac{M_0+Ma_0}{B}$$

$$b_3=\frac{1}{6}\,\frac{N_0+Na_0}{B}\cdot\frac{M_0+Ma_0}{B}$$

$$c_4=\frac{1}{3}\left[\frac{Ga_0}{m}\left\{\frac{G_0+Ga_0^2}{m}+\frac{1}{2}\left(\frac{H}{m}+\frac{N_0+Na_0}{B}\right)\right\}\right.$$
$$\left.-\frac{1}{4}g\,\frac{\cos\tau_0}{V_0}\cdot\frac{H}{m}\right]V_0\cdot\frac{M_0+Ma_0}{B}$$

$$a_4=\frac{1}{6}\left[\frac{Ga_0}{m}\cdot\frac{H_0+Ha_0}{m}+\frac{1}{4}\,\frac{H}{m}\left(\frac{H}{m}+\frac{N_0+Na_0}{B}+g\frac{\sin\tau_0}{V_0}\right)\right]\frac{M_0+Ma_0}{B}$$

$$b_4=-\frac{1}{24}\left[\frac{M}{B}+\left(\frac{N_0+Na_0}{B}\right)^2\right]\cdot\frac{M_0+Ma_0}{B}$$

....

.... (40a)

The approximate equations (32) are applicable to any symmetric motion of the aeroplane, by choosing properly the values of a_0, b_0, c_0, and b_1. If the scheme is, for instance, to apply these equations to the investigation of the motion of a machine which dives from a horizontally propelled, straight flight at the angle of incidence a_0, by sudden shut off of the motor, the initial conditions must be taken as follows:

$$\tau_0=0, \qquad V_0=U_0^2,$$

$$\theta_0=-a_0, \qquad b_0=\left(\frac{d\theta}{ds}\right)_{s=0}=0,$$

where the value of U_0 is to be found from the equilibrium condition in the preliminary horizontal flight.

The numerical values of constants c_1, a_1, etc. in *eq* (32) are then calculated by means of *eq* (40). Thus we can follow the motion by *eq* (32) having numercial coefficients.

Applications of these equations are, however, limited within certain small fractions of a second in order to retain sufficient accuracy.

Therefore we are forced to follow the motion successively by repeating the calculations. This method seems tedious; and in fact a very laborious calculation is made in obtaining *eq* (32) for each step of repetition. But, for the sake of accuracy, this method will be applied in the present investigation. The results will thus retain much greater accuracy than those of successive calculations by equations as usually constructed under assumptions of constancy of velocity and angles during a time of slight duration.

Before entering into numerical applications, the problem of the pitching moment must be settled. As we have not sufficient experimental data for the pitching moments of aeroplanes with the elevator set at various angles, we must calculate the necessary values.

If we take the gliding at the angle of incidence 10° (this angle of incidence will be taken as the basis for comparison in all the numerical calculations herewith presented) as the final condition of disturbed motions, we shall have :—first, to find the angle of the elevator at which the aeroplane can glide steadily at the angle of incidence 10°; and then, to calculate pitching moments of the machine having the elevator set at this same angle. The curve Me in Fig. 6 is an example of a curve of pitching moments derived from experiments on the Clark plane, representing the change of the pitching moment when the horizontal tail surface is set at the angle $-5°$ relative to the wings. This arrangement of the horizontal tail was adopted by Hunsaker in his stability investigation of the same aeroplane. From this curve we find that the Clark plane with the tail set at the angle $-5°$ flying at the angle of incidence 10° has the pitching moment,

$Me = -\cdot 257$ lb. ft. per 1 ft. sec. velocity. This moment may be counterbalanced by the opposing moment produced by changing the angle of the elevator, which may be found by repeated calculations to be about 6°, so that the angle of the elevator becomes $-11°$ relative to the wings.

Taking $-11°$ as the fixed angle of the elevator we will calculate values of Me corresponding to various angles of incidence of the machine under the assumption that the "wash" of the stabilizer has no effect upon the elevator. Of course, this assumption is quite incorrect, because the flow of air would be considerably deviated after having passed over the stabilizer, so that the actual angle of incidence of the elevator to the wind would be different. Furthermore, at a higher angle of the elevator relative to the stabilizer, we may infer the generation of certain eddy currents near the corner of intersection of the two surfaces, as the leading edge of the elevator is usually situated close to the rear edge of the stabilizer. Therefore certain corrections of calculated result under such an assumption would be necessary.

The "wash" of the wings has also certain effects on the tail surfaces. By tests on wings put in tandem Eiffel found that the "wash" of the front wing would change the flow relative to the rear wing. As has been done by Klemin, we will make here corrections for deviation of stream due to "wash" of the wings to obtain the actual angle of incidence of the tail surfaces, under the assumption.

$$\text{deviation of stream} = \left(\frac{a}{2}+1\right) \text{ in degrees,}$$

where a is the angle of incidence of the wings; this empirical formula having been derived by Klemin* from data by Eiffel.

The calculations for obtaining the pitching moment under these assumptions are shown in tabular form as fallows:

TABLE A

α in degrees	γ_0 in degrees	γ_0' in degrees	$R_0=A\sqrt{K_x^2+K_y^2}$ in pound per 1 ft. sec. vel.	K_y/K_x ratio of lift to drift	l_0 in feet.	$M_0=R_0l_0$ in lb. ft. per 1 ft. sec. vel.
10	5	−1	.00251	3·3	17·78	.045
8	3	−2	.00571	5·0	18·41	.105
4	−1	−4	.01132	6·5	18·72	.212
2	−3	−5	.01277	7·1	18·82	.240
0	−5	−6	.01452	7·6	18·86	.274
−2	−7	−7	.01646	7.7	18·93	.312
−4	−9	−8	.01938	7·3	18·94	·367

* Course in Aerodynamics and Aeroplane Design, Part 1, Section 9; Aviation and Aeronautical Engineering, Dec. 1, 1916.

TABLE B

α in degrees	γ in degrees	γ' in degrees	$R=A\sqrt{K_x^2+K_y^2}$ in pound per 1 ft. sec. vel.	K_y/K_x ratio or lift to drift	l in feet	$M=Rl$ in lb. ft. per 1 ft. sec. vel.
10	−1	−7	·01646	7·7	18·62	·306
8	−3	−8	·01938	7·3	18·63	·361
4	−7	−10	·02283	5·1	18·60	·425
2	−9	−11	·02414	4·5	18·57	·448
0	−11	−12	·02536	4·1	18·55	·471
−2	−13	−13	·02660	3·9	18·58	·494
−4	−15	−14	·02792	3·6	18·57	·519

In Table A, for various angles of incidence α of the wings, shown in the first column, we have corresponding values of γ_0, the angle of the elevator relative to the horizontal when it is at the neutral position, *i·e·*, $-5°$ relative to the wing; γ_0' is the actual angle of incidence of the elevator at the same position, corrected for the deviation of the stream. The air resistances upon the elevator, $R_0=A\sqrt{K_x^2+K_y^2}$, are obtained from experimental data on flat plates,* assuming the elevator to be a flat plate with aspect ratio 3 : 1 and having the area, $A=16$ square feet. l_0 is the perpendicular distance from the centre of gravity of the aeroplane to the line of action of R_0. The values of l_0 at various angles of incidence of the machine may be obtained by means of the ratio of the lift to drift. In this case the distance between the centre of gravity of the machine and the centre of pressure on the elevator is assumed constant and to be 19 feet, neglecting small shifts of the centre of pressure due to the change of the angle of incidence of the elevator. Then multiplying together the values of R_0 and l_0, we obtain the moments contributing to the pitching moments of the machine by the elevator at the neutral position.

In Table B, γ, γ' $R, l,$ and M represent corresponding values in the case when the elevator is set at the angle $-11°$ relation to the wings

* Course in Aerodynamics and Aeroplane Design, Part 1, Section 4; Aviation, Sep. 15, 1916.

The difference of these moments,

$$M_e' = M - M_0$$

would be the change of the pitching moment of the aeroplane, owing to the change of the angle of elevator from $-5°$ to $-11°$ relative to the wings. Therefore we may obtain required values of the pitching moment of the machine with the elevator set at the angle $-11°$, by adding the values of M_e' calculated above and shown in the second column of Table *C*, to the corresponding values of M_e which are indicated in Fig 6. The third column in the same table shows these moments. If we plot these values

TABLE C.

a	M_e'	$M_e + M_e'$	$M_e + \cdot 257$
10°	·261	·004	0
8°	·256	·044	·045
4°	·213	·113	·166
2°	·208	·200	·249
0°	·197	·288	·348
−2°	·182	·303	·378
−4°	·152	·348	·454

as ordinates on the same abscissa as in Fig. 6, the curve thus obtained will be nearly a straight line and will cut the axis of abscissa near 10° of a.

However, to use these values as actual pitching moments of the aeroplane a considerable inaccuracy would be introduced into our calculations of motions, because of errors more or less serious in values of M in Table *B*, which must be expected from neglecting effects of the "wash" of the stabilizer. Hence, certain corrections should be made upon the values of M; but, without reliable experimental data, no reasonable corrections can be obtained in the present knowledge of hydromechanics. We understand that Mr. Klemin is performing arduous experiments concerning this subject. When these experiments are accomplished we shall obtain valuable data.

It may not be acceptable to use values of the pitching moment formed by adding calculated values of M_e', without correction from reliable data, to values of M_e from Fig. 2 and Fig. 6. Therefore we will add a certain constant moment (·257 lb. ft. per 1 ft. sec. velocity for Clark aeroplane and ·193 lb. ft. per 1 ft. sec. velocity for the Curtiss *JN*2) in the following numerical calculations of motions.

Figures thus obtained for the Clark machine are shown in the fourth column of Table C. These pitching moments would represent those when the stabilizer is set at a negative angle slightly higher.

Inaccuracy caused by applying these assumed values for pitching moments would be tolerable in obtaining comparative figures for the study of the behaviors of the machines in their flattening-out motions.

Tests* on a model of a tail plane with the elevator attached were performed in the National Physical Laboratory. The model used was made to a scale of $\frac{1}{16}$ full size from drawings of an aeroplane of the $BE-2_a$ pattern. The lift, drift, and pitching moment were measured at angles of inclination of the chord of the tail plane, from $-6°$ to $+6°$ by $2°$ steps, and at angles of the elevator planes to the chord of the tail plane from $-45°$ to $+45°$, by $15°$ steps. The curves of the lift, drift, and pitching moment have been plotted with the angle of incidence as abscissa, separate curves being drawn for each angle of the elevator planes.

The lift curves at angles, $0°$, $15°$, and $-15°$, of the elevator are nearly straight and parallel to each other. The drift curves and the pitching moment curves for these angles show very similar characteristics. These results of experiments give us confirmation that our assumption is by no means absurd.

V Application of Successive Calculation to Curtiss *JN*2.

The principal dimensions of the aeroplane used in the stability discussion of Hunsaker were as follows:

Weight full load 1,800 lbs.

° Technical Report of the Committee for Aeronautics for the Year 1912–13, No. 74.

Total wing area (including aelerons)	384 sq. ft.
Area of fixed tail	23 sq. ft.
Area of horizontal rudder	19 sq. ft.
Area of vertical rudder	7·8 sq. ft.
Span of wings	36 ft.
Chord of Wings	5·3 ft.
Gap of Wings	5·3 ft.
Length of body	26 ft.
Radius of gyration about *oy*-axis	5.83 ft.

Fig. 2 shows the curves of the lift, drift, pitching moment, and damping moment of this machine. These curves are plotted so as to indicate K_x, K_y, M_e, and M_q in the foregoing equations. All these quantities were computed by using the experimental data of Hunsaker.

Taking this aeroplane as an actual example, the motion at flattening-out after a steep glide will be traced by the application of successive calculations.

Before going into these, as a preliminary calculation, we must determine the conditions of motion of the aeroplane in the gliding attitude at a possible steep angle.

The features of the gliding motion are expressed by the general equations of motion (4). In a steady gliding, there is no pitching moment to swing the machine about its transverse axis, and no components of force either to deviate it from the straight path or to accelerate it along the path.

Hence, X, Z, and mM should be zero. In such a condition the pitching moment of the machine, indicated by the curve M_e in Fig. 2, should be counterbalanced by the moment created by the elevator settled at a certain definite angle.

The first two conditions give us the relations,

$$\left.\begin{aligned} D=K_xU^2=W\sin\tau \\ L=K_yU^2=W\cos\tau \end{aligned}\right\} \tag{41}$$

from which we have:

$$\frac{K_x}{K_y}=\tan\tau. \tag{42}$$

This gives the gliding angle τ at a particular angle of incidence at which the drift and lift coefficient are K_x and K_y.

In the "Official Specification for Army Hydroaeroplane of U. S. A., it is required that:—"Dive with longitudinal axis of the aeroplane at an angle at least 50° to the horizon and hold this approximate angle for between one and two seconds, then pull out reasonably quick." If the machine glides at 50° to the horizon, from *eq* (42) it should be:

$$\frac{K_x}{K_y}=\tan 50^\circ = 1.1918$$

The same value of the ratio, $\frac{K_x}{K_y}$, may be found from curves of K_x and K_y in Fig 2, at $\alpha = -.0436$ radian; so that

$$K_x = \cdot 0315$$

and

$$K_y = \cdot 02645$$

Hence, if

$$U_0^2 = \frac{W \sin\tau}{K_x} = \frac{1800 \times \cdot 766}{\cdot 0315} \doteqdot 43700$$

or

$$U_0 \doteqdot 209 \text{ feet per second}$$

or

$$\doteqdot 142{\cdot}5 \text{ miles per hour,}$$

the machine will glide steadily at this angle.

This velocity seems enormously high, and it might be difficult to maintain the steep gliding, long enough to attain such a high velocity. But, as it is well known, a gliding aeroplane has greater longitudinal stability than a horizontally propelled aeroplane.

The coefficients of the biquadratic equation (18) and Routh's discriminant at this gliding attitude of the present machine are computed by *eq* (19), from the given data; and are as follows:

$$A_1 = \cdot 0722$$
$$B_1 = \cdot 001336$$
$$C_1 = \cdot 000{,}00191$$
$$E_1 = \cdot 000{,}000{,}00183$$
$$\text{Routh's discriminant} = \cdot 000{,}000{,}171$$

These figures satisfy the conditions of longitudinal stability. In this calculation it has been assumed that the damping at this angle of incidence has no change from that at 1°, failing the existence of experimental data at such a low angle of incidence; and the same assumption will be made in our future calculations. This assumption will not, however, induce any material error in the result, for the change of the damping would be slight in the vicinity of 0° of the angle of incidence.

The numerical biquadratic equation here is:

$$D^4+\cdot 0722D^3+\cdot 001336D^2+\cdot 000{,}00191D+\cdot 000{,}000{,}00183=0.$$

Multiplying each term of this equation by U^4 or $(U_0+\delta U)^4$, we have:

$$(D\delta U)^4+.0722\,(U_0+\delta U)(D\delta U)^3+\cdot 001336\,(U_0+\delta U)^2((D\delta U)^2$$
$$+\cdot 000{,}00191\,(U_0+\delta U)^3(D\delta U)+\cdot 000{,}000{,}00183\,(U_0+\delta U)^4=0$$

or
$$D'^4+\cdot 0722\,(U_0+\delta U)D'^3+\cdot 001336\,(U_0+\delta U)^2D'^2$$
$$+\cdot 000{,}00191\,(U_0+\delta U)^3D'+\cdot 000{,}000{,}00183(U_0+\delta U)^4=0,$$

where D' is the new operator which indicates differentiation with regard to time, so that

$$DU=\frac{d}{ds}\cdot\frac{ds}{dt}=\frac{d}{dt}=D'.$$

Neglecting small variation of U_0, we have approximately:

$$D'^4+15\cdot 1D'^3+58\cdot 4D'^2+17\cdot 5D'+3\cdot 49=0.$$

As Bairstow has done, the biquadratic equation is factored approximately as follows:

$$\left[D'^2+15\cdot 1D'+58\cdot 4\right]\left[D'^2+\left(\frac{17\cdot 5}{58.4}-\frac{15\cdot 1\times 3\cdot 49}{58\cdot 4^2}\right)D'+\frac{3\cdot 49}{58\cdot 4}\right]=0.$$

The first factor reduces to

$$D'=-7\cdot 55\pm 1\cdot 225\ i,$$

where $i=\sqrt{-1}$.

This represents a short oscillation of:

$$\text{period}=\frac{2\pi}{1\cdot 225}=5\cdot 13\ \text{seconds};$$

and the amplitude is damped to one half in:

$$\text{time}=\frac{\log_e 2}{7\cdot 55}=\cdot 092\ \text{second}.$$

This oscillation dies out so rapidly that it may be left out of consideration.

The second factor reduces to:

$$D'^2+\cdot2835D'+\cdot0598=0$$

$$D'=-\cdot1418\pm\cdot199\,i.$$

This represents a long oscillation of:

$$\text{period}=\frac{2\pi}{\cdot199}=31\cdot6 \text{ seconds},$$

and $$\text{time to damp 50 per cent}=\frac{\log e2}{.1418}=4\cdot9 \text{ seconds}.$$

The long oscillations, which form the important factor in stability problems, are here more easily and shortly damped out than in the horizontal flight at high speed (compare with figures in Hunsaker's stability discussion). Thus we have assured ourselves that this aeroplane can glide very steadily at the angle 50° at full speed. Taking as the initial condition this glide at the angle 50° in the equilibrium condition, we will calculate the motions at flattening-out caused by two different controls, as will be presented in Case I and Case II.

Case I.—Suppose the elevator suddenly turned up to such a position that the aeroplane shall come to its new steady flight at the angle of incidence of 10°. Taking as the starting point the instant at which the equilibrium condition of glide is broken by a control, we have the initial values of V, τ, θ, and $\left(\frac{d\theta}{ds}\right)$,

$$V_0=43700,$$

$$\tau_0=\cdot8727,$$

$$\theta_0=\cdot9163,$$

and $$\left(\frac{d\theta}{ds}\right)_{s=0}=0$$

By putting in *eq* (40a) the values:

$$G_0+Ga_0^2=\cdot0304+\cdot579\times(-\cdot0436)^2=\cdot0315$$

$$H_0+Ha_0=\cdot104+1\cdot778\times(-\cdot0436)=\cdot02645$$

$$M_0+Ma_0=\cdot2776+(-\cdot513)\times(-\cdot0436)=\cdot300$$

$$N_0+Na_0=72\cdot7+0$$

which are picked out from curves in Fig. 2, and taking:

$$m=\frac{1800}{32{\cdot}2}=55{\cdot}9$$

$$B=55{\cdot}9\times(5{\cdot}83)^2=1900\,;$$

we can evaluate constants of *eq* (32), thus:

$$\tau=\cdot 8727-\cdot 000{,}00084\,s^3+\cdot 000{,}000{,}01484\,s^4+\cdots\cdots$$

$$\theta=\cdot 9163-\cdot 000{,}079\,s^2+\cdot 000{,}001008\,s^3-\cdot 000{,}000{,}00788\,s^4+\cdots\cdots$$

$$V=43700+\cdot 00204\,s^3-\cdot 000{,}0457\,s^4+\cdots\cdots \qquad (32_1)$$

By putting serial values of s in feet in these equatians we get the corresponding values of τ, θ, and V.

s	τ	θ	V
10	·8720	·9093	43700
20	·8684	·8915	43709
30	·8620	·8662	43718
....			
....			

We carry the calculation to only 30 feet of s during the first stage of solution of *eq* (32_1) in order to retain great accuracy; and have:

$$V_0=43718,$$

$$\tau_0=\cdot 8620,$$

$$\theta_0=.8662,$$

$$a_0=-\cdot 0042,$$

and

$$\left(\frac{d\theta}{ds}\right)_{30}=-\cdot 000{,}079\,(2\times 30)+\cdot 000{,}001008\,(3\times 30^2)$$

$$-\cdot 000{,}000{,}00788\,(4\times 30^3)+\cdots\cdots$$

$$\doteqdot-\cdot 00286$$

as initial conditions of fhe second stage of calculation.

Similarly we can get the numerical equations for the second stage. Thus:

$$\tau=\cdot 8620-\cdot 00124\,s-\cdot 000{,}02532\,s^2+\cdot 000{,}000{,}0843\,s^3+\cdots\cdots$$

$$\theta=\cdot 8662-\cdot 00286\,s-\cdot 000{,}0187\,s^2+\cdot 000{,}000{,}347\,s^3+\cdots\cdots$$

$$V=33718+\cdot 456\,s-\cdot 01612\,s^2-\cdot 00492\,s^3+\cdots\cdots \qquad (32_2)$$

Continuing the same way we get the successive amounts of $\tau, \theta,$ and V, corresponding to serial values of s. These are tabulated in Table I. The fifth column in the same table gives the velocity U in feet per second at each point.

From the relation, $t=\int \frac{ds}{U}$, we can get the time elapsed on running a length s, by integrating $\frac{1}{U}$ with regard to s. This time t is given in the last column of Table I. Curves of these values are shown in Fig. 3, in which points are plotted indicating these values for each value of s.

Case II.—Let us make a slight modification upon control in Case I; supposing that the elevator on its way to full control is settled for a while in an intermediate position at which a righting moment, say $M_e = \cdot 193$ lb. ft., is caused to arise instead of full moment ·3 lb. ft., and after 30 feet run it is turned up to the final position.

In this case the values of $\tau, \theta, U,$ and t are calculated in the same way as in the previous case, except that the pitching moment in this case is developed in the machine in two steps. The calculated results are shown in Table II, and are plotted in Fig. 4.

As will be seen in Fig 3 and Fig 4, the aéroplane has great residual velocities when the path has come into the horizon, such that:

$$U = 191{\cdot}8 \text{ ft. per sec. in Case I,}$$
$$U = 191{\cdot}9 \text{ ft. per sec. in Case II.}$$

Hence, after that, the aeroplane mill climb up on account of its own kinetic energy, if the elevator is left in the same position. Indeed, we may find the fact that the machine has its velocity of 111 ft. per sec., even when it has stalled high up to $\tau = -1{\cdot}0821$ or $\doteqdot -62°$, in case II. It would be then on the brink of danger. If the elevator be left untouched the aeroplane would be capsized, because the decrease of gravity component along the normal to the path makes the stall much easier. On the other hand, this velocity seems not great enough to lead the machine to a smooth somersault. But study of these conditions are out of the scope of the present work.

At angle of incidence of 10°, the aeroplane has the lift and drift coefficients, $K_x = \cdot 0621$ and $K_y = \cdot 378$;

and consequently the gliding angle corresponding to this angle of incidence is:

$$\tau_f = \tan^{-1}\frac{K_x}{K_y} = \cdot 1627 \text{ radian,}$$

and $\theta_f = \cdot 1627 - \cdot 1745 = -\cdot 0118$ radian, where θ_f is the angle of the longitudinal axis of the aeroplane relative to the ground in this glade.

If we take the instant at which the longitudinal axis arrives first at this angle, $\theta_f = -\cdot 0118$, as the datum for comparison of time T required for flattening-out, then we can read this time as follows:

$T = 1{\cdot}57$ seconds. in Case I.

$T = 1{\cdot}65$ seconds. in Case II.

Of course, the time which elapsed throughout the entire oscillatory motion until the machine come again to its final steady condition is much longer.

The loading on the wings through the motion is calculated by means of the second equation in (12′). Since $\frac{d\tau}{ds}$ in the equation represents the curvature at any point of the line of flight, $-mU^2\frac{d\tau}{ds}$ indicates the centrifugal force due to the curved path. Hence the expression,

$$(H_0 + Ha)\,U^2 = -m\frac{d\tau}{ds}U^2 + mg\cos\tau,$$

gives the values of the total loading on the wings in pounds, which resists the centrifugal force and the gravity component normal to the path.

As is seen in Fig 3 and Fig 4, the increase of α, or $\tau-\theta$, is very slow after its rapid change in the first short run, while the velocity falls off quickly for further run after passing its maximum point. Accordingly the maximum point of the lift can easily be found by plotting a few points indicating the product of α and U^2 for a short length of path; or more accurately by plotting the value of the actual lift which may be picked up from the lift curve in Fig. 2.

The maximum loadings L thus obtained for two cases are as follows:

Case I, at $s \doteqdot 180$ feet,

$$L = K_y U^2 = \cdot 208 \times 42180 = 8773 \text{ pounds};$$

Case II, at $s \doteqdot 200$ feet

$$L = K_y U^2 = \cdot 205 \times 42025 = 8615 \text{ pounds.}$$

And the load facter R are computed as follows:

$$R = \frac{8773}{1800} \doteqdot 4\cdot 9 \quad \text{in Case I}$$

$$R = \frac{8615}{1800} \doteqdot 4\cdot 8 \quad \text{in Case II.}$$

Case III.—We here consider the flattening-out of the same aeroplane from gliding at the angle 60° for sake of comparison with Case I and Case II.

For steady glide the angle of incidence must be:

$$a = -\cdot 048 \text{ radian.}$$

so that
$$\frac{K_x}{K_y} = \frac{\cdot 0319}{.0184} = 1\cdot 7321 \doteqdot \tan 60° ;$$

and
$$U^2 = \frac{1800}{\cdot 0319} \doteqdot 4800$$

or
$$U = 220\cdot 9 \text{ feet per second.}$$

The biquadratic equation in this case is:

$$D^4 + \cdot 07191 D^3 + \cdot 001326 D^2 + \cdot 000{,}00192 D + \cdot 000{,}000{,}000{,}234 = 0,$$

and
$$\text{Routh's discriminant} = \cdot 000{,}000{,}000{,}178.$$

We know, therefore, that the aeroplane in this condition has a stability not greatly different from that of the glide at angle 50°. Assuming the same angle of the elevator after the control, as in Case I and in case II, the results of calculation are shown in Table III, and illustrated in Fig. 5.

On the same datum of comparison of time as before, we obtain:

$$T = 1\cdot 765 \text{ seconds.}$$

Maximum loading on the wings is also obtained, at $s = 260$ feet to be:

$$L = K_y \times U^2 = \cdot 211 \times 45736 = 9650 \text{ pounds,}$$

so that
$$R = \frac{9650}{1800} \doteqdot 5\cdot 4$$

VI Application of Successive Calculation to the Clark Plane.

In his "Dynamical Stability of Aeroplanes" Hunsaker says:—"The curtiss machine is a practical aeroplane with powerful control, which does not pretend to possess any particular degree of stability. The Clark aeroplane, on the other hand, was designed to be inherently stable while departing as little as possible from the lines of the ordinary military aeroplane as typified by the Curtiss *JN*2." It will be interesting to compare these two aeroplanes as regards their flattening-out motion under similar conditions.

The principal dimensions of the Clark aeroplane are as follows:

Whole weight	1600 lb.
Total wing area (including aelerons)	464 sq. ft.
Area of Siabilizer	16·1 sq. ft.
Area of elevator	16·0 sq. ft.
Area of vertical rudder	9·35 sq. ft.
Span of wings	41 ft.
Gap of wings	6·37 ft.
Length of body	24·5 ft.
Radius of gyration about *oy*-axis	4·65 ft.

Fig. 6 gives the necessary data for our calculation to the same scale as those in Fig. 2.

When this aeroplane is steadily gliding at angle 50°, the angle of incidence will be:

$$a = -\cdot 0378 \text{ radian},$$

at which

$$K_x = \cdot 03814$$

$$K_y = \cdot 032,$$

and

$$U^2 = \frac{1600 \times 7\cdot 66}{\cdot 03814} \div 32100$$

or

$$U = 179 \text{ feet per second.}$$

The biquadratic equation here is:

$$D^4 + \cdot 131 D^3 + \cdot 00419 D^2 + \cdot 000{,}01085 D + \cdot 000{,}000{,}0111 = 0,$$

and Routh's discriminant$=\cdot000,000,00565$.

Substituting the new operator D' which indicates differentiation with regard to time t, we have:

$$D'^4+23\cdot45D'^3+134\cdot6D'^2+62\cdot4D'+11\cdot42=0,$$

which is divided into two factors approximately:

$$\left[D'^2+23\cdot45D'+134\cdot6\right]\left[D'^2+\left(\frac{62\cdot4}{134\cdot6}-\frac{23\cdot45\times11\cdot42}{134\cdot6^2}\right)D'+\frac{11\cdot42}{134\cdot6}\right]=0$$

or $$[D'^2+23\cdot45D'+134\cdot6][D'^2+\cdot4488D'+\cdot0849]=0.$$

The short oscillations are not important. The long oscillations which are indicated by the second factor in the above equation have:

$$\text{period}=\frac{2\pi}{\cdot1863}=33\cdot7 \text{ seconds},$$

and

$$\text{time to damp out 50 percent}=\frac{\log_e 2}{\cdot2244}=3\cdot1 \text{ seconds}.$$

We see that the Clark machine in this glide is much more stable dynamically than the Curtiss JN2 under the same condition.

Case IV.—Suppose this aeroplane gliding at angle 50° to be recovered by the same control as in Case I of the Curtiss JN2.

The flattening-out motion obtained by the method of successive calculations is shown in Table IV, and these are plotted on curves in Fig. 7.

At angle of incidence of 10°, this machine has:

$$K_x=\cdot0628$$
$$K_y=\cdot5074$$

so that $$\tau_f=\tan^{-1}\frac{\cdot0628}{\cdot5074}=\tan^{-1}\cdot1238=\cdot1232 \text{ radian},$$

and $$\theta_f=\cdot1232-\cdot1745=-\cdot0513 \text{ radian}.$$

The time required for the machine to arrive at this θ_f is:

$$T=1\cdot614 \text{ second}.$$

The maximum wing loading, arising at $s=200$ feet, is:

$$L=\cdot219\times30072=6586 \text{ pounds},$$

and the load factor here is:

$$R = \frac{6586}{1600} \doteqdot 4{\cdot}1.$$

VII Summary.

To trace the actual path of the centre of gravity of the aeroplane will facilitate the comparison of various motions. To do this we assume that a small length of path coincides with a circular arc of radius equal to the radius of curvature of the actual path at the point concerned. This assumption may be good enough in our case, if special care is taken in plotting the first part of the path, in which the change of curvature is appreciable. At a short distance from the starting point, the curvature of the path becomes nearly constant. The chordal length of each segment of the path is expressed by $2r \sin \dfrac{\Delta\tau}{2}$, in which r is the radius of curvature and $\Delta\tau$ the small angle at the centre of curvature suspended on the segment. Taking the horizontal and vertical axes with origin at the position of the centre of graviry at start, we can get the rectangular co-ordinates of any end-point of the segment by adding the projections of all the segments previous to it upon these axes. The results of calculations are shown in Fig. 8.

For the benefit of an obvious comparison, the results obtained above in various cases are concentrated in the following table:

	H feet	S feet	T second	R
Case I } Curtiss $JN2$	153	321	1·57	4·9
Case II } Curtiss $JN2$	162	338	1·64	4·8
Case III } Curtiss $JN2$	204	378	1·765	5·4
Case IV Clark Plane	123	286	1·614	4·1

where

H is the maximum vertical distance of the centre of gravity from its starting point, in the first flattening-out of flight;

S is the length of path traversed by the centre of gravity, until the longitudinal axis of the machine first assumes its final inclination relative to the ground;

T is the time elapsed for running the length S mentioned above;

R is the ratio of the maximum wing loading through the motion to the normal wing loading in the horizontal flight.

The flattening-out in Case II of the Curtiss *JN*2 is slower and less sharp than in Case I; and, consequently the wings are slightly relieved from the excessive loading in Case I. These differences in results were caused by a slight modification of control. If the recovering of the aeroplane from a steep glide is carried on more gently by a gradual turning of the elevator, the excessive loading on the wings would be considerably diminished. Moveover, by a gentle control, the greater part of the excess velocity dies out through the longer run, until the machine comes to its first flattened flight; and consequently the swinging up of the machine would be considerably diminished. As a result, a violent oscillation or capsizing may be averted.

Even in such a steep gliding as in Case III the flight may be easily flattened-out by a careful control without imposing upon the wings a load in excess of that allowed by the factor of safety. Of course, the vertical distance H of the fall of the machine through the sharp flattening-out is increased by the gradual turning of the elevator. Also, this height is much longer when the initial gliding is steaper. There is, therefore, some necessary allowance in altitude previous to the manoeuvering, corresponding to particular conditions.

The Clark aeroplane flattens out in a considerably shorter distance of H, and the wing loading through the motion is much lighter than that of Curtiss *JN*2 under the same conditions. Thus the results are all favorable to the Clark aeroplane.

Loading on the wings of the Clark machine is light on account of its low velocity throughout the manoeuvering, the sharpness of its longitudinal turning being rather superior to that in other cases. The lower gliding velocity at a certain angle is obtained by the greater head resistance or by a poor lift-drift ratio in the corresponding angle of incidence. The quick recovering from a steep dive, on the other hand, is obtained by a rapid increase of the lift with angle of incidence.

By comparing the lift and drift curves of the Clark aeroplane in Fig. 7 with those of the Curtiss *JN*2, we find that the drift coefficient of the former machine is considerably greater than that of the latter, especially in lower angles of incidence; and that the lift curve of the former is much steeper than that of the latter. The steepness of the lift curve and the greater drift coefficient are both effected by the large area of wings, or, in other words, by the light wing load per unit area at normal flight.

The wing loads per unit area of the two machines are as follows:

$$\frac{1800}{384}=4{\cdot}69 \text{ pounds per sq. ft.}$$

for Curtiss *JN*2

$$\frac{1600}{464}=3{\cdot}45 \text{ pounds per sq. ft.}$$

for Clark plane.

The former is about the average for biplanes in present practice, and the latter is much smaller. Thus, light loading corresponding with large wing areas give a larger margin of safety against excessive loadings due to manoeuvering. On the other hand, the greater head resistance due to large wing areas necessitates in general large engine powers to fly at a given speed.

TABLE I.

s feet	τ radian	θ radian	V (ft./sec.)^2	U ft./sec.	$\frac{1}{U}$ sec./ft.	t seconds
0	·8727	·9163	43700	209·05	·004783	0
20	·8684	·8915	43709	209·06	·004782	·09566
40	·8472	·8361	43718	209·09	·004782	·19130
60	·8050	·7734	43680	209·00	·004784	·28696
80	·7537	·7104	43591	208·78	·004789	·38268
100	·6968	·6479	43467	208·49	·004797	·47854
120	·6369	·5882	43240	207·94	·004808	·57458
140	·5785	·5241	42962	207·24	·004824	·67090
160	·5180	·4641	42597	206·44	·004844	·76758
180	·4615	·4042	42181	205·38	·004869	·86470
200	·4015	·3441	41701	204·21	·004896	·96234
220	·3432	·2847	41180	202·93	·004927	1·06056
240	·2841	·2257	40586	201·46	·004964	1·15946
260	·2250	·1659	39910	199·77	·005006	1·25916
280	·1666	·1074	39177	197·91	·005053	1·35974
300	·1090	·0484	38390	195·93	·005104	1·46130
320	·0505	−·0091	37570	193·75	005161	1·56394
340	−·0072	−·0680	36670	191·49	·005221	1·66776
360	−·0638	−·1237	35760	189·10	·005288	1·77284
380	−·1204	−·1810	34820	186·60	·005359	1·87930

TABLE II.

s feet.	τ radian.	θ radian.	V(ft./sec.)^2	U ft./sec.	$\frac{1}{U}$ sec./ft.	t seconds
0	·8727	·9163	43700	209·05	·004783	0
20	·8699	·8976	43706	209·06	·004783	·09566
40	·8551	·8594	43725	209·11	·004782	·19132
60	·8210	·8060	43743	209·15	·004781	·28696
80	·7768	·7460	43685	209·01	·004784	·38260
100	·7262	·8653	43551	208·68	·004791	·47834
120	·6735	·6241	43349	208·30	·004800	·57424
140	·6169	·5657	43093	207·59	·004816	·67040
160	·5599	·5078	42789	206·85	·004834	·76690
180	·5024	·4488	42440	206·01	·004854	·86378
200	·4452	·3899	42025	205·00	·004878	·96110
220	·3874	·3224	41559	203·86	·004905	1·05892
240	·3290	·2732	41027	202·55	·004937	1·15734
260	·2715	·2157	40429	201·07	·004973	1·25644
280	·2127	·1578	39758	199·46	·005013	1·35630
300	·1538	·1000	39050	197·61	·005060	1·45702
320	·0974	·0421	38283	195·66	·005111	1·55872
340	·0380	−·0180	37463	193·55	·005167	1·66150
360	−·0174	−·0749	36600	191·25	·005228	1·76544
380	−·0737	−·1326	35664	188·85	·005295	1·87066
400	−·1291	−·1885	34696	186·27	·005368	1·97728

TABLE III.

s feet	τ radian	θ radian	V (ft./sec.)^2	U ft./sec.	$\frac{1}{U}$ (sec./ft.)	t second
0	1·0472	1·0952	48800	220·91	·004526	0
20	1·0430	1·0701	48814	220·94	·004524	·09050
40	1·0220	1·0164	48853	221·00	·004523	·18098
60	·9787	·9526	48836	220·99	·004524	·27144
80	·9282	·8899	48779	220·81	·004527	·36194
100	·8720	·8296	48670	220·61	·004532	·45252
120	·8177	·7710	48517	220·27	·004539	·54322
140	·7580	·7065	48320	219·82	·004549	·63410
160	·7000	·6447	48057	219·22	·004561	·72520
180	·6372	·5841	47724	218·46	·004578	·81658
200	·5789	·5232	47314	217·52	·004597	·90832
220	·5190	·4625	46830	216·40	·004620	1·00048
240	·4561	·4014	46322	215·23	·004646	1·09314
260	·3970	·3375	45736	213·87	·004675	1·18634
280	·3359	·2778	45102	212·37	·004708	1·28016
300	·2760	·2170	44420	210·76	·004744	1·37468
320	·2162	·1578	43679	209·00	·004784	1·46996
340	·1541	·0989	42870	207·01	·004830	1·56610
360	·0982	·0402	42008	205·00	·004878	1·66318
380	·0404	−·0186	41093	202·71	·004933	1·76128
400	−·0181	−·0771	40125	200·31	·004992	1·86052
420	−·0756	−·1354	39100	197·74	·005057	1·96100
440	−·1337	−·1938	38031	195·02	·005127	2·06284

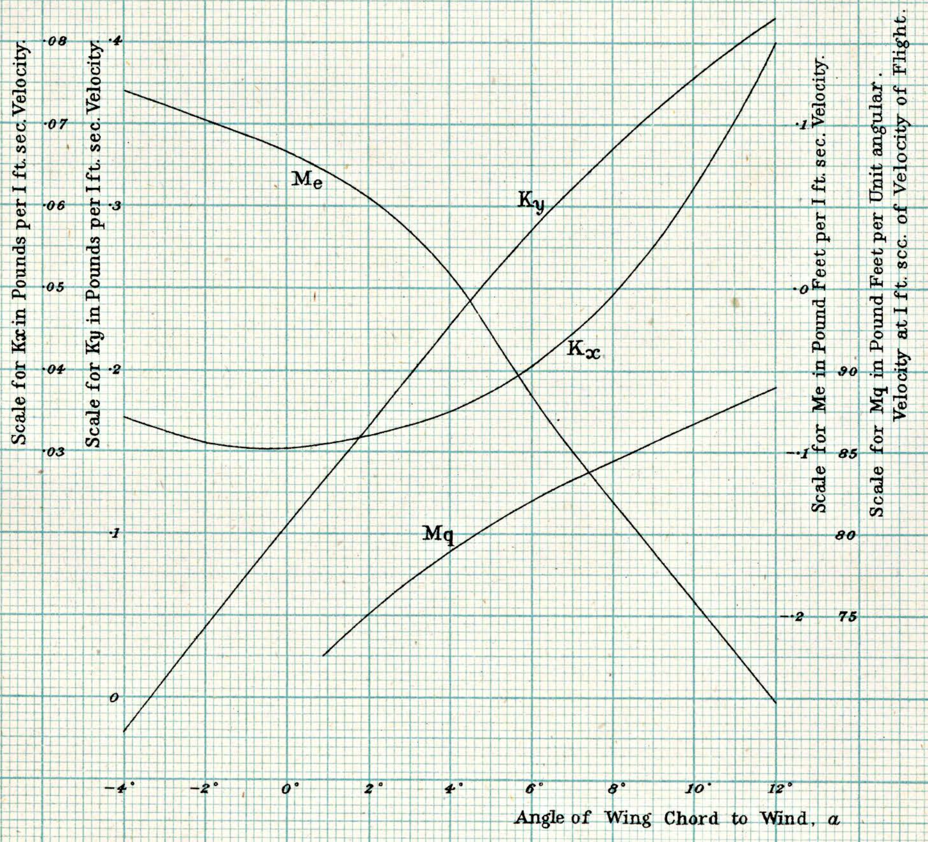
TABLE IV.

s feet	τ radian	θ radian	V(ft./sec.)^2	U ft./sec.	$\frac{1}{U}$ sec./ft.	t second
0	·8727	·9105	32100	179·16	·005581	0
20	·8594	·8617	32139	179·27	·005578	·11160
40	·8102	·7935	32174	179·37	·005575	·22312
60	·7485	·7219	32143	179·31	·005576	·33462
80	·6836	·6520	32056	179·03	·005585	·44622
100	·6165	·5841	31898	178·60	·005598	·55804
120	·5480	·5142	31672	177·97	·005618	·67020
140	·4778	·4447	31373	177·12	·005645	·78282
160	·4088	·3778	31004	176·08	·005678	·89604
180	·3422	·3097	30576	174·86	·005718	1·01000
200	·2766	·2401	30072	173·41	·005766	1·12484
220	·2065	·1726	29480	171·70	·005823	1·24072
240	·1375	·1046	28840	169·82	·005888	1·35782
260	·0735	·0371	28129	167·72	·005962	1·47632
280	·0065	−·0319	27361	165·41	·006045	1·59638
300	−·0637	−·1004	26508	162·81	·006141	1·71824
320	−·1301	−·1686	25614	160·04	·004248	1·84212

TABLE V.

	Curtiss $JN2$						Clark	
	Case I		Case II		Case III		Case IV	
s	x	y	x	y	x	y	x	y
10	6·444	7.647	6·437	7·651	5·018	8·650	6·517	7·585
20	19·364	22·913	19·335	22·937	15·090	25·928	19·575	22·735
40	32·606	37·893	32·459	38·029	25·522	42·992	33·365	37·223
60	46·470	52·309	46·089	52·667	36·682	59·590	48·019	54 835
80	61·054	66·037	60·355	66·687	48·668	75·602	63·525	63·467
100	76·394	78·871	75·309	79·967	61·534	90·914	79·843	75·029
120	92·474	90 765	90·944	92·441	75·212	105·506	96·915	85·447
140	109·220	101·701	107·258	104·009	89·736	119·256	114·649	94·643
160	126·596	111·605	124·204	114·631	105·032	132·140	133·001	102·593
180	144·502	120·511	141·732	124·249	121·112	144·040	151·843	109·301
200	162·912	128·327	159·784	132·861	137·854	154·982	171·083	114·763
220	181·746	135·057	178·302	140·417	155·220	164 902	190·657	118·865
240	200·944	140·661	197·230	146·879	173·176	173·712	210·469	121·607
260	220·440	145·123	216·496	152·243	191·620	181·446	230·415	123·077
280	240·164	148·435	236·044	156·467	210·502	188·040	250·415	123·207
300	260·046	150·611	255·808	159·531	229·746	193·490	270·377	121·933
320	280·020	151·619	275·714	161·475	249·280	197·780	290·207	119·339
340	300·020	151·475	295·700	162·235	269·042	200·850		
360	319·980	150·199	315·696	161·887	288·946	202·810		
380	339·836	147·797	335·646	160·413	308·930	203·618		
400			355·488	157·839	328·926	203·256		
420			375·144	154·145	348·768	201·744		
440					368·590	199·078		

Fig. 2. Curves Kx, Ky, Me, and Mq for Curtiss $JN2$.



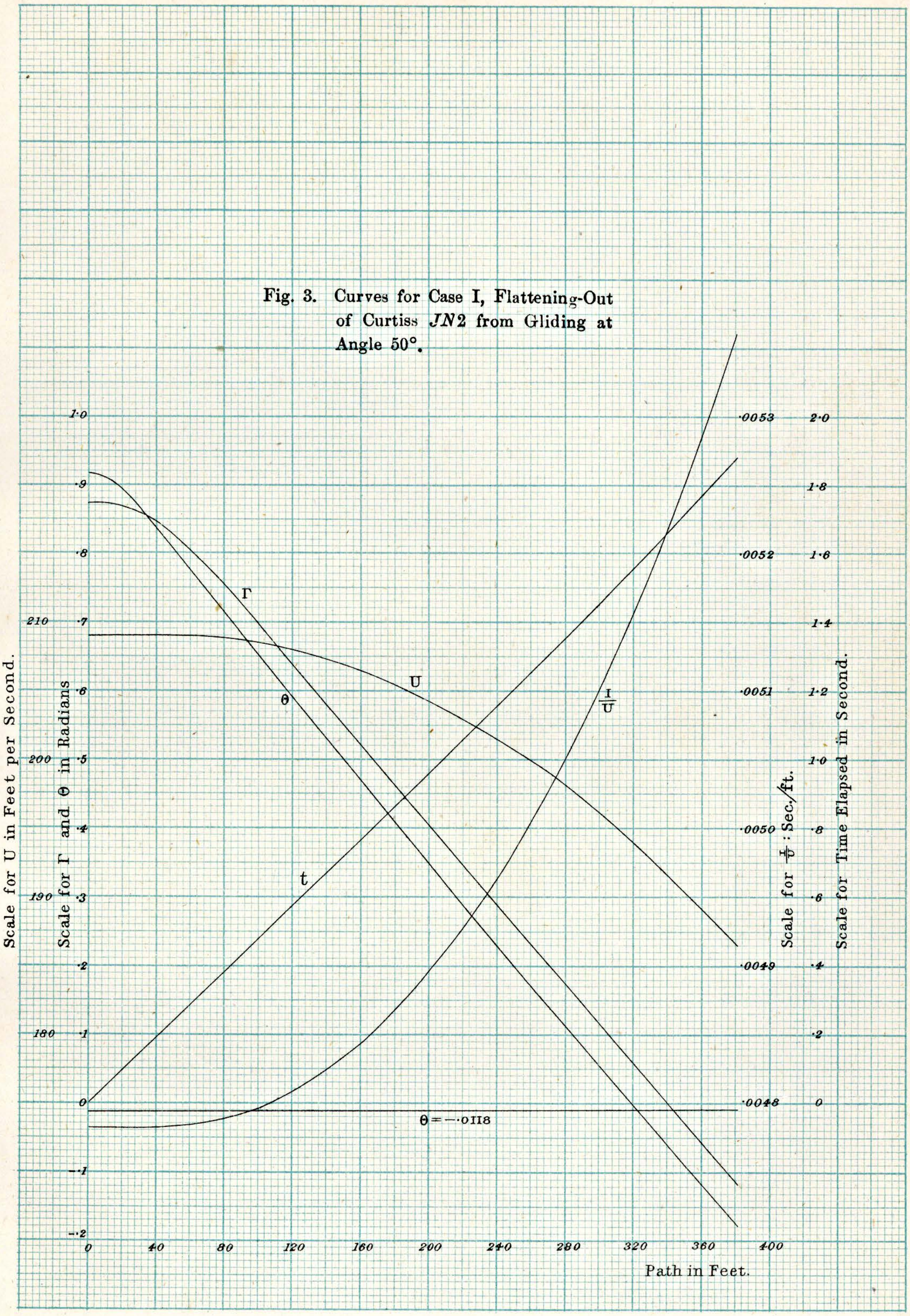


Fig. 3. Curves for Case I, Flattening-Out of Curtiss JN2 from Gliding at Angle 50°.
Scale for U in Feet per Second.
Scale for Γ and Θ in Radians
Scale for 1/U : Sec./ft.
Scale for Time Elapsed in Second.
Path in Feet.
Γ
θ
U
t
1/U
θ = −·0118

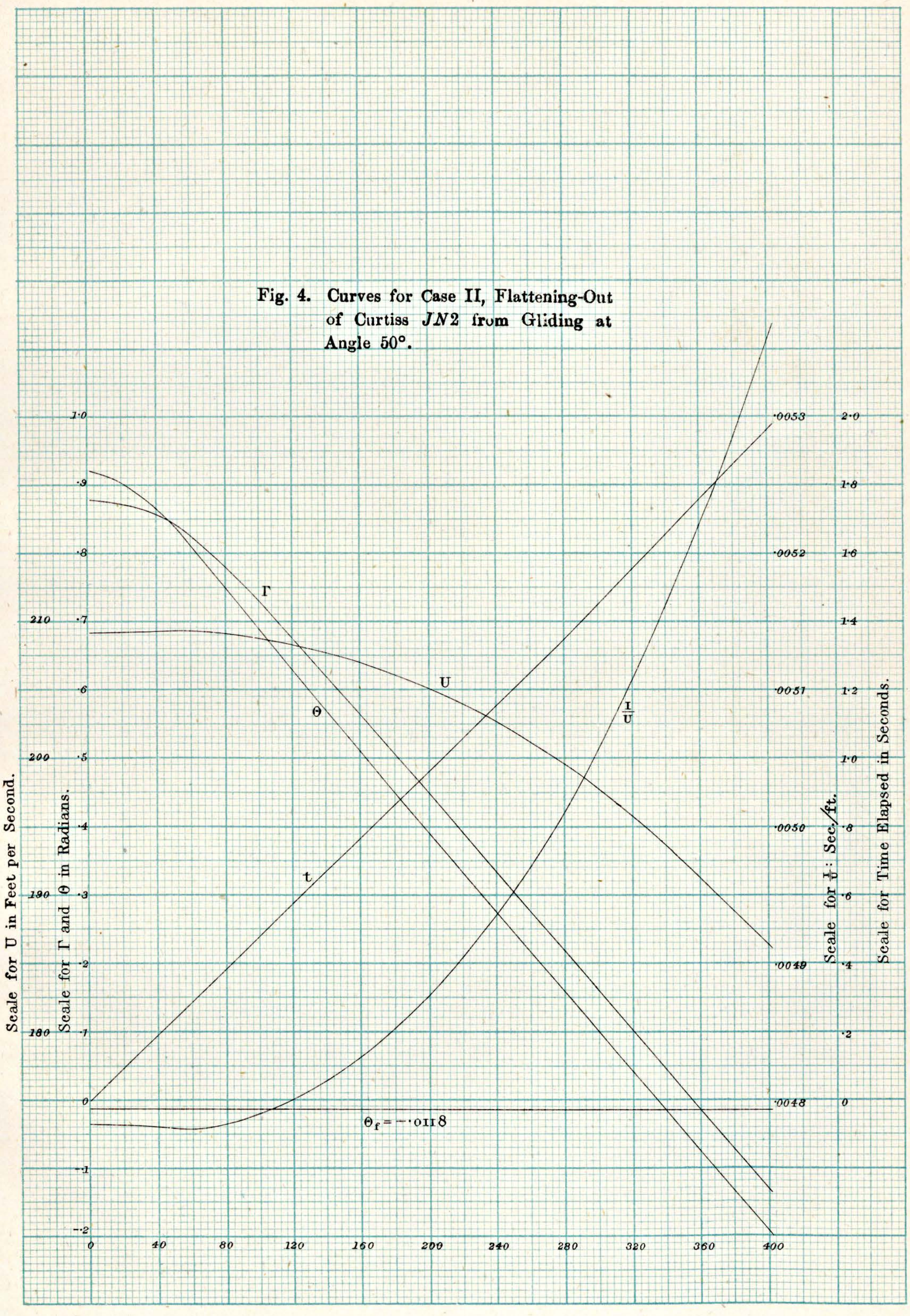


Fig. 4. Curves for Case II, Flattening-Out of Curtiss *JN2* from Gliding at Angle 50°.
Γ
Θ
U
t
1/U
$\Theta_f = -\cdot 0118$
Scale for U in Feet per Second.
Scale for Γ and Θ in Radians.
Scale for 1/U : Sec./ft.
Scale for Time Elapsed in Seconds.
210
200
190
180
1·0
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
-·1
-·2
·0053
·0052
·0051
·0050
·0049
·0048
2·0
1·8
1·6
1·4
1·2
1·0
·8
·6
·4
·2
0
0
40
80
120
160
200
240
280
320
360
400

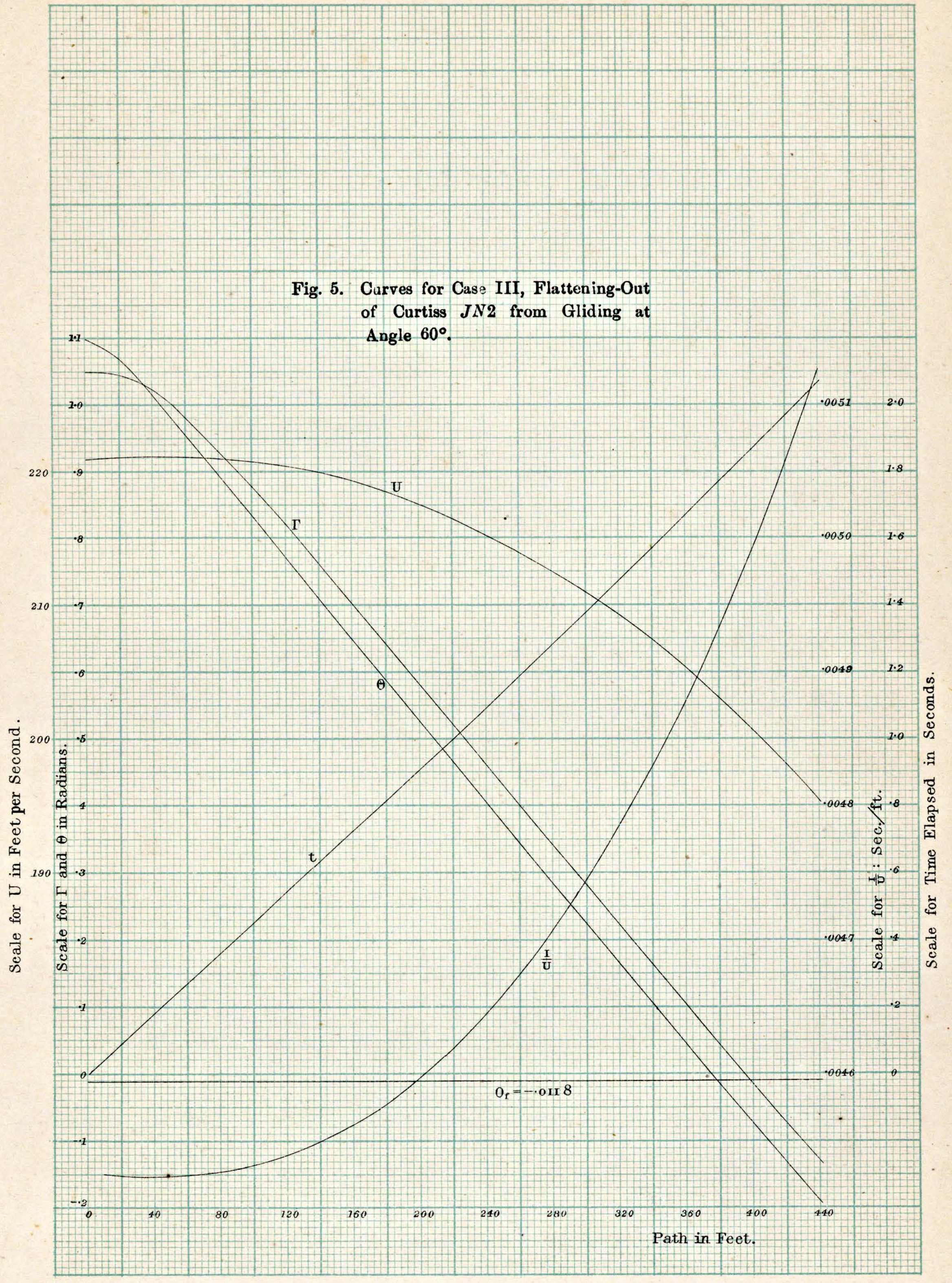


Fig. 5. Curves for Case III, Flattening-Out of Curtiss *JN2* from Gliding at Angle 60°.

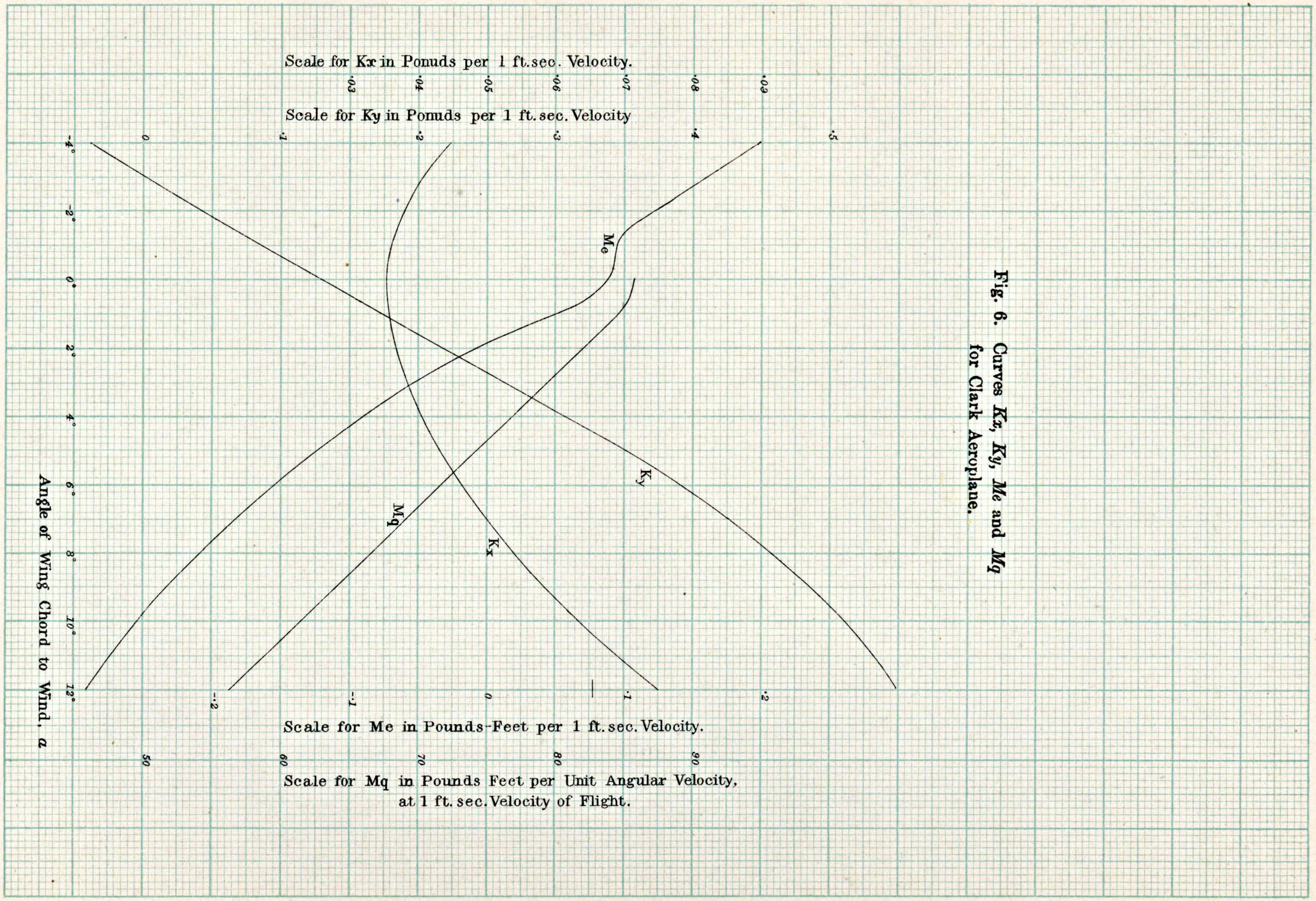


Fig. 6. Curves *Kx*, *Ky*, *Me* and *Mq* for Clark Aeroplane.

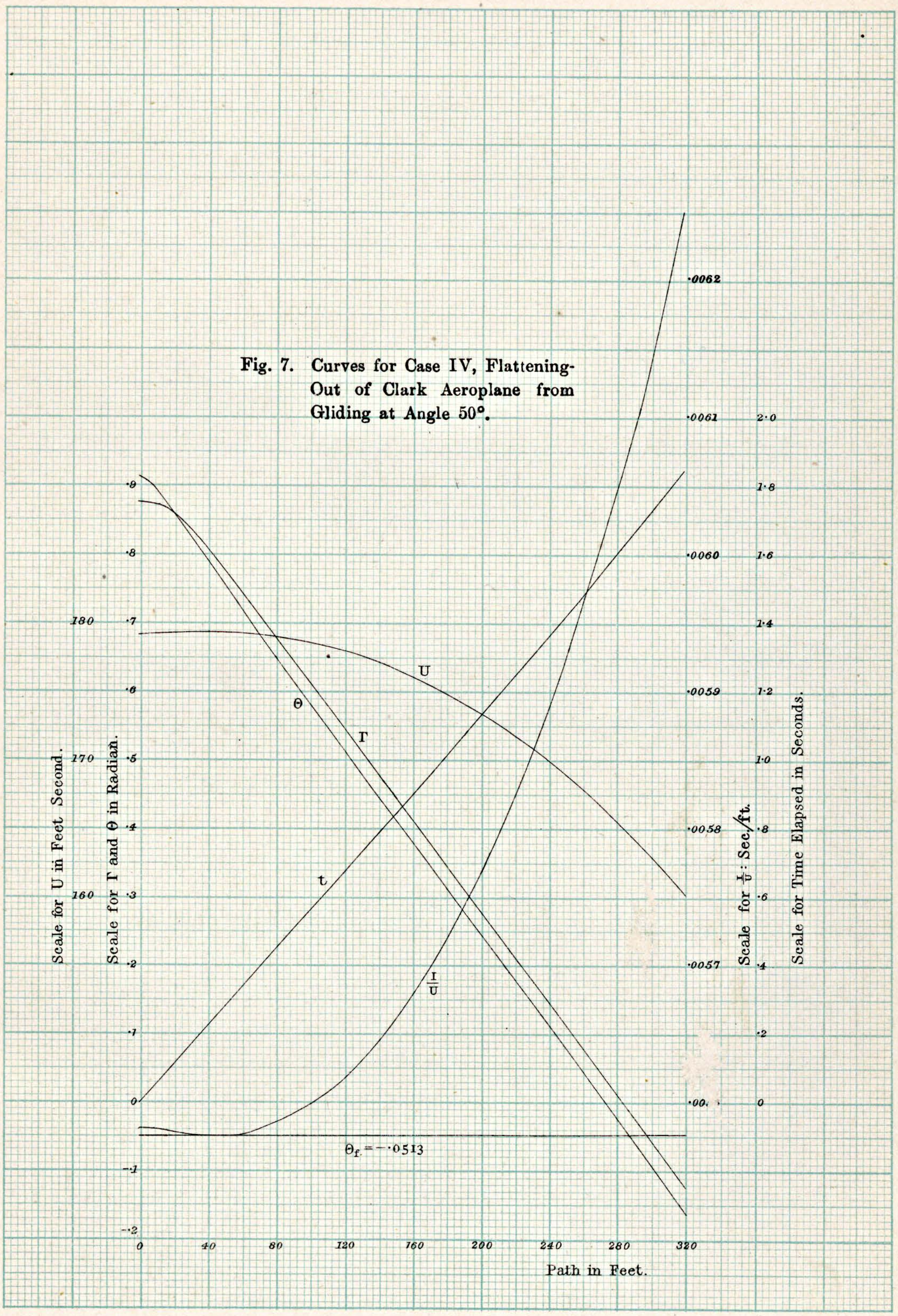


Fig. 7. Curves for Case IV, Flattening-Out of Clark Aeroplane from Gliding at Angle 50°.
Scale for U in Feet Second.
Scale for Γ and Θ in Radian.
Scale for $\frac{1}{U}$: Sec/ft.
Scale for Time Elapsed in Seconds.
Path in Feet.
U
Θ
Γ
t
$\frac{1}{U}$
$\Theta_f = -\cdot 0513$
180
170
160
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
-·1
-·2
0
40
80
120
160
200
240
280
320
·0062
·0061
·0060
·0059
·0058
·0057
2·0
1·8
1·6
1·4
1·2
1·0
·8
·6
·4
·2
0

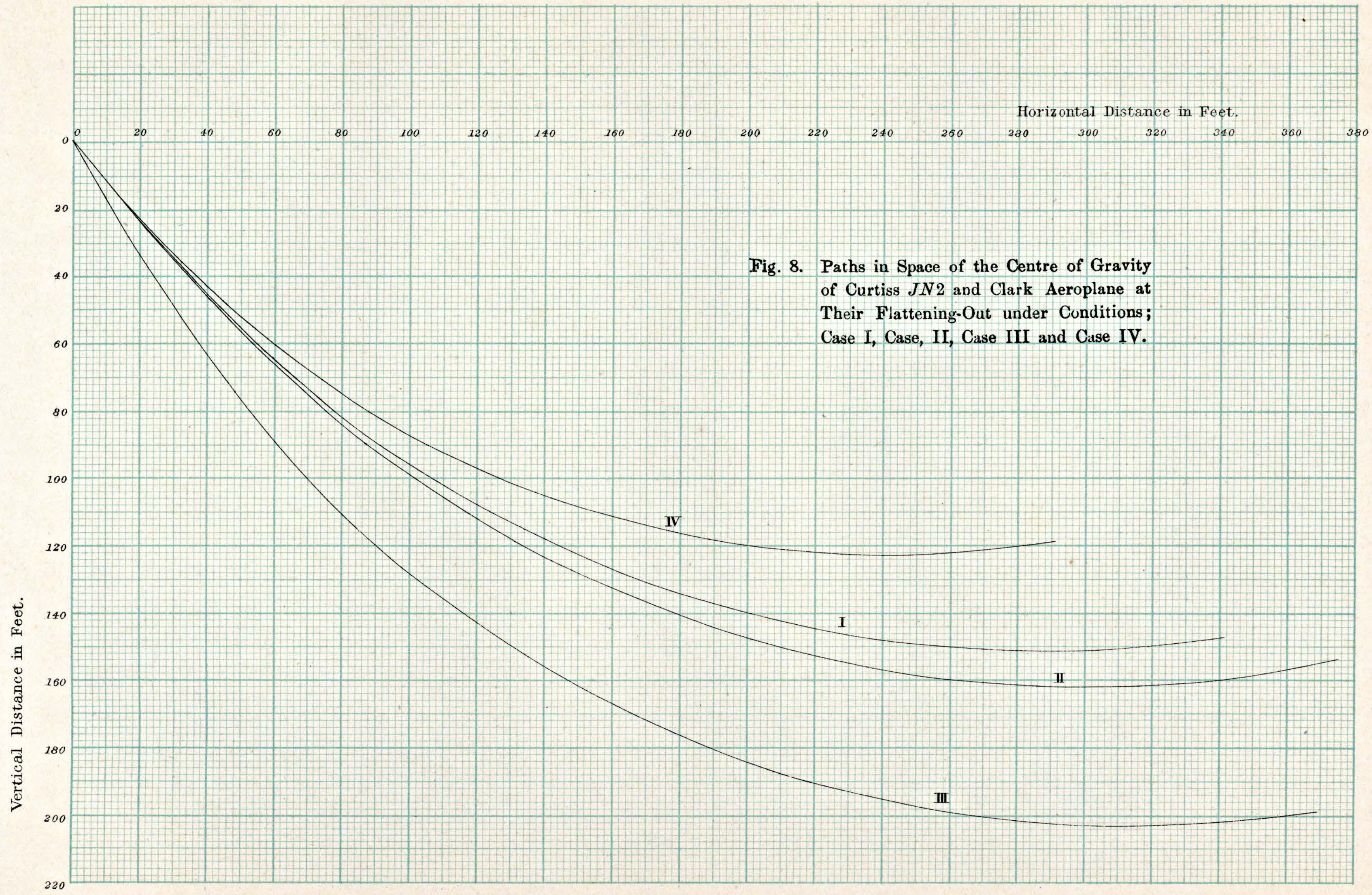


Fig. 8. **Paths in Space of the Centre of Gravity of Curtiss *JN*2 and Clark Aeroplane at Their Flattening-Out under Conditions; Case I, Case, II, Case III and Case IV.**