Critical depth and its relation to water surface curve in an open channel.

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Contents.

Introductory.

Chapter I. Energy curve and Critical depth.

Chapter II. Equation of water surface curve.

Chapter III. Several forms of water surface curve.

Chapter IV. Momentum curve and Hydraulic jump.

Chapter V. Application to hydraulic constructions.

Introductory.

By the term "open channel" we include all rivers and artificial canals of whatever section in which the water flows with free surface exposed to the atmosphere.

The force producing the flow, consequently, can not be produced by any external head, but is solely due to the slope or gradient of the channel.

Flow of water in an open channel may be classified as follows:

1. Steady flow.
2. Non-steady or Variable flow.

Steady flow is such a one in which the same quantity per unit time passes through each cross section, so that the mean velocity in a given section is always constant. The steady flow is divided also into two classes.

a) Uniform flow b) Non-uniform flow.

When all the water cross sections are equal and the slope of the water surface is parallel to that of the bed of the channel, the flow is said to be uniform. If the sections vary, the flow is said to be non-uniform, although the condition of steady flow is fulfilled.

Uniform flow occurs mostly in artificial canal, but in natural river the flow is generally non-uniform. In the former case, the mean velocity in any section is constant, in the latter however, it varies being either increased or decreased.

Non-steady or variable flow is such a one in which the mean velocity in a given section and its sectional area change with the time, having a varying quantity of flowing water per unit time.

In the following, we will discuss only the steady flow, generally the case of non-uniform flow treating the uniform flow as a particular case of it.

Chapter I.

Energy curve and Critical depth.

Energy head at any point in an open channel means the sum of the velocity head and the static head at that point. In considering the energy of flow for the whole cross section in a channel with regard to the change in water surface, it is convenient to take the energy head in that section, which is the sum of the velocity head and the static head at a point on the bottom of the channel axis. This relation may be expressed thus:

$$H=\frac{v^2}{2g}+y\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

in which H is the energy head in the section and $\frac{v^2}{2g}$ the velocity head, v heing the mean velocity of the section, g being the acceleration of gravity and y is the static head or water depth at the channel axis.

The velocity head due to the velocity v is obviously equal to $\frac{v^2}{2g}$, but the velocity in every point is not constant through the cross section, so for the entire cross section there must arise some difference between the velocity head due to the mean velocity and the true velocity head due to the individual velocities in the section. The ratio of these two depends upon the condition of the distribution of velocity in the section and it has been investigated by several authorities such as Jasmund, St. Venant, Boussinesq and others.

Now, if we denote $a=\frac{\gamma^2}{v^2}$, $\frac{\gamma^2}{2g}$ being the true velocity head in the section, the value of a determined experimentally ranges from 1.0851 to 1.1380, and 1.11 is generally accepted as a mean.

Thus the above equation must be improved as follows

$$H=\frac{av^2}{2g}+y\dots\dots\dots\dots\dots\dots\dots\dots(1),$$

Let

Q, quantity of flowing water in unit time,

F, area of whole cross section,

b, breadth of water surface in the section.

then $v=\frac{Q}{F}$,

eq. (1) becomes

$$H=\frac{aQ^2}{2gF^2}+y\text{........................}(2)$$

When the channel has a rectangular section, the water breadth b is always constant being independent of water depth, but when it has a parabolic section, b varies with the change in water level. In the latter case, we assume the section may be represented by the following equation

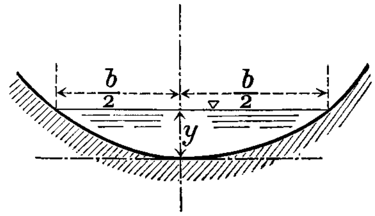


Fig. 1.

$$\frac{b^2}{4}=\delta y\text{.....................}(3)$$

where δ, parameter of the parabola,

y, greatest depth at the center of the channel section.

And from properties of the parabola, we have

$$\left.\begin{aligned}F&=\frac{2}{3}by\\&=\frac{4}{3}\delta^{\frac{1}{2}}y^{\frac{3}{2}}\end{aligned}\right\}\text{........................}(4)$$

$$b=2\delta^{\frac{1}{2}}y^{\frac{1}{2}}\text{...................................}(5)$$

Hence, for the rectangular section,

$$F=b.\,y.$$

$$H=\frac{aQ^2}{2gb^2y^2}+y$$

$$=\frac{k_1}{y^2}+y\text{.................................}(6)$$

or $\quad y^3-Hy^2+k_1=0\text{..............................}(6)_1$

where $\quad k_1=\frac{aQ^2}{2gb^2}=\text{constant}.$

For the parabolic section,

$$F=\frac{2}{3}by=\frac{4}{3}\delta^{\frac{1}{2}}y^{\frac{3}{2}},$$

$$H=\frac{aQ^2}{2g\times\frac{16}{9}\delta y^3}+y$$

$$=\frac{9aQ^2}{32\delta gy^3}+y$$

$$=\frac{k_2}{y^3}+y \dots\dots\dots\dots\dots\dots\dots\dots (7)$$

or $$y^4-Hy^3+k_2=0 \dots\dots\dots\dots\dots\dots (7)_1$$

where

$$k_2=\frac{9}{32}\frac{aQ^2}{\delta g}=\text{constant.}$$

These equations (6) and (7) give the relation between water depth y and energy head H in a section of the channel, the former for rectangular and the latter for parabolic section. Each equation may be called the "equation of energy curve," since it represents the energy of flow corresponding to varions depths.

Since eq. $(6)_1$ is a cubic equation of y, there should exist three roots, of these, however, one comes out negatively and is useless for practical purposes, leaving two real positive roots. Equation $(7)_1$ is a fourth degree equation of y and it has four roots generally, but in this case only two real positive roots remain for practical use.

Thus every equation of the energy curve has always two positive roots and for a given depth there is also one other depth having an equal value of energy head in a given section. We call these the "alternate energy stages," one lower, the other higher.

Now, let

y_1, water depth corresponding to lower energy stage,

y_2, water depth corresponding to higher energy stage.

Then, for rectangular section,

by $(6)_1$ $$y_1^3-Hy_1^2+k_1=0,$$

$$y_2^3-Hy_2+k_1=0.$$

Eliminating k_1 from these two, we have

$$y_1^3-Hy_1^2-y_2^3+Hy_2^2=0,$$

$$H(y_2^2-y_1^2)=y_2^3-y_1^3,$$

$$H=\frac{y_2^3-y_1^3}{y_2^2-y_1^2}=\frac{y_2^2+y_1y_2+y_1^2}{y_2+y_1},$$

when $$y_1=y_2=y,$$

$$H=\frac{3y^2}{2y}=\frac{3}{2}y.$$

Substitute this value of H in $(6)_1$, then

$$y^3-\frac{3}{2}y^3+k_1=0,$$

$$y^3=2k_1=\frac{aQ^2}{gb^2},$$

$$y=\sqrt[3]{\frac{aQ^2}{gb^2}}.$$

For the parabolic section, by $(7)_1$

$$y_1^4-Hy_1^3+k_2=0,$$

$$y_2^4-Hy_2^3+k_2=0.$$

Eliminating k_2 as before, we have

$$y_1^4-Hy_1^3-y_2^4+Hy_1^3=0,$$

$$H(y_2^3-y_1^3)=(y_2^4-y_1^4),$$

$$H(y_2-y_1)(y_2^2+y_1y_2+y_1^2)=(y_2-y_1)(y_2+y_1)(y_2^2+y_1^2),$$

$$H=\frac{(y_2+y_1)(y_2^2+y_1^2)}{y_2^2+y_1y_2+y_1^2},$$

when $$y_1=y_2=y,$$

$$H=\frac{4y^3}{3y^2}=\frac{4}{3}y.$$

Substitute this value of H in $(7)_1$, then

$$y^4-\frac{4}{3}y^4+k_2=0,$$

$$y^4=3k_2=\frac{27}{32}\frac{aQ^2}{\delta g},$$

$$y=\sqrt{\frac{27}{32}\frac{aQ^2}{\delta g}}.$$

We know that, consequently, in an open channel of whatever section, when the ratio between the energy head of flow and water depth at a section, reaches some finite value, two alternate energy stages merge into one. Such water depth is usually called "critical depth" and is denoted by y_c.

Thus, for the rectangular section,

$$y_c=\frac{2}{3}H\dots\dots\dots\dots(8)$$

$$y_c=\sqrt{2k_1}=\sqrt[3]{\frac{aQ^2}{gb^2}}\dots\dots\dots\dots(9)$$

for the parabolic section

$$y_c=\frac{3}{4}H\dots\dots\dots\dots(10)$$

$$y_c=\sqrt[4]{3k_2}=\sqrt[4]{\frac{27}{32}\frac{Qa^2}{\delta g}}\dots\dots\dots\dots(11)$$

The critical depth is quite independent of roughness and inclination of the channel bed, only varying with the sectional form for constant quantity of flowing water.

From the equations of energy curve represented by (6) ann (7), we see that, when water depth y becomes smaller and smaller, the energy head H becomes larger and larger and approaches infinity as y approaches zero, or when y becomes larger and larger, H becomes also larger and larger and at the limit it becomes infinity.

Hence there is no maximum finite value of H for any value of y.

To determine the minimum value of H, if any,

put $$\frac{dH}{dy}=0.$$

Differentiating the equations (6) and $(6)_1$

$$\frac{dH}{dy}=1-\frac{2k_1}{y^3}$$

$$=3-\frac{2H}{y}$$

differentiating the equations (7) and $(7)_1$

$$\frac{dH}{dy}=1-\frac{3k_2}{y^4}$$

$$=4-\frac{3H}{y}.$$

Thus, for the rectangular section,

$$\frac{dH}{dy}=0,\qquad 3-\frac{2H}{y}=0,\qquad y=\frac{2}{3}H,$$

$$\text{or}\qquad 1-\frac{2k_1}{y^3}=0,\qquad y=\sqrt[3]{2k_1}=\sqrt[3]{\frac{aQ^2}{gb^2}},$$

for the parabolic section,

$$\frac{dH}{dy}=0, \qquad 4-\frac{3H}{y}=0, \qquad y=\frac{3}{4}H,$$

$$\text{or} \qquad 1-\frac{3k_2}{y^4}=0, \qquad y=\sqrt[4]{3k_2}=\sqrt[4]{\frac{27}{32}\frac{aQ^2}{\delta g}}.$$

In both cases, the water depth y corresponds to the critical depth yc, so we may say that the energy head has its minimum value at the critical depth.

Since for the rectangular section,

$$\frac{dH}{dy}=1-\frac{2k_1}{y^3}=1-\frac{y_c^3}{y^3},$$

and for the parabolic section,

$$\frac{dH}{dy}=1-\frac{3k_2}{y^4}=1-\frac{y_c^4}{y^4},$$

so when $\quad y<y_c, \quad \frac{dH}{dy}$ is negative,

$\quad y>y_c, \quad \frac{dH}{dy}$ is positive,

and since the absolute value of $\frac{dH}{dy}$ for the same positive value of y_c-y and $y-y_c$ is always greater when $y<y_c$ than when $y>y_c$, the form of energy curve must be steeper in the former than in the latter case. The physical meaning is as follows:

When water in an open channel flows decreasing its velocity along the direction of flow with a water depth initially less than the critical depth, the depth increases on the downstream in accordance with the change of velocity, a part of the kinetic energy being converted to potential energy and as until the depth reaches the critical, the rate of decrease of kinetic energy is greater than the rate of increase of potential energy, so total energy of flow becomes smaller and smaller, at the limit it has minimum value and beyond the critical depth the rate of decrease of kinetic energy is less than the rate of increase of potential energy, thus causing the total energy of flow to become larger and larger with less rate of change than the former.

When water flows increasing its velocity, the relations are quite the reverse.

For the reasons above mentioned, if we plot generally the energy curve for a given condition of flow and channel, taking y as abscissa and H as ordinate, it may be shown as in the figure.

$$Hmin=\frac{3}{2}y_c$$

for the rectangular section.

$$Hmin=\frac{4}{3}y_c$$

for the parabolic section.

Lower energy stage $y_1 < y_c$.

Higher energy stage $y_2 > y_c$.

$$y_c - y_1 < y_2 - y_c.$$

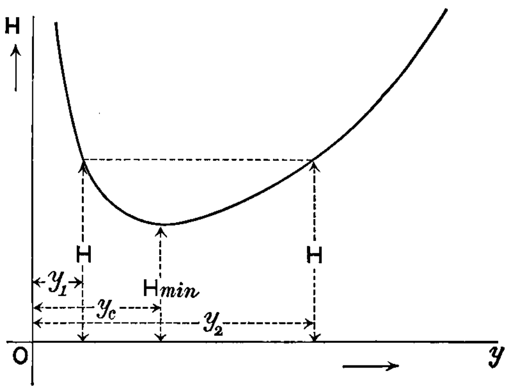


Fig. 2.

When the water depth changes without any loss of energy, it should be transferred from one of alternate energy stage to the other.

If we know one value of either alternate energy stage y_1 or y_2, the other value can be determined graphically by plotting the energy curve or it may be calculated in the following way.

For the rectangular section :—

by eq. (6) $$H=\frac{k_1}{y_1^2}+y_1=\frac{k_1}{y_2^2}+y_2,$$

$$y_2-y_1=k_1\left(\frac{1}{y_1^2}-\frac{1}{y_2^2}\right)=k_1\frac{y_2^2-y_1^2}{y_1^2y_2^2}=k_1\frac{(y_2-y_1)(y_2+y_1)}{y_1^2y_2^2},$$

$$y_1^2y_2^2-k_1y_1-k_1y_2=0,$$

$$y_1=\frac{k_1}{2y_2^2}\pm\sqrt{\frac{k_1^2}{4y_2^4}+\frac{k_1}{y_2}}=\frac{k_1}{2y_2^2}\left(1\pm\sqrt{1+\frac{4y_2^3}{k_1}}\right),$$

or $$y_2=\frac{k_1}{2y_1^2}\pm\sqrt{\frac{k_1^2}{4y_1^4}+\frac{k_1}{y_1}}=\frac{k_1}{2y_1^2}\left(1\pm\sqrt{1+\frac{4y_1^3}{k_1}}\right).$$

The negative sign which precedes the radical may be omitted as y_1 or y_2 must be always positive.

$$\therefore \qquad y_1=\frac{k_1}{2y_2^2}+\sqrt{\frac{k_1^2}{4y_2^4}+\frac{k_1}{y_2}}=\frac{k_1}{2y_2^2}\left(1+\sqrt{1+\frac{4y_2^3}{k_1}}\right)$$

$$\text{or} \qquad y_2=\frac{k_1}{2y_1^2}+\sqrt{\frac{k_1^2}{4y_1^4}+\frac{k_1}{y_1}}=\frac{k_1}{2y_1^2}\left(1+\sqrt{1+\frac{4y_1^3}{k_1}}\right) \quad \ldots\ldots(12)$$

For the parabolic section :—

by eq. (7)

$$H=\frac{k_2}{y_1^3}+y_1=\frac{k_2}{y_2^3}+y_2,$$

$$y_2-y_1=k_2\left(\frac{1}{y_1^3}-\frac{1}{y_2^3}\right)=k_2\frac{y_2^3-y_1^3}{y_1^3\,y_2^3}=k_2\frac{(y_2-y_1)(y_2^2+y_1y_2+y_1^2)}{y_1^3\,y_2^3},$$

$$y_1^3\,y_2^3-k_2y_1^2-k_2y_1y_2-k_2y_2^2=0,$$

$$y_1^3-\frac{k_2}{y_2^3}y_1^2-\frac{k_2}{y_2^2}y_1-\frac{k_2}{y_2}=0.$$

To solve this equation, put $y_1=x+\frac{k_2}{3y_2^3}$, then

$$x^3-\left(\frac{k_2^2}{3y_2^6}+\frac{k_2}{y_2^2}\right)x-\left(\frac{2k_2^3}{27y_2^9}+\frac{k_2^2}{3y_2^5}+\frac{k_2}{y_2}\right)=0,$$

$$x^3-px-q=0$$

$$x=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}+\sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}},$$

where

$$p=\frac{k_2^2}{3y_2^6}+\frac{k_2}{y_2^2},$$

$$q=\frac{2k_2^3}{27y_2^9}+\frac{k_2^2}{3y^5}+\frac{k_2}{y_2},$$

and in this case

$$\frac{q^2}{4}-\frac{p^3}{27}>0,$$

$$\frac{q}{2}>\sqrt{\frac{q^2}{4}-\frac{p^3}{27}},$$

hence the equation, $x^3-px-q=0$ has only one real positive root, leaving two imaginary roots.

$$\therefore \qquad y_1=\frac{k_2}{3y_2^3}+\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}+\sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}} \quad \ldots\ldots\ldots\ldots(13)$$

If we use y_1 for y_2 and y_2 for y_1, the value of y_2 can be obtained in a similar way.

$$y_2=\frac{k_2}{3y_1^3}+\sqrt[3]{\frac{q_1}{2}+\sqrt{\frac{q_1^2}{4}-\frac{p_1^3}{27}}}+\sqrt[3]{\frac{q_1}{2}-\sqrt{\frac{q_1^2}{4}-\frac{p_1^3}{27}}} \quad \ldots\ldots\ldots\ldots(13)_1$$

Chapter II.

Equation of water surface curve.

Change of water depth in the direction of flow with constant quantity of flowing water should be accompained by change of velocity, which may be caused by one or more factors such as changes in slope of the bed, shape and dimension or roughness of the channel.

Bernouilli's theorem shows that when water flows in an open channel from one point to another along the direction of flow, the sum of the velocity head, the static head and the height from any datum line at the first point is equal to the sum of the same quantities at the second point and the friction head employed to overcome the frictional resistance in flowing between two points.

Consider the points on the bed along the stream axis in two sections. Taking a horizontal line passing through the point on the bed in down stream section as datum and considering the mean velocity of the whole section for each point, the theorem may be expressed as follows:

$$\frac{av_1^2}{2g}+y_1+l\sin\theta$$
$$=\frac{av_2^2}{2g}+y_2+f \text{......(14)}$$

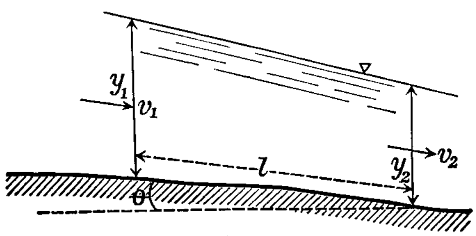


Fig 3.

where

v_1, mean velocity at upstream section,

v_2, mean velocity at down stream section,

y_1, water depth at up-stream section along the stream axis,

y_2, water depth at down-stream section along the stream axis,

l, distance between two sections along the stream axis,

θ, inclination of channel bed, being positive for downward slope as in figure and negative for upward slope,

f, friction head between two sections.

By eq. $(1)_1$ $H_1=\frac{av_1^2}{2g}+y_1$ and $H_2=\frac{av_2^2}{2g}+y_2$,

then eq. (14) becomes

$$H_2-H_1=l\sin\theta-f \quad\text{......................(15)}$$

i. e. change in energy head at two sections is equal to the difference of bottom height at one section above the other and the friction head between the two sections.

In the case of uniform flow, where the section is throughout constant, the friction head for the distance l is generally exprssed by the following equation,

$$f=\frac{\sigma v^2}{c^2F}l \quad\text{..................................(16)}$$

where

σ, wetted perimeter of the section,

v, mean velocity of the section,

F, sectional area,

c, velocity coefficient depending upon roughness, slope of channel, shape and dimension of section.

Now, if we take two sections indefinitely near with distance dx along the direction of flow, the above equation (16) may be considered as being also applicable even in the case of non-uniform flow, *i.e.*,

friction head $df=\frac{\sigma_m v_m}{c_m F_m}dx$ ....$(16)_1$

Subscript m signifies the mean value in the two sections and here we consider that these mean values remain constant for a short distance dx. Therefore, change in energy head

$$dH=dx\sin\theta-df \quad\text{....................}(15)_1$$

To deduce the equations of water surface curve, let it be assumed that the coefficient, c, is sensibly constant.

Rectangular section :—

$$\sigma_m=\frac{1}{2}\{b+2y+b+2(y+dy)\}=b+2y+dy.$$

$$v_m=\frac{Q}{2}\left(\frac{1}{by}+\frac{1}{b(y+dy)}\right)=\frac{Q}{2}\,\frac{2y+dy}{by(y+dy)}$$

$$F_m=\frac{1}{2}\{by+b(y+dy)\}=b\left(y+\frac{dy}{2}\right)$$

$$\frac{v_m^2}{F_m}=\frac{Q^2}{4}\frac{(2y+dy)^2}{b^2y^2(y+dy)^2\times b\left(y+\frac{dy}{2}\right)}$$

$$=\frac{Q^2(y^2+ydy)}{b^3y^5+\frac{5b^3y^4}{2}dy},$$ neglecting differentials of the second order.

$$df=\frac{\sigma_m}{c^2}\frac{Q^2(y^2+ydy)}{b^3y^5+\frac{5}{2}b^3y^4dy}dx,$$

$$dH=\left(\sin\theta-\frac{\sigma_m}{c^2}\frac{Q^2(y^2+ydy)}{b^3y^5+\frac{5}{2}b^3y^4dy}\right)dx$$

$$=\left(\sin\theta-\frac{\sigma_m}{c^2}\frac{Q^2}{b^3y^3}\right)dx\text{..........................}(a)$$

neglecting differentials of the second order.

But, by eq. (6), $H=\frac{k_1}{y^2}+y,\quad k_1=\frac{aQ^2}{2gb^2},$

$$dH=dy-\frac{2k_1}{y^3}dy=\left(1-\frac{aQ^2}{gb^2y^3}\right)dy\text{...............}(b)$$

Equating (a) and (b),

we have $\left(\sin\theta-\frac{\sigma_m}{c^2}\frac{Q^2}{b^3y^3}\right)dx=1-\frac{aQ^2}{gb^2y^3}dy,$

$$dy=\frac{y^3\sin\theta-\frac{\sigma_m}{c^2}\frac{Q^2}{b^3}}{y^3-\frac{aQ^2}{gb^2}}dx\text{...........................}(17)$$

Putting $\sigma_m=b+2y+dy$ and neglecting differentials of the second order, we have

$$\left.\begin{aligned}dy&=\frac{y^3\sin\theta-\frac{2Q^2}{c^2b^3}y-\frac{Q^2}{c^2b^2}}{y_3+\frac{aQ^2}{gb^2}}dx,\\ \text{or}\quad dx&=\frac{y^3-\frac{aQ^2}{gb^2}}{y^3\sin\theta-\frac{2Q^2}{gb^2}y-\frac{Q^2}{c^2b^2}}dy\end{aligned}\right\}\text{................}(18)$$

This differential equation gives generally the relation between changes of water depth and distance in an open channel whose section is rectangular and we must integrate the equation between proper limits in order to determine the amount of change. However, as the integration is too complicated, it is wiser to use a simple equation for practical purposes, which will be mentioned later.

As a special case, when the channel bed is horizontal, $\sin\theta$ must be equal to zero. Then eq. (18) becomes

$$dy=\frac{\frac{2Q^2}{c^2b^3}y+\frac{Q^2}{c^2b^2}}{\frac{aQ^2}{gb^2}-y^3}dx=\frac{\frac{2Q^2}{c^2b^3}y+\frac{Q^2}{c^2b^2}}{y_c^3-y^3}dx,$$

or $$dx=\frac{y_c^3-y^3}{\frac{2Q^2}{c^2b^3}y+\frac{Q^2}{c^2b^2}}dy \qquad \text{.......(19)}$$

Integrating this equation between proper limits,

$$\int_{x_1}^{x_2}dx=\int_{y_1}^{y_2}\frac{y_c^3-y^3}{\frac{2Q^2}{c^2b^3}y+\frac{Q^2}{c^2b^2}}dy,$$

$$x_2-x_1=\frac{c^2b^3}{2Q^2}\left\{-\frac{y_2^3-y_1^3}{3}+\frac{b}{4}(y_2^2-y_1^2)-\frac{b^2}{4}(y_2-y_1)+\left(y_c+\frac{b^3}{8}\right)log\left(\frac{2y_2+b}{2y_1+b}\right)\right\} \qquad \text{.......(20)}$$

or $$x_2-x_1=l=\phi(y_2)-\phi(y_1) \qquad \text{.......(20)}$$

when $$\phi(y)=\frac{c^2b^3}{2Q^2}\left\{-\frac{1}{3}y^3+\frac{b}{4}y^2-\frac{b^2}{4}y+\left(y_c+\frac{b^3}{8}\right)log(2y+b)\right\}$$

This is an equatien of water surface curve showing the change of water depth corresponding to the change of distance in an open horizontal channel with rectangular section.

Parabolic section :—

$$v_m=\frac{Q}{2}\left(\frac{1}{\frac{2}{3}by}+\frac{1}{\frac{2}{3}(b+db)(y+dy)}\right)=\frac{3Q}{4}\frac{2by+ydb+bdy}{b^2y^2+by^2db+b^2ydy},$$

$$F_m=\frac{1}{2}\left\{\frac{2}{3}by+\frac{2}{3}(b+db)(y+dy)\right\}=\frac{2}{3}\left(by+\frac{bdy+ydb}{2}\right),$$

$$\frac{v_m^2}{F_m}=\frac{9Q^2}{16}\frac{4b^2y^2+4by^2db+4b^2ydy}{b^4y^4+2b^3y^4db+2b^4y^3dy}\times\frac{3}{2}\frac{1}{by+\dfrac{bdy+ydb}{2}}$$

$$=\frac{27}{32}Q_2\frac{4b^2y^2+by^2db+4b^2ydy}{b^5y^5+\dfrac{5}{2}b^5y^4dy+\dfrac{5}{2}b^4y^5db},$$

$$df=\frac{\sigma_m v_m^2}{c^2F_m}dx=\frac{27Q}{8}\frac{\sigma_m}{c^2}\frac{b^2y^2}{b^5y^5}dx=\frac{27}{8}\frac{\sigma_m Q^2}{c^2b^3y^3}dx,$$

$$dH=dx\sin\theta-df=\left(\sin\theta-\frac{27}{8}\frac{Q^2}{c^2}\frac{\sigma_m}{b^3y^3}\right)dx\ldots\ldots\ldots\ldots(c)$$

But, by eq. (7), $H=\dfrac{k_2}{y^3}+y,\qquad k_2=\dfrac{9}{32}\dfrac{\alpha Q^2}{\delta g},$

$$dH=\left(1-\frac{3k_2}{y^4}\right)dy=\left(1-\frac{27}{32}\frac{\alpha Q^2}{\delta g y^4}\right)dy\ldots\ldots\ldots\ldots\ldots(d)$$

Equating (c) and (d), we have

$$\left(\sin\theta-\frac{27}{8}\frac{Q^2}{c^2}\frac{\sigma_m}{b^3y^3}\right)dx=\left(1-\frac{27}{32}\frac{\alpha Q^2}{\delta g y^4}\right)dy.$$

Substituting $4\delta y$ for b^2 by eq. (3),

$$dy=\frac{\sin\theta-\dfrac{27}{32}\dfrac{Q^2}{c^2\delta}\dfrac{\sigma_m}{by^4}}{1-\dfrac{27}{32}\dfrac{\alpha Q^2}{\delta g y^4}}dx$$

$$=\frac{y^4\sin\theta-\dfrac{27}{32}\dfrac{Q^2}{c^2\delta}\dfrac{\sigma_m}{b}}{y^4-\dfrac{27}{32}\dfrac{\alpha Q^2}{\delta g}}dx,$$

or $$dx=\frac{y^4-\dfrac{27}{32}\dfrac{\alpha Q^2}{\delta g}}{y^4\sin\theta-\dfrac{27}{32}\dfrac{Q^2}{c^2}\dfrac{\sigma_m}{\delta b}}dy,$$

but $\sigma_m=\dfrac{1}{2}\Big\{\sigma+(\sigma+d\sigma)\Big\}=\sigma+\dfrac{d\sigma}{2},$

$\dfrac{\sigma_m}{b}=\dfrac{\sigma}{b}+\dfrac{d\sigma}{2b}=\dfrac{\sigma}{b},$ since the second term will vanish when the differentials of the second order are neglected.

And although both b and σ are dependent on water depth y, we may assume the ratio $\frac{\sigma}{b}$ to be constant without any appreciable error for practical use. This is true for the channel having greater water breadth compared with water depth.

Thus, integrating the equation,

$$dx=\frac{y^4-\frac{27}{32}\frac{aQ^2}{\delta g}}{\sin\theta y^4-\frac{27}{32}\frac{Q^2\sigma}{c^2\delta b}}dy \qquad \text{(21)}$$

between proper limits, we have

$$\int_{x_1}^{x^2}dx=\int_{y_1}^{y_2}\frac{y^4-A}{\sin\theta y^4-B}dy$$

$$x_2-x_1=\frac{y_2-y_1}{\sin\theta}+\frac{B-A\sin\theta}{\sqrt[4]{\sin^5\theta B^3}}\left\{\frac{1}{4}\log\frac{\left(y_2-\sqrt{\frac{B}{\sin\theta}}\right)\left(y_1+\sqrt{\frac{B}{\sin\theta}}\right)}{\left(y_1-\sqrt{\frac{B}{\sin\theta}}\right)\left(y_2+\sqrt{\frac{B}{\sin\theta}}\right)}\right.$$

$$\left.-\frac{1}{2}\left(\tan^{-1}\frac{y_2}{\sqrt[4]{\frac{B}{\sin\theta}}}-\tan^{-1}\frac{y_1}{\sqrt[4]{\frac{B}{\sin\theta}}}\right)\right\} \qquad \text{(22)}$$

or $$x_2-x_1=l=\phi(y_2)-\phi(y_1) \qquad \text{(22)}_1$$

$$\phi(y)=\frac{y}{\sin\theta}+\frac{B-A\sin\theta}{\sqrt[4]{\sin^5\theta B^3}}\left\{\frac{1}{4}\log\frac{y-\sqrt{\frac{B}{\sin\theta}}}{y+\sqrt{\frac{B}{\sin\theta}}}-\frac{1}{2}\tan^{-1}\frac{y}{\sqrt[4]{\frac{B}{\sin\theta}}}\right\}$$

in which

$$A=\frac{27}{32}\frac{aQ^2}{\delta g}=y_c^4, \qquad \text{see eq. (11).}$$

$$B=\frac{27}{32}\frac{Q^2}{c^2\delta}\frac{\sigma}{b}.$$

This is also an equation of water surface curve showing the change of water depth corresponding to the change of distance in an open channel having a parabolic section.

As a special case, when the channel bed is horizontal, $\sin\theta=0$. Then eq. (21) becomes

$$dx = \frac{\frac{27}{32}\frac{aQ^2}{\delta g} - y^4}{\frac{27}{32}\frac{Q^2}{c^2\delta}\frac{\sigma}{b}}dy$$

$$= \frac{32}{27}\frac{c^2\delta b}{Q^2\sigma}(y_c^4 - y^4)dy \quad \text{.................................(23)}$$

By integration, we have

$$\int_{x_1}^{x_2} dx = \frac{32}{27}\frac{c^2\delta b}{Q^2\sigma}y_c^4\int_{y_1}^{y_2} dy - \frac{32}{27}\frac{c^2\delta b}{Q^2\sigma}\int_{y_1}^{y_2} y^4 dy$$

$$x_2 - x_1 = \frac{32}{27}\frac{c^2\delta b}{Q^2\sigma}\left\{(y_2 - y_1) - \frac{1}{5}(y_2^5 - y_1^5)\right\} \quad \text{...................(24)}$$

or $$x_2 - x_1 = l = \phi(y_2) - \phi(y_1) \quad \text{.......................................(24)}_1$$

$$\phi(y) = \frac{32}{27}\frac{c^2\delta b}{Q^2\sigma}\left(y - \frac{1}{5}y^5\right).$$

This equation shows the relation between changes of water depth and distance in an open horizontal channel having a parabolic section.

For steady uniform flow, as the water depth at every cross section is constant, the water surface being parallel to the bottom slope, dy must be equal to zero. Hence, for the rectangular section, by eq. (17) and $\sigma_m = \sigma$,

$$y^3\sin\theta - \frac{\sigma}{c^2}\frac{Q^2}{b^3} = 0 \quad \text{.....................................(e)}$$

$$\frac{Q^2}{b^2y^2} = c^2\frac{by}{\sigma}\sin\theta,$$

but $\quad F = b.y.$

Hydraulic mean radius, $\quad R = \frac{F}{\sigma} = \frac{by}{\sigma}.$

Inclination of bed, $\quad s = \sin\theta.$

$$\therefore \quad \frac{Q^2}{b^2y^2} = v^2 = c^2\mathrm{RS},$$

$$v = c\sqrt{RS}.$$

Similarly for the parabolic section, by eq (21),

$$y^4\sin\theta - \frac{27}{32}\frac{Q^2}{c^2\delta}\frac{\sigma}{b} = 0 \quad \text{..............................}(f)$$

$$\frac{Q^2}{\frac{16}{9}\delta y^3}=c^2\frac{\frac{2}{3}by}{\sigma}\sin\theta,$$

but $$F=\frac{2}{3}by, \qquad F^2=\frac{16}{6}\delta y^3, \qquad \text{see eq. (4).}$$

$$\therefore \quad \frac{Q^2}{F^2}=c^2RS$$

$$v=c\sqrt{RS}.$$

Both arrive at the same result $v=c\sqrt{RS}$ which is the well-known Chezy's formula for uniform flow.

And when $dy=0, \quad dH=0,$ see (b), (d).

$$dx\sin\theta-df=0 \qquad \text{see eq. } (15)_1.$$

$$dx\sin\theta=df.$$

It shows that, in the case of uniform flow, the entire fall due to bed incluation is employed in overcoming the frictional resistance, and the energy of flow in every section is always constant.

Let y_n be the water depth which with given quantity of flowing water and bottom slope in the channel, would maintain a condition of uniform flow, and we call it "neutral depth."

Next we will discuss the relations between water depth, neutral depth and critical depth in the case of non-uniform flow.

Let, y_n, neutral depth,

σ_n, wetted perimeter when $y=y_n$.

Rectangular section :—

$$y_n^3\sin\theta-\frac{\sigma_n Q^2}{c^2b^3}=0 \qquad \text{see eq. (e).}$$

$$y_n=\sqrt{\frac{\sigma_n Q^2}{c^2b^3\sin\theta}} \quad \cdots\cdots\cdots\cdots(25)$$

Now let us assume $\sigma_m=\sigma=\sigma_n$ in eq. (17) for the sake of simplicity, this is true for greater breadth compared with water depth as in the case of most natural channels, and substitute $y_n^3\sin\theta$ for $\frac{\sigma Q^2}{c^2b^3}$ and y_c for $\frac{\alpha Q^2}{gb^2}$.

Then we have

$$dy=\frac{y^3\sin\theta-y_n^3\sin\theta}{y^3-y_c^3}dx=\sin\theta\frac{y^3-y_n^3}{y^3-y_c^3}dx$$

or $$dx=\frac{1}{\sin\theta}\cdot\frac{y^3-y_c^3}{y^3-y_n^3}dy \qquad \text{.............................(26)}$$

But, by equation (25), $\sin\theta=\frac{\sigma_n Q^2}{c^2b^3y_n^3}$,

and by eq. (9) $\frac{Q^2}{b^2}=\frac{gy_c^3}{a}$,

$$\sin\theta=\frac{g}{ac^2}\cdot\frac{\sigma_n}{b}\cdot\frac{y_c^3}{y_n^3}.$$

$$\therefore \quad dy=\frac{g}{ac^2}\cdot\frac{\sigma_n}{b}\cdot\frac{\frac{1}{y_n^3}(y^3-y_n^3)}{\frac{1}{y_c^3}(y^3-y_c^3)}dx$$

$$=\frac{g}{ac^2}\cdot\frac{\sigma_n}{b}\cdot\frac{\frac{y^3}{y_n^3}-1}{\frac{y^3}{y_c^3}-1}dx,$$

or $$dx=\frac{ac^2}{g}\cdot\frac{b}{\sigma_n}\cdot\frac{\frac{y^3}{y_c^3}-1}{\frac{y^3}{y_n^3}-1}dy \qquad \text{..........................(26)}_1$$

The differential equation (26) or $(26)_1$ gives the relation between water depth and distance for the rectangular channel section by the terms of critical depth and neutral depth.

The value of y_n, however, can not be determined directly from equation (25), since σ_n is dependent on y_n.

$$\sigma_n=2y_n+b,$$

$$y_n^3\sin\theta-\frac{(2y_n+b)Q^2}{c^2b^3}=0,$$

$$c^2b^3\sin\theta y_n^3-2Q^2y_n-bQ^2=0.$$

$$y_n^3-\frac{2Q^2}{c^2b^3\sin\theta}y_n-\frac{Q^2}{c^2b^2\sin\theta}=0,$$

$$y_n^3-py_n-q=0.$$

$$\therefore \quad y_n=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}-\sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}} \quad \ldots\ldots(27)$$

where p stands for $\dfrac{2Q^2}{c^2b^3\sin\theta}$ and q for $\dfrac{Q^2}{c^2b^2\sin\theta}$.

Thus the value of y_n may be determined by this equation.

Integrating eq. (26) between proper limits, we have

$$\int_{x_1}^{x^2} dx=\frac{1}{\sin\theta}\int_{y_1}^{y_2}\frac{y^3-y_c^3}{y^3-y_n^3}dy,$$

$$x_2 \dot{-} x_1=$$

$$\frac{y_2-y_1}{\sin\theta}+\frac{y_n}{\sin\theta}\left(1-\frac{y_c^3}{y_n^3}\right)\left\{\frac{1}{6}\log\frac{(y_2-y_n)^2}{(y_1-y_n)^2}\;\frac{(y_1^2+y_ny_1+y_n^2)}{(y_2^2+y_ny_1+y_n^2)}\right.$$

$$\left.-\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{2y_2+y_n}{\sqrt{3}\,y_n}-\tan^{-1}\frac{2y_1+y_n}{\sqrt{3}\,y_n}\right)\right\}\ldots\ldots(28)$$

This equation is too complicated for practical use and therefore it is more convenient to integrate the equation (26) or $(26)_1$ by putting z for $\dfrac{y}{y_n}$.

$$y=y_nz,$$

$$dy=y_ndz,$$

$$dx=\frac{1}{\sin\theta}\;\frac{\dfrac{y^3}{y_n^3}-\dfrac{y_c^3}{y_n^3}}{\dfrac{y^3}{y_n^3}-1}dy$$

$$=\frac{y_n}{\sin\theta}\;\frac{z^3-\dfrac{y_c^3}{y_n^3}}{z^3-1}dz.$$

$$\int_{x_1}^{x_2}dx=\frac{y_n}{\sin\theta}\int_{z_1}^{z_2}\frac{z^3-\dfrac{y_c^3}{y_n^3}}{z^3-1}dz,$$

$$x_2-x_1=\frac{y_n}{\sin\theta}(z_2-z_1)+\frac{y_n}{\sin\theta}\left(1-\frac{y_c^3}{y_n^3}\right)\left\{\frac{1}{6}\log\frac{(z_2-1)^2(z_1^2+z_1+1)}{(z_1-1)^2(z_2^2+z_2+1)}\right.$$

$$\left.-\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{2z_2+1}{\sqrt{3}}-\tan^{-1}\frac{2z_1+1}{\sqrt{3}}\right)\right\}\ldots(28)_1$$

and
$$\frac{y_c^3}{y_n^3}=\frac{ac^2b}{g\sigma_n}\sin\theta,$$

or
$$x_2-x_1=\phi(z_2)-\phi(z_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)_2$$

where
$$\phi(z)=\frac{y_n z}{\sin\theta}+y_n\left(\frac{1}{\sin\theta}-\frac{ac^2b}{g\sigma_n}\right)\left\{\frac{1}{6}\log\frac{(z-1)^2}{z^2+z+1}\right.$$
$$\left.-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}\right\}.$$

If we prepare the tables giving the values of $\frac{1}{6}\log\frac{(z-1)^2}{z^2+z+1}$ and $\frac{1}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}$ computed for various values of z, the value of $\phi(z)$ may be determined conveniently.

Thus, equations (28), $(28)_1$ and $(28)_2$ serve for determining the change in water depth corresponding to change in distance referring to neutral depth which is the water depth as being a uniform flow, or the distance between two cross sections having given water depths in non-uniform flow and so these are equations of water surface curve in an open channel with rectangular cross section.

Parabolic section :—

Let b_n, breadth of water surface when $y=y_n$.

$$y_n^4\sin\theta-\frac{27}{32}\frac{\sigma_n}{b_n}\frac{Q^2}{c^2\delta}=0 \qquad \text{see eq. } (f).$$

$$y_n=\sqrt[4]{\frac{27}{32}\frac{\sigma_n}{b_n}\frac{Q^2}{c^2\delta\sin\theta}} \qquad \text{.........................(29)}$$

Since the ratio $\frac{\sigma}{b}$ may be assumed as a constant, the value of y_n can be determined by this equation, after the value of $\frac{\sigma_n}{b_n}$ is found for any water depth in the given section. When $\frac{y}{b}$ is small, the approximate value of σ may be computed by the following equations.

$$\sigma=b+\frac{8}{3}\frac{y^2}{b}-\frac{32}{5}\frac{y^4}{b^3},$$

or
$$\sigma=b+\frac{8}{3}\frac{y^2}{b}.$$

Substituting $y_n^4\sin\theta$ for $\frac{27}{32}\frac{\sigma}{b}\frac{Q^2}{c^2\delta}$ and y_c^4 for $\frac{27}{32}\frac{aQ^2}{\delta g}$ in eq. (21), we have

$$dx = \frac{y^4 - y_c^4}{\sin\theta(y^4 - y_n^4)} dy,$$

$$= \frac{1}{\sin\theta} \frac{y^4 - y_c^4}{y^4 - y_n^4} dy. \quad \ldots\ldots\ldots\ldots\ldots\ldots(30)$$

But, by eq. (29), $\sin\theta = \frac{27}{32} \frac{\sigma_n}{b_n} \frac{Q^2}{c^2 \delta y_n^4},$

and by eq. (11), $\frac{Q^2}{\delta} = \frac{32}{27} \frac{g y_c^4}{a},$

$$\sin\theta = \frac{g}{ac^2} \frac{\sigma_n}{b_n} \frac{y_c^4}{y_n^4},$$

$$dx = \frac{ac^2 b_n}{g\sigma_n} \frac{\frac{1}{y_c^4}(y^4 - y_c^4)}{\frac{1}{y_n^4}(y^4 - y_n^4)} dy$$

$$= \frac{ac^2 b_n}{g\sigma_n} \frac{\frac{y^4}{y_c^4} - 1}{\frac{y^4}{y_n^4} - 1} dy. \quad \ldots\ldots\ldots\ldots\ldots\ldots(30)_1$$

The differential equations (30) and $(31)_1$ for the parabolic section have quite the same form as that for the rectangular section, differing only in index, and they also give the relation between water depth and distance by the terms of critical depth and neutral depth.

Intergrating eq. (30) between proper limits, we have

$$\int_{x_1}^{x_2} dx = \frac{1}{\sin\theta} \int_{y_1}^{y_2} \frac{y^4 - y_c^4}{y^4 - y_n^4} dy,$$

$$x_2 - x_1 = \frac{y_2 - y_1}{\sin\theta} + \frac{y_n}{\sin\theta}\left(1 - \frac{y_c^4}{y_n^4}\right)\left\{\frac{1}{4}\log\frac{(y_2 - y_n)(y_1 + y_n)}{(y_1 - y_n)(y_2 + y_n)} - \frac{1}{2}\left(\tan^{-1}\frac{y_2}{y_n} - \tan^{-1}\frac{y_1}{y_n}\right)\right\} \quad \ldots(31)$$

As before, put $z = \frac{y}{y_n}$.

Then $$dx = \frac{1}{\sin\theta} \frac{\frac{y^4}{y_n^4} - \frac{y_c^4}{y_n^4}}{\frac{y^4}{y_n^4} - 1} dy$$

$$=\frac{y_n}{\sin\theta}\;\frac{z^4-\dfrac{y_c^4}{y_n^4}}{z^4-1}dz.$$

By integration,

$$x_2-x_1=\frac{y_n}{\sin\theta}(z_2-z_1)+\frac{y_n}{\sin\theta}\left(1-\frac{y_c^4}{y_n^4}\right)\left\{\frac{1}{4}\log\frac{(z_2-1)(z_1+1)}{(z_1-1)(z_2+1)}\right.$$
$$\left.-\frac{1}{2}(\tan^{-1}z_2-\tan^{-1}z_1)\right\}\ldots\ldots\ldots(31)_1$$

$$\frac{y_c^4}{y_n^4}=\frac{ac^2b_n}{g\sigma_n}\sin\theta,$$

$$x_2-x_1=l=\phi(z_2)-\phi(z_1)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)_2$$

$$\phi(z)=\frac{y_n}{\sin\theta}z+y_n\left(\frac{1}{\sin\theta}-\frac{ac^2b_n}{g\sigma_n}\right)\left\{\frac{1}{4}\log\frac{z-1}{z+1}-\frac{1}{2}\tan^{-1}z\right\}.$$

In this case, it is also convenient to prepare the tables of $\frac{1}{4}\log\frac{z-1}{z+1}$ and $\frac{1}{2}\tan^{-1}z$ for computation of $\phi(z)$.

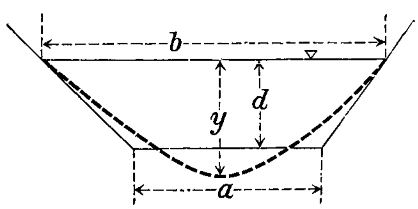
Thus, equations (31), $(31)_1$ and $(31)_2$ serve for determining the change in water depth corresponding to change in distance referring to neutral depth, or the distance between two sections having given water depths in non-uniform flow and so these are equations of water surface curve in an open channel with parabolic cross section.

All the discussions investigated above are wholly restricted to the channel whose cross section is either rectangular or parabolic. If it be desired to apply the theory above-mentioned to a channel of trapezoidal cross section, we can replace it approximately by a parabolic section having the same area of cross section and the same width at water surface, i. e.,

sectional area $F=\frac{1}{2}(a+b)d=\frac{2}{3}by,$

$$y=\frac{3}{4}\;\frac{(a+b)d}{b},$$

and $\delta=\dfrac{b^2}{4y}.$



When the channel has greater water surface width compared with water depth as in the case of most natural streams, it may be replaced by a rectangular section whose width should be the water surface width and water depth should be the average depth obtained by dividing the sectional area by the water surface width.

For any other sectional form, if it can be expressed by a simple mathematical equation, we can deduce all the relations in the similar way as already described.

Chapter III.

Several forms of water surface curve.

The surface curves corresponding to several kinds of flow will now be investigated.

A. Channel bottom sloping downward.

For the rectangular channel section,

$$\frac{dy}{dx}=\sin\theta\frac{y^3-y_n^{\,3}}{y^3-y_c^{\,3}} \qquad \text{(26)}$$

$$=\frac{g}{ac^2}\ \frac{\sigma_n}{b}\ \frac{\dfrac{y^3}{y_n^{\,3}}-1}{\dfrac{y^3}{y_c^{\,3}}-1} \qquad \text{(26)}_1$$

For the parabolic channel section,

$$\frac{dy}{dx}=\sin\theta\frac{y^4-y_n^{\,4}}{y^4-y_c^{\,4}} \qquad \text{(30)}$$

$$=\frac{g}{ac^2}\ \frac{\sigma_n}{b_n}\ \frac{\dfrac{y^4}{y_n^{\,4}}-1}{\dfrac{y^4}{y_c^{\,4}}-1} \qquad \text{(30)}_1$$

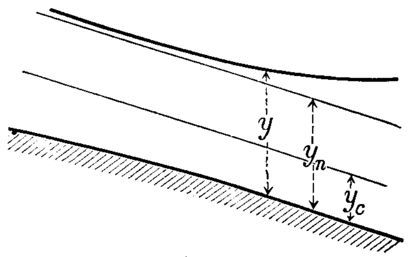
1) $y_n > y_c$.

a) $y > y_n$.

In above equations both numerator and denominator on right hand side being positive, dy is always positive when dx be taken as positive along the direction of flow. Hence water depth increases to downstream, velocity being reduced.

When the water depth increase indefinitely toward the right, both fractions $\frac{y^3-y_n^{\,3}}{y^3-y_c^{\,3}}$ and $\frac{y^4-y_n^{\,4}}{y^4-y_c^{\,4}}$ ultimately tend to the limit unity, i.e., $\frac{dy}{dx}$ will approach $\sin\theta$, the inclination of the channel bed. It follows that the water surface curve on the down stream side will approach a horizontal line.

In the other direction, toward the left, as y is always decreasing, it will gradually approach y_n and at the limit $\frac{dy}{dx}$ becomes zero, so that the surface curve tends to be assymptotic to the line y_n.

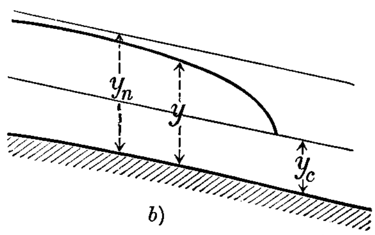


a)

This case is perhaps the form of backwater curve most frequently encountered in engineering practice, and corresponds to the curve created above a dam or other obstruction in open channel.

b) $y<y_n$ but $y>y_c$.

The numerator being negative and the denominator positive on the right hand side of the equations, dy is always negative, the water depth decreases to downstream, velocity being increased. Toward the right as y approaches y_c, the absolute value of $\frac{dy}{dx}$ becomes greater or the surface curve becomes steeper and at the limit when $y=y_c$, $\frac{dy}{dx}$ becomes $-\infty$ so that the curve becomes vertical at that point.



b)

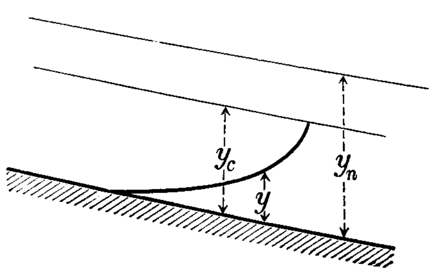
Toward the left as y approaches y_n, the absolute value of $\frac{dy}{dx}$ becomes less and at the limit when $y=y_n$, $\frac{dy}{dx}$ must be equal to zero so that the curve becomes assymptotic to the line y_n.

This case corresponds to the water surface curve created just above a sudden drop in the bottom of channel.

c) $y>y_c$.

Both numerater and denominator being negative, dy is always positive and hence the water depth increases to downstream.

Toward the right as y approaches y_c, the curve becomes continnously steeper and approaches a condition of verticality.



c)

Toward the left as y decreases, $\frac{dy}{dx}$ approaches a constant value $\left(=\frac{g}{ac^2}\,\frac{\sigma_n}{b} \text{ or } \frac{g}{ac^2}\,\frac{\sigma_n}{b_n}\right)$ so that at the limit when $y=0$, the curve intersects the bottom of channel at a definite angle. Hence the curve is definitely limited at both ends. Such a curve exists when water emerges from a sluice way with a relatively high velocity of efflux.

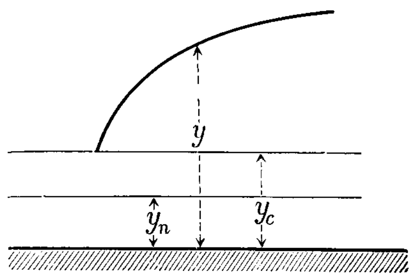
2) $y_n < y_c$.

d) $y > y_c$.

Both numerator and denominator being positive, dy is always positive.

As y increases indefinitely toward the right, the curve approaches a horizontal direction, it becoming flatter.

In the other direction as y approaches y_c, the curve becoming steeper it becomes vertical at the limit when $y=y_c$.



d)

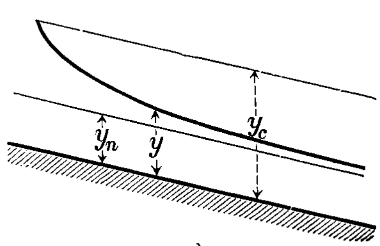
This case corresponds to the surface curve obtained where an underwater obstruction such as a dam is placed across a stream with a rapid slope.

e) $y < y_c$, but $y > y_n$.

The numerator being pesitive and the denominator negative, dy is negative.

As y decreases indefinitely toward the right, the curve becomes flatter and at the limit when $y=y_n$, $\frac{dy}{dx}$ becoming equal to zero, the curve becomes assymptotic to the line y_n.

Toward the left, as y approaches y_c the curve becomes steeper and at the limit when $y=y_c$, $\frac{dy}{dx}$ being $-\infty$, the curve becomes vertical.



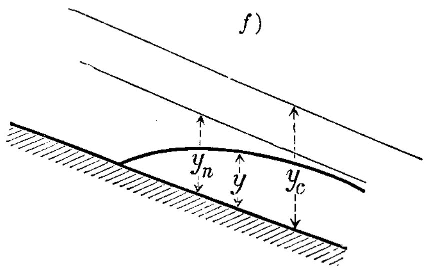
e)

This case is seen when a sudden rush of water occurs, such as may be produced by the bursting of an embankment.

f) $y < y_n$.

Both numerator and denominator being negative, dy is positive.

As y increases toward the right, the numerator tends to vanish faster than the denominator and at the limit when $y=y_n$, $\frac{dy}{dx}=0$, i. e., the curve becomes assymptotic to the line y_n.



Toward the left, as y approaches zero, the curve will intersect the bottom of channel at a definite acute angle.

This state is attained at a sluice in a stream having a slope such as

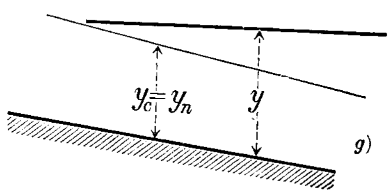
for the rectangular channel section $\sin\theta > \frac{g\sigma_n}{ac^2b}$,

for the parabolic channel section $\sin\theta > \frac{g\sigma_n}{ac^2b_n}$.

3) $y_n = y_c$.

g) $y > y_c$.

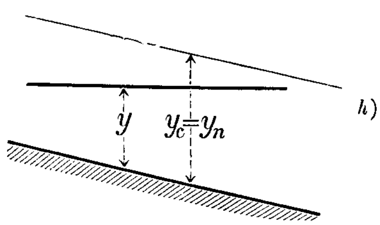
Both numerator and denominator being positive, dy is positive. For any value of y, $\frac{dy}{dx}$ has always a constant value equal to $\sin\theta$, so the surface curve makes a horizontal straight line.



h) $y < y_c$.

Both numerator and denominator being negative, dy is positive.

By the same reasoning as in the preceding case, the curve makes a horizontal line.



B. Channel bottom level.

Here, as $\sin\theta$ is equal to zero, y_n can not enter into the equations.

For the rectangular channel section,

$$\frac{dy}{dx} = \frac{\frac{2Q^2}{c^2b^3}y + \frac{Q^2}{c^2b^2}}{y_c^3 - y^3} = \frac{Q^2}{c^2b^2}\,\frac{\frac{2}{b}y+1}{y_c^3-y^3} \quad \text{............(19)}$$

$$= \frac{gy_c^3}{c^2a}\,\frac{\frac{2}{b}y+1}{y_c^3-y^3} \quad \text{............(19)}_1$$

For the parabolic channel section,

$$\frac{dy}{dx}=\frac{27}{32}\frac{Q^2\sigma}{c^2\delta b(y_c^4-y^4)} \quad \text{.............................(23)}$$

$$=\frac{g\sigma}{c^2ab}\,\frac{y_c^4}{y_c^4-y^4} \quad \text{..............................(23)}_1$$

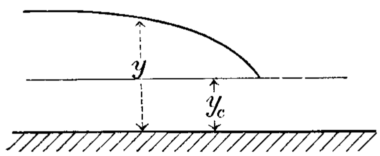
i) $y>y_c$.

The numerator being always positive and the denominator negative, dy is negative.

As y decreases toward the right, the curve becomes steeper and at the limit when $y=y_c$ it becomes vertical.

Toward the left as y increases indefinitely, the curve approaches a horizontal direction it becoming flatter.

i)

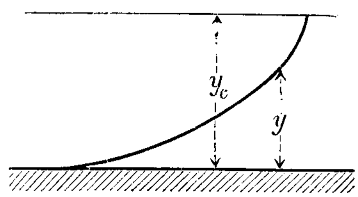


j) $y<y_c$.

Both numerator and denominator being positive, dy is positive.

Toward the right, as y increases, the curve becomes steeper and at the limit when $y=y_c$ it becomes vertical.

Toward the left, as y decreases indefinitely, the curve becomes flatter and at the limit when $y=0$, $\frac{dy}{dx}$ has a constant value ($=\frac{g}{ac^2}$ for the rectangular section; $\frac{g}{ac^2}\frac{\sigma}{b}$ for the parabolic section) so that the curve intersects the bottom of channel at a definite angle. Hence the curve has limited length as in case *c*.



j)

This case and the preceding one corresponds closely to cases *c* and *b*, if in the latter, y_n has become infinite.

This is true since as the slope of channel becomes smaller and smaller y_n becomes indefinitely greater.

C. Channel bottom sloping upward.

Here $\sin\theta$ is negative and y_n must be put out of consideration as it has no physical significance.

For the rectangular channel section,

$$\frac{dy}{dx}=\frac{y^3\sin\theta-\frac{2Q^2}{c^2b^3}y-\frac{Q^2}{c^2b^2}}{y^3-\frac{aQ^2}{gb^2}} \qquad \text{(18)}$$

$$=\frac{y^3\sin\theta-\frac{gy_c^3}{c^2a}\left(\frac{2}{b}y+1\right)}{y^3-y_c^3} \qquad \text{(18)}_1$$

For the parabolic channel section,

$$\frac{dy}{dx}=\frac{y^4\sin\theta-\frac{27}{32}\frac{Q^2\sigma}{c^2\delta b}}{y^4-\frac{27}{32}\frac{aQ^2}{\delta g}} \qquad \text{(21)}$$

$$=\frac{y^4\sin\theta-\frac{g\sigma}{c^2ab}y_c^4}{y^4-y_c^4} \qquad \text{(21)}_1$$

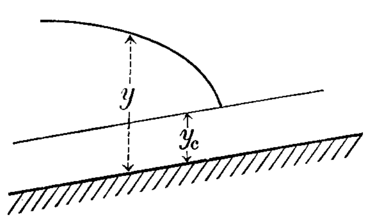
k) $y>y_c$.

The numerator being negative and the denominator positive, dy is negative.

Toward the right as y decreases, the curve becomes steeper and approaches verticality as it approaches the depth y_c.

Toward the left, as y increases steadily, the curve gradually approaches the horizontal.

k)

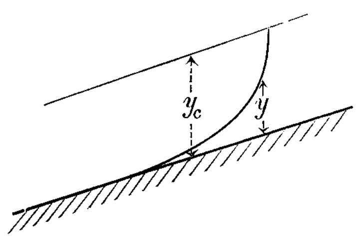


l) $y<y_c$.

Both numerator and denominator being negative, dy is positive.

Toward the right, as y increases, the curve becomes steeper and at the limit when $y=y_c$, it becomes vertical.

Toward the left, as y approaches zero, $\frac{dy}{dx}$ approaches a constant value $\left(=\frac{g}{c^2a}\text{ or }\frac{g\sigma}{c^2ab}\right)$ and at the limit, the curve intersects the bottom at a definite angle.



l)

This curve has definite length also as in the cases *c* and *j*.

Chapter IV.

Momentum curve and Hydraulic jump.

When a shallow stream moving with a high velocity strikes water of sufficient depth there is commonly produced an abrupt rise of water level which is called "hydraulic jump" or "standing wave."

We see that from the several water surface curves produced in accordance with the conditions of flow, when the water depth reaches the critical depth there occurs neassarily a change in the nature of the curve.

With an increasing velocity, the water depth being reduced along the direction of flow, the critical depth may he passed through smoothly. With a decreasing velocity, however, the water depth being increased, the critical depth can not be passed through without a heavy internal disturbance accompanying the hydraulic jump, except when the neutal depth and the critical depth are equal as represented in g and h in the preceding chapter. In general, the flow at a critical depth may be considered as a possible cource of danger, and hence worthy of special consideration.

As already mentioned there exist two alternate stages of flow with constant energy head in an open channel and if we assume no energy loss occurs during the flow, the water surface should be transferred from a lower to a higher energy stage at the hydraulic jump.

This assumption is not correct, however, since the hydraulic jump is accompanied by a great tumbling of the commingling water and the production of a white formy condition throughout the moving mass, which are caused by the dissipation of energy. Hence, at the hydraulic jump the water surface, is actually transferred from a lower stage to another a little lower, owing to loss of energy.

If we can find the amount of such loss of energy, then the height of hydraulic jump and its location may be determined.

For the investigation of the hydraulic jump it is convenient to apply Newton's second law of motion, i. e. the momentum theory.

According to this law, change produced in momentum by the change of velocity must be equal to the unbalanced force acting on the moving mass to retard its motion.

Now, let

v_1, mean velocity at section 1 entering the hydraulic jump,

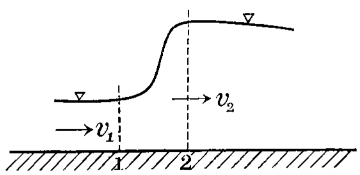
v_2, mean velocity at section 2 leaving the hydraulic jump,

Q, quanitity of flowing water in unit time,

w, weight of water in unit volume,

g, acceleration of gravity,

a, correction coefficient for square of mean velocity.



Then

mass of water flowing in unit time $=\frac{wQ}{g}$,

change in velocity $=v_1-v_2$,

change in momentum $=\frac{awQ}{g}(v_1-v_2)$.

Also, let

P_1, hydrostatic pressure on the plane of section 1,

P_2, hydrostatic pressure on the plane of section 2,

f, frictional resistance along the wetted perimeter or any other external force applied between two sections.

Then

the unbalanced force in the two sections $=P_2-P_1+f$.

But frictional resistance is generally so small that it may be neglected in this case and here we assume that no external force is being applied between the two sections.

Therefore, Newton's 2nd law shows

$$\frac{awQ}{g}(v_1-v_2)=P_2-P_1,$$

$$\frac{awQ}{g}v_1+P_1=\frac{awQ}{g}v_2+P_2$$

i. e., the sum of the momentum and the static pressure remains constant in sections both entering and leaving the hydraulic jump.

For convenience, we call here the sum of the momentum and the static pressure in a section the "momentum head" and in the following we will discuss the relation between the momentum head and the water depth in an open channel.

Let M be the momentum head and F the sectional area, then

$$M=\frac{awQ}{g}v+P,$$

$$v=\frac{Q}{F},$$

$$M=\frac{awQ^2}{Fg}+P \quad \text{.......................................(32)}$$

For the rectangular section,

water depth $=y$, width of section $=b$,

$F=by$,

and $P=\frac{1}{2}wby^2$,

$$\therefore \quad M=\frac{awQ^2}{byg}+\frac{wby^2}{2} \quad \text{.................................(33)}$$

$$y^3-\frac{2M}{bw}y+\frac{2aQ^2}{gb^2}=0 \quad \text{..............................(33)}_1$$

For the parabolic section,

greatest depth at centre of the section $=y$,

breadth of water surface $=b$,

$$F=\frac{4}{3}\delta^{\frac{1}{2}}y^{\frac{3}{2}}, \qquad b=2\delta^{\frac{1}{2}}y^{\frac{1}{2}}.$$

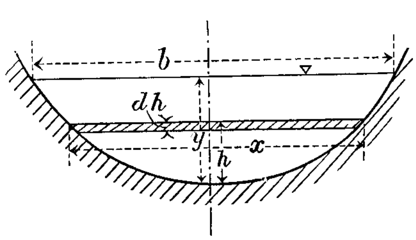
To find the static pressure for the whole section, consider the elementary area with breadth x, depth dh at distance $y-h$ below the water surface.

Static pressure for elementary area

$$xdh=w(y-h)xdh,$$

static pressure for whole section

$$P=\int_0^y w(y-h)xdh,$$



but from the equation of parabola

$$x=2\delta^{\frac{1}{2}}h^{\frac{1}{2}},$$

$$\therefore \quad P = wy\int_0^y xdh - w\int_0^y hxdh$$

$$= 2\delta^{\frac{1}{2}} wy\int_0^y h^{\frac{1}{2}}dh - 2w\delta^{\frac{1}{2}}\int_0^y h^{\frac{3}{2}}dh$$

$$= \frac{4}{3}\delta^{\frac{1}{2}}wy^{\frac{5}{2}} - \frac{4}{5}\delta^{\frac{1}{2}}wy^{\frac{5}{2}} = \frac{8}{15}\delta^{\frac{1}{2}}wy^{\frac{5}{2}}.$$

Hence, by equation (32)

$$M = \frac{3}{4}\ \frac{awQ^2}{\delta^{\frac{1}{2}}gy^{\frac{3}{2}}} + \frac{8}{15}\delta^{\frac{1}{2}}wy^{\frac{5}{2}} \quad \text{..................}(34)$$

$$y^4 - \frac{15}{8}\ \frac{M}{\delta^{\frac{1}{2}}w}y^{\frac{3}{2}} + \frac{45}{32}\ \frac{aQ^2}{\delta g} = 0 \quad \text{..............}(34)_1$$

Both equations (33) and (34) show the relation between the water depth and the momentum head, the former being for the rectangular section and the latter for the parabolic section.

We call these equations the "equation of momentum curve," since it represents the momentum head in a section corresponding to various depths of flow.

In examining the equation of momentum curve we see that when the water depth y becomes smaller and smaller, the momentum head M becomes larger and larger and approaches infinity as y approaches zero, or when y becomes larger and larger, M becomes also larger and larger and at the limit it becomes infinity. Hence there is no maximum finite value of M for any value of y.

However, there might be a minimum value of M for a finite value of y. Differentiating eq. (33), we get

$$3y^2dy - \frac{2M}{bw}dy - \frac{2y}{bw}dM = 0,$$

$$\frac{dM}{dy} = \frac{3y^2 - \frac{2M}{bw}}{\frac{2y}{bw}}.$$

For the minimum value of M, if any, $\frac{dM}{dy}$ must be equal to zero,

$$3y^2-\frac{2M}{bw}=0,$$

$$M=\frac{3}{2}bwy^2,$$

substituting the value of M to eq. $(33)_1$,

$$y^3-3y^3+\frac{2aQ^2}{gb^2}=0,$$

$$y^3=\frac{aQ^2}{gb^2},$$

$$y=\sqrt[3]{\frac{aQ^2}{gb^2}}.$$

This value of y just corresponds to the critical depth y_c (see eq. 9).

Differentiating eq. $(34)_1$, we get

$$4y^3dy-\frac{45}{16}\frac{M}{w\delta^{\frac{1}{2}}}y^{\frac{1}{2}}dy-\frac{15}{8}\frac{y^{\frac{3}{2}}}{w\delta^{\frac{1}{2}}}dM=0,$$

$$\frac{dM}{dy}=\frac{4y^3-\frac{45}{16}\frac{M}{w\delta^{\frac{1}{2}}}y^{\frac{1}{2}}}{\frac{15}{8}\frac{y^{\frac{3}{2}}}{w\delta^{\frac{1}{2}}}}.$$

For the minimum value of M, if any, $\frac{dM}{dy}=0$,

$$4y^3-\frac{45}{16}\frac{M}{w\delta^{\frac{1}{2}}}y^{\frac{1}{2}}=0,$$

$$M=\frac{64}{45}w\delta^{\frac{1}{2}}y^{\frac{5}{2}},$$

substituting this value of M to eq. $(34)_1$,

$$y^4-\frac{8}{3}y^4+\frac{45}{32}\frac{aQ^2}{\delta g}=0,$$

$$y^4=\frac{27}{32}\frac{aQ^2}{\delta g},$$

$$y=\sqrt[4]{\frac{27}{32}\frac{aQ^2}{\delta g}}.$$

This value of y is also equal to the critical depth y_c. see eq. (11).

Thus we may say that in an open channel of whatever section, when the water depth reaches the critical, the momentum head has its minimum value.

And the minimum value of the momentum head is expressed by the following equations.

for the rectangular channel section, $M=\frac{3}{2}bwy_c^2$ ....(35)

for the parabolic channel section, $M=\frac{64}{45}w\delta^{\frac{1}{2}}y_c^{\frac{5}{2}}$ .....(36)

Equation $(33)_1$ is a cubic equation of y and it may have at most two positive roots. Let us assume y_1' and y_2' to be such positive roots.

$$y^3-\frac{2M}{bw}y+\frac{2aQ^2}{gb^2}=0,$$

put $A=\frac{2M}{bw}$, $B=\frac{2aQ^2}{gb^2}$,

then $y_1'^3-Ay_1'+B=0$ ....(a)

$y_2'^3-Ay_2'+B=0$ ....(b)

Eliminating B by subtracting (b) from (a), we have

$$(y_1'^3-y_2'^3)-A(y_1'-y_2')=0,$$

$$y_1'^2+y_1'y_2'+y_2'^2-A=0 \quad \text{..............................}(c)$$

$$y_1'=-\frac{y_2'}{2}\pm\sqrt{\frac{y_2'^2}{4}-y_2'^2+A}$$

$$=-\frac{y_2'}{2}\pm\sqrt{A-\frac{3}{4}y_2'^2}.$$

Similarly $y_2'=-\frac{y_1'}{2}\pm\sqrt{A-\frac{3}{4}y_1'^2}$,

but $A=\frac{2M}{bw}$

$$=\frac{2aQ^2}{gb^2y}+y^2. \qquad \text{by eq. (33).}$$

$$\therefore \quad y_1'=-\frac{y_2'}{2}\pm\sqrt{\frac{2aQ^2}{gb^2y_2'}+y_2'^2-\frac{3}{4}y_2'^2}$$

$$=-\frac{y_2'}{2}\pm\sqrt{\frac{2aQ^2}{gb^2y_2'}+\frac{y_2'^2}{4}}$$

$$=-\frac{y_2'}{2}\pm\frac{y_2'}{2}\sqrt{\frac{8aQ^2}{gb^2y_2'^3}+1}$$

$$=-\frac{y_2'}{2}\pm\frac{y_2'}{2}\sqrt{\frac{8y_c^3}{y_2'^3}+1}$$

and
$$y_2'=-\frac{y_1'}{2}\pm\frac{y_1'}{2}\sqrt{\frac{8y_c^3}{y_1'^3}+1}\ .$$

For any positive value of y, $\sqrt{\frac{8y_c^3}{y^3}+1}$ is always positive and greater than 1.

Thus y_1' and y_2' are both positive, provided we are using only the positite sign which precedes the radical, and hence the equation of the momentum curve for the rectangular section has two positive roots, i. e.

$$\left.\begin{aligned} y_1'&=\frac{y_2'}{2}\left(\sqrt{\frac{8y_c^3}{y_2'^3}+1}-1\right)\\ y_2'&=\frac{y_1'}{2}\left(\sqrt{\frac{8y_c^3}{y_1'^3}+1}-1\right)\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

When $y_1'=y_2'=y$,

by (c) $3y^2=A=\frac{2aQ^2}{gb^2y}+y^2,$

$$y^3=\frac{aQ^2}{gb^2},$$

$$y=\sqrt[3]{\frac{aQ^2}{gb^2}}=y_c.$$

i. e. at the critical depth, two values of y merge into one.

The equation of the momentum curve (34) for the parabolic section is much too complicated to determine how many positive roots it should have. In this case or for any other sectional form of channel, it is convenient to use the graphical method.

Since for the rectangular section,

$$M=\frac{awQ^2}{bgy}+\frac{wby^2}{2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33)$$

$$\frac{dM}{dy}=-\frac{awQ^2}{bgy^2}+wby=-\frac{wby_c^3}{y^2}+wby=wby\left(1-\frac{y_c^3}{y^3}\right),$$

and for the parabolic section,

$$M=\frac{3}{4}\ \frac{awQ^2}{\delta^{\frac{1}{2}}gy^{\frac{3}{2}}}+\frac{8}{15}\delta^{\frac{1}{2}}wy^{\frac{5}{2}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(34)$$

$$\frac{dM}{dy} = -\frac{9}{8}\frac{awQ^2}{\delta^{\frac{1}{2}}gy^{\frac{5}{2}}} + \frac{4}{3}\delta^{\frac{1}{2}}wy^{\frac{3}{2}}$$

$$= \frac{4}{3}\delta^{\frac{1}{2}}wy^{\frac{3}{2}}\left(1 - \frac{27}{32}\frac{aQ^2}{\delta g y^4}\right)$$

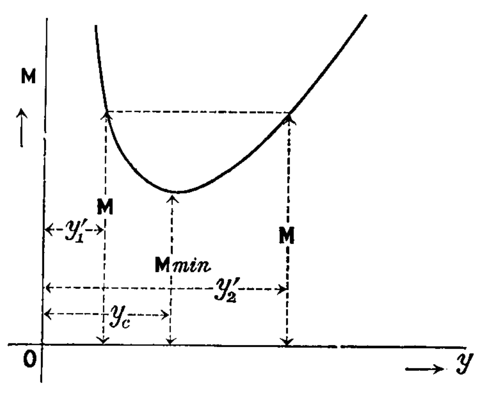
$$= \frac{4}{3}\delta^{\frac{1}{2}}wy^{\frac{3}{2}}\left(1 - \frac{y_c^4}{y^4}\right),$$

so when $y < y_c$, $\frac{dM}{dy}$ is negative,

$y > y_c$, $\frac{dM}{dy}$ is positive,

and since the absolute value of $\frac{dM}{dy}$ for the same positive value of $y_c - y$ and $y - y_c$ is always greater when $y < y_c$ than when $y > y_c$, the form of the momentum curve should be steeper in the former than in the latter. The physical meaning is a follows:

When water in an open channel flows decreasing its velocity along the direction of flow with a water depth initially less than the critical depth, the depth increases on the downstream in accordance with change of velocity, a part of the momentum being converted to static pressure and as until the depth reaches the critical, the rate of decrease of momentum is greater than the rate of increase of static pressure, so the momentum head becomes smaller and smaller, at the limit it has mininum value and beyond the critical depth the rate of decrease of momentum is less than the rate of increase of static pressure, thus causing the momentum to become larger and larger with less rate of change than the former. When water flows increasing its velocity, the relations are quite the reverse.



For the reasons above mentioned, if we plot generally the momentum curve for a given condition of flow and channel taking y (water depth) as abscissa and M (momentum head) as ordinate it may be shown as in the figure.

$$\text{Mmin.} = \frac{3}{2} bwy_c^2 \quad \text{for the rectangular section,}$$

$$= \frac{64}{45} w\delta^{\frac{1}{2}} y_c^{\frac{5}{2}} \quad \text{for the parabolic section.}$$

Form of momentum curve is quite analogous to that of energy curve but the former is situated above the latter, since M is always greater than H (energy head) for same water depth.

Thus, the equation of momentum curve has always two positive roots and for a given depth there is always one other depth having an equal value of momentum head, this point falling in all cases beyond the critical depth. We call these the "alternate momentum stages," one lower, the other higher. The flow may change from one stage to the other without the intervention of an external force and at critical depth these two stages merge into one.

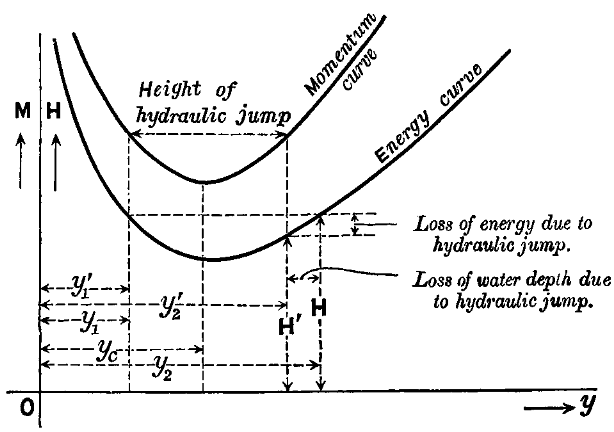
Lower momentum stage $y_1' < y_c$,

higher momentum stage $y_2' > y_c$.

At hydraulic jump the momentum head remains constant in the sections both entering and leaving the jump and the water surface should be changed from lower momentum stage to higher momentum stage.

Such a change, however, requires a change in the energy of flow.

If we plot both momentum and energy curves in the same figure we can determine the height of hydraulic jump and loss of energy due to the jump easily for a condition of flow and cannel.



Now, let

y_1', lower momentum stage,

y'_2, higher momentum stage,

y_1, lower energy stage,

y_2, higher energy stage,

H, energy head corresponding to water depth y_2,

H', energy head corresponding to water depth y'_2.

Then,

height of hydraulic jump $=y'_2-y'_1$,

loss of energy head due to hydraulic jump $=H-H'$,

loss of water depth due to hydraulic jump $=y_2-y'_2$

We may calculate these values in the following way.

Assume the channel section to be rectangular.

By eq. (37),
$$y'_2-y'_1=y'_2-\frac{y'_2}{2}\left(\sqrt{\frac{\delta y_c^3}{y_2'^3}+1}-1\right)$$
$$=\frac{3y'_2}{2}-\sqrt{\frac{2y_c^3}{y'_2}+\frac{y_2'^2}{4}}\ \ldots\ldots\ldots\ldots\ldots\ldots(38)$$

or
$$y'_2-y'_1=\frac{y_1'^2}{2}\left(\sqrt{\frac{8y_c^3}{y_1'^3}+1}-1\right)-y'_1$$
$$=\sqrt{\frac{2y_c^3}{y'_1}+\frac{y_1'^2}{4}}-\frac{3y'_1}{2}\ \ldots\ldots\ldots\ldots\ldots\ldots(38)_1$$

Hence, if we know either y'_1 or y'_2, the water depth at the section entering or leaving the hydraulic jump, then the height of the jump can be determined bv one of these equations.

By eq. (6) and (9),
$$H=\frac{k_1}{y_2^2}+y_2=\frac{k_1}{y_1^2}+y_1=\frac{y_c^3}{2y_1^2}+y_1,$$
$$H'=\frac{y_c^3}{2y_2'^2}+y'_2,$$

by eq. (37),
$$y'_2=\frac{y_1}{2}\left(\sqrt{\frac{8y_c^3}{y_1^3}+1}-1\right)$$
$$=\sqrt{\frac{2y_c^3}{y'_1}+\frac{y_1'^2}{y}}-\frac{y'_1}{2}$$
$$y_2'^2=\frac{2y_c^3}{y_1}+\frac{y_1'^2}{2}-y'_1\sqrt{\frac{2y_c^3}{y'_1}+\frac{y_1'^2}{4}}.$$

Equating y'_1 and y_1,

$$H-H'=\frac{y_c^3}{2y_1^2}+y_1-\frac{y_c^3}{\frac{4y_c^3}{y_1}+y_1^2-2y_1\sqrt{\frac{2y_c^3}{y_1}+\frac{y_1^2}{4}}}-\sqrt{\frac{2y_c^3}{y_1}+\frac{y_1^2}{4}}+\frac{y_1}{2}$$

$$=\frac{y_c^3}{2y_1^2}+\frac{3y_1}{2}-\frac{y_c^3}{\frac{4y_c^3}{y_1}+y_1^2-2y^1\sqrt{\frac{2y_c^3}{y_1}+\frac{y_1^2}{4}}}-\sqrt{\frac{2y_c^3}{y_1}+\frac{y_1^2}{4}} \quad \text{.................(39)}$$

Change of water level in the flow with constant energy head

$$=y_2-y_1.$$

Change of water level in the flow with constant momentum head

$$=y'_2-y'_1.$$

Equating y'_1 and y_1, $y^2-y'_2$ is loss of water depth due to the hydraulic jump.

By eq. (12) and (9),

$$y_2=\frac{y_c^3}{4y_1^2}\left(1+\sqrt{1+\frac{8y_1}{y_c^3}}\right),$$

and by eq. (37),

$$y'_2=\frac{y_1}{2}\left(\sqrt{1+\frac{8y_c^3}{y_1^3}}-1\right),$$

$$\therefore \quad y_2-y'_2=\frac{y_c^3}{4y_1^2}+\frac{y_c^3}{4y_1^2}\sqrt{1+\frac{8y_1^3}{y_c^3}}-\frac{y_1}{2}\sqrt{1+\frac{8y_c^3}{y_1^3}}+\frac{y_1}{2}$$

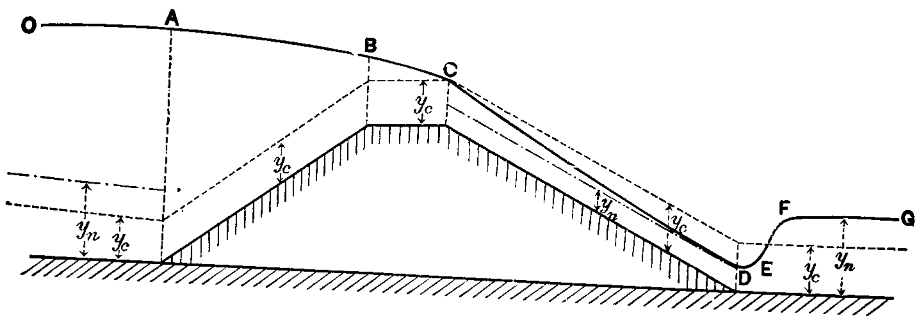
$$=\frac{y_c^3-2y_1^3}{4y_1^2}+\frac{y_c^3}{4y_1^2}\sqrt{1+\frac{8y_1^3}{y_c^3}}-\frac{y_1}{2}\sqrt{1+\frac{8y_c^3}{y_1^3}} \quad \text{.....(40)}$$

Chapter V.

Application to hydraulic constructions.

Several cases of water surface curve were discussed already and each case is one part of a water surface curve formed by practical engineering construction. In most cases, the actual water surface forms a transition curve composed of two or more of such special curves. If the conditions of flow are known at a particular place in a given open channel, the type of water surface curve existing at the place can be readily determined. The critical depth is computed from the quantity of water flowing in unit time and the width of water surface and the neutral depth from the quantity of flow, sectional form, slope of channel bed and its roughness. A comparison of critical depth, neutral depth and actual water depth will determine immediately what curve in all possible cases is obtaining.

An example of a transition curve which is very frequently encountered in practice will now be shown.



The figure shown represents the flow over a wide crested dam constructed across the channel whose bed indination is slight. Before the construction of the dam the water was flowing with the neutral depth y_n which is always greater than the critical depth y_c. Since the neutral depth is quite independent of the bed slope and of the roughness of the channel, it has a constant value for a given condition of flow, so the surface line representing the critical depth should be parallel to the bed slope over which the water flows.

In the figure, the continuous line $OABCDEFG$ shows the actual water surface curve. Until point A from up-stream side the water has decreasing velocity, from A to D, it has increasing velocity and from there the velocity being reduced, after causing the hydraulic jump, the water flows down with neutral depth as in the original state.

From O to A the surface curve has been general form of case a, from A to B, case k, from B to C, case i, from C to D, case e, from D to E, case c and at point E water jumping up suddenly to point F, from there the water flows down uniformly. Thus the actual water surface is a transition curve formed by a combination of several curves in special cases.

In this case, the actual surface curve should cross the critical depth at point C which is the intersection of the critical depth lines for two different bed slopes, since above this point the curve being convex upward (case i), the water depth must be greater than the critical depth and below the point the curve being concave upward (case e), the water depth must be less than the critical depth. With an increasing velocity, the water depth being reduced, the critical depth may be passed through smoothly as at point C. With a decreasing velocity, however, such as exists at EF in the figure, the water depth being increased, the critical depth can not be passed through without heavy internal disturbance accompanying the hydraulic jump, except in the case where the neutral depth and the critical depth conicide as represented in cases g and h.

The condition where the water surface crosses the critical depth at point C, determines the water depth at that point. If we know generally the water depth at a particular point the change in water depth at any other place can be determined by using the equations of water surface curve according to the condition of flow.

Here, the water depth at C, y_c, is found by eq. (9) or (11), then the water depth at B is determined by eq. (20) or (24), the water depth at A by eq. (28) or (31), assuming for the moment y_n to stand for the depth at which the given quantity of water would flow uniformly in the reverse direction and using the negative sign for $\sin\theta$, and the water depth at O and D by eq. (28) or (31).

To determine the location of the hydraulic jump in this case, the neutral depth y_n at the down-stream side of the dam is given and taking it as the higher momentum stage y'_2, the height of the jump may be found from eq. (38).

Then the lower momentum stage y'_1 or the water depth at E is determined. And since the water depth at D is known, with two known water depths at D and E, the distance between these two points may be computed by eq. (28) or (31).

In the case of a wide-crested dam, if we observe the water depth at the outer edge of the crest (point C in figure) the quantity of flowing water in unit time may be computed by eq. (9) or (11). Thus for the rectangular section, by eq. (9),

$$y_c=\sqrt[3]{\frac{aQ^2}{gb^2}},$$

$$Q=\left(\frac{g}{a}\right)^{\frac{1}{2}}by_c^{\frac{3}{2}}$$

Taking $a=1.11$, $g=32.2$ ft. per sec. per sec.

then $Q=5.39by_c^{\frac{3}{2}}$ in second foot ....(41)

The Francis formula for the weir without end contraction.

$$Q=3.33bh^{\frac{3}{2}}$$

h being the head measured at up stream side a little apart from the weir, avoiding the water surface drop on the top of weir. Comparing these fornulas, we get

$$Q=5.39by_c^{\frac{3}{2}}=3.33bh^{\frac{3}{2}}$$

$$0.618h^{\frac{3}{2}}=y_c^{\frac{3}{2}},$$

$$0.725h=y_c,$$

i. e. y_c is 27.5% less than h and this gives approximately the amount of water surface drop on the top of the weir.

As regards other hydraulic problems, the critical depth may be used conveniently as a means of determining the change in the water surface curve.

Loss of energy is necessarily accompanied by the occurrence of the hydraulic jump and so we may utilize such phenomena by proper construction as a means of securing the desired elimination of kinetic energy due to the water flowing at high velocity. In practice we observe very frequently the phenomenon of the hydraulic jump below many high spillway dams and thus we may reasonably conjecture the scouring power of the overflowing water at the apron of the dam being moderately reduced.

END