Theory of Single Phase Generator.

By

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§ 1. Fundamental equations and their solution.

If the field poles be non-salient and laminated and the field and armature winding be so arranged that the flux distribution circumferentially along the air gap is sinusoidal, then the fundamental equations of the single phase generator are

$$ax + b\frac{dx}{dt} + c\frac{d}{dt}(y\cos\omega t) = d\sin\omega t \quad \text{.................}(1)$$

$$a_1 y + b_1\frac{dy}{dt} + c\frac{d}{dt}(x\cos\omega t) = 0 \quad \text{.......................}(2)$$

(where a and b are the resistance and inductance of the armature circuit, a_1 and b_1 those of the field circuit, c the maximum mutual inductance between the field and armature winding, $d = cI_f\omega$ the amplitude of the induced E.M.F. due to the exciting current I_f, and x and y the armature and the field current)

which, regarding the permanent phenomena, can be solved as follows :—

Put $x = \sum_{n=1} X_n\sqrt{2}\sin(n\omega t - \phi_n)$ and $y = \sum_{n=1}^{\infty} Y_n\sqrt{2}\sin(n\omega t - \theta_n)$

Then the equation (1) becomes

$$a\sum_{n=1}^{\infty} X_n\sqrt{2}\sin(n\omega t - \phi_n) + b\omega\sum_{n=1}^{\infty} nX_n\sqrt{2}\cos(n\omega t - \phi_n)$$
$$+\frac{1}{2}c\frac{d}{dt}\left[\sum_{n=1}^{\infty} Y_n\sqrt{2}\sin(\overline{n+1}\,\omega t - \theta_n) + \sum_{n=1}^{\infty} Y_n\sqrt{2}\sin(\overline{n-1}\,\omega t - \theta_n)\right]$$
$$= d\sin\omega t$$

that is

$$a\sum_{n=1}^{\infty} X_n\sqrt{2}\sin(n\omega t - \phi_n) + b\omega\sum_{n=1}^{\infty} nX_n\sqrt{2}\cos(n\omega t - \phi_n)$$
$$+\frac{1}{2}c\omega\left[\sum_{n-2}^{\infty} nY_{n-1}\cos(n\omega t - \theta_{n-1}) + \sum_{n=1}^{\infty} nY_{n+1}\sqrt{2}\cos(n\omega t - \theta_{n+1})\right]$$
$$= d\sin\omega t$$

which, when multiplied by $\cos n\omega t$ and integrated between $\omega t = -\pi$ and $+\pi$, gives

$$-aX_n \sin\phi_n + bn\omega X_n \cos\phi_n + \frac{1}{2} cn\omega \left(Y_{n-1}\cos\theta_{n-1} + Y_{n+1}\cos\theta_{n+1}\right) = 0$$

when $n \neq 1$......(3)

$$-aX_1 \sin\phi_1 + b\omega X_1 \cos\phi_1 + \frac{1}{2} c\omega Y_2 \cos\theta_2 = 0 \quad \text{when} \quad n = 1 \text{ ....(4)}$$

and when multiplied by $\sin n\omega t$ and integrated between $\omega t = -\pi$ and $+\pi$ gives

$$aX_n \cos\phi_n + bn\omega X_n \sin\phi_n + \frac{1}{2} cn\omega \left(Y_{n-1}\sin\theta_{n-1} + Y_{n+1}\sin\theta_{n+1}\right) = 0$$

when $n \neq 1$ ......(5)

$$aX_1 \cos\phi_1 + b\omega X_1 \sin\phi_1 + \frac{1}{2} c\omega Y_2 \sin\theta_2 = \frac{1}{\sqrt{2}} d \quad \text{when} \quad n = 1 \text{ ....(6)}$$

Now add equation (3) multiplied by $\frac{2}{cn\omega}$ to equation (5) multiplied by $\sqrt{-1}\,\frac{2}{cn\omega}$ that is $j\,\frac{2}{cn\omega}$. Then we have

$$\frac{2b}{c} X_n (\cos\phi_n + j\sin\phi_n) + j\,\frac{2a}{nc\omega} X_n (\cos\phi_n + j\sin\phi_n)$$
$$+ Y_{n-1}(\cos\theta_{n-1} + j\sin\theta_{n-1}) + Y_{n+1}(\cos\theta_{n+1} + j\sin\theta_{n+1}) = 0$$

that is

$$\left(\frac{2b}{c} + j\,\frac{2a}{nc\omega}\right) X_n \,.\, \epsilon^{j\phi_n} + Y_{n-1} \,.\, \epsilon^{j\theta_{n-1}} + Y_{n+1} \,.\, \epsilon^{j\theta_{n+1}} = 0$$

and similarly from equations (4) and (6) we have

$$\left(\frac{2b}{c} + j\,\frac{2a}{c\omega}\right) X_1 \,.\, \epsilon^{j\phi_1} + Y_2 \,.\, \epsilon^{j\theta_2} = j\,\frac{d\sqrt{2}}{c\omega}.$$

Next, similarly equation (2) gives

$$\left(\frac{2b_1}{c} + j\,\frac{2a_1}{nc\omega}\right) Y_n \,.\, \epsilon^{j\theta_n} + X_{n-1} \,.\, \epsilon^{j\phi_{n-1}} + X_{n+1} \,.\, \epsilon^{j\phi_{n+1}} = 0$$

and

$$\left(\frac{2b_1}{c}+j\,\frac{2a_1}{c\omega}\right) Y_1\,.\,\epsilon^{j\theta_1}+X_2\,.\,\epsilon^{j\phi_2}=0$$

Therefore, denoting

$$\left(\frac{2b}{c}+j\,\frac{2a}{c\omega}\right) \text{ by } t_1,\ \left(\frac{2b}{c}+j\,\frac{2a}{2c\omega}\right) \text{ by } t_2,\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\left(\frac{2b}{c}+j\,\frac{2a}{nc\omega}\right) \text{ by } t_n\ldots\ldots\ldots\ldots$$

$$\left(\frac{2b_1}{c}+j\,\frac{2a_1}{c\omega}\right) \text{ by } \tau_1,\ \left(\frac{2b_1}{c}+j\,\frac{2a_1}{2c\omega}\right) \text{ by } \tau_2,\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\left(\frac{2b_1}{c}+j\,\frac{2a_1}{nc\omega}\right) \text{ by } \tau_n\ldots\ldots\ldots\ldots$$

$$X_1\,.\,\epsilon^{j\phi_1} \text{ by } \alpha_1,\ X_2\,.\,\epsilon^{j\phi_2} \text{ by } \alpha_2,\ \ldots\ldots\ldots\ldots\ X_n\,.\,\epsilon^{j\phi_n} \text{ by } \alpha_n\ \ldots\ldots\ldots\ldots$$

$$Y_1\,.\,\epsilon^{j\theta_1} \text{ by } \beta_1,\ Y_2\,.\,\epsilon^{j\theta_2} \text{ by } \beta_2,\ \ldots\ldots\ldots\ldots\ Y_n\,.\,\epsilon^{j\theta_n} \text{ by } \beta_n\ \ldots\ldots\ldots\ldots$$

and $j\,\dfrac{d\sqrt{2}}{c\omega}$ that is $jI_f\sqrt{2}$ by A

we have

$$\left.\begin{array}{r} t_1\alpha_1+\beta_2=A \\ \beta_1+t_2\alpha_2+\beta_3=0 \\ \beta_2+t_3\alpha_3+\beta_4=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \text{ and } \left.\begin{array}{r} \tau_1\beta_1+\alpha_2=0 \\ \alpha_1+\tau_2\beta_2+\alpha_3=0 \\ \alpha_2+\tau_3\beta_3+\alpha_4=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

that is

$$\left.\begin{array}{r} t_1\alpha_1+\beta_2=A \\ \alpha_1+\tau_2\beta_2+\alpha_3=0 \\ \beta_2+t_3\alpha_3+\beta_4=0 \\ \alpha_3+\tau_4\beta_4+\alpha_5=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \text{ and } \left.\begin{array}{r} \tau_1\beta_1+\alpha_2=0 \\ \beta_1+t_2\alpha_2+\beta_3=0 \\ \alpha_2+\tau_3\beta_3+\alpha_4=0 \\ \beta_3+t_4\alpha_4+\beta_5=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

that is

$$\left.\begin{aligned} t_1+\frac{\beta_2}{\alpha_1}&=\frac{A}{\alpha_1}\\ \frac{\alpha_1}{\beta_2}+\tau_2+\frac{\alpha_3}{\beta_2}&=0\\ \frac{\beta_2}{\alpha_3}+t_3+\frac{\beta_4}{\alpha_3}&=0\\ \frac{\alpha_3}{\beta_4}+\tau_4+\frac{\alpha_5}{\beta_4}&=0\\ \cdots\cdots&\cdots\cdots \end{aligned}\right\}\quad\text{and}\quad \left.\begin{aligned} \tau_1+\frac{\alpha_2}{\beta_1}&=0\\ \frac{\beta_1}{\alpha_2}+t_2+\frac{\beta_3}{\alpha_2}&=0\\ \frac{\alpha_2}{\beta_3}+\tau_3+\frac{\alpha_4}{\beta_3}&=0\\ \frac{\beta_3}{\alpha_4}+t_4+\frac{\beta_5}{\alpha_4}&=0\\ \cdots\cdots&\cdots\cdots \end{aligned}\right\}$$

Therefore

$$\left.\begin{aligned} &\frac{A}{\alpha_1}=s_1 \quad \text{where } s_1=t_1-\frac{1}{\tau_2-}\,\frac{1}{t_3-}\,\frac{1}{\tau_4-}\,\frac{1}{t_5-}\text{etc.}\\ &\frac{-\alpha_1}{\beta_2}=s_2 \quad \text{where } s_2=\tau_2-\frac{1}{t_3-}\,\frac{1}{\tau_4-}\,\frac{1}{t_5-}\,\frac{1}{\tau_6-}\text{etc.}\\ &\frac{-\beta_2}{\alpha_3}=s_3 \quad \text{where } s_3=t_3-\frac{1}{\tau_4-}\,\frac{1}{t_5-}\,\frac{1}{\tau_6-}\,\frac{1}{t_7-}\text{etc.}\\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\}$$

and

$$\left.\begin{aligned} &\frac{0}{\beta_1}=\sigma_1 \quad \text{where } \sigma_1=\tau_1-\frac{1}{t_2-}\,\frac{1}{\tau_3-}\,\frac{1}{t_4-}\,\frac{1}{\tau_5-}\text{etc.}\\ &\frac{-\beta_1}{\alpha_2}=\sigma_2 \quad \text{where } \sigma_2=t_2-\frac{1}{\tau_3-}\,\frac{1}{t_4-}\,\frac{1}{\tau_5-}\,\frac{1}{t_6-}\text{etc.}\\ &\frac{-\alpha_2}{\beta_3}=\sigma_3 \quad \text{where } \sigma_3=\tau_3-\frac{1}{t_4-}\,\frac{1}{\tau_5-}\,\frac{1}{t_6-}\,\frac{1}{\tau_7-}\text{etc.}\\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\}$$

Thus

$$\beta_1=\alpha_2=\beta_3=\alpha_4=\cdots\cdots\cdots=0$$

and

$$\alpha_1=\frac{A}{s_1},\quad \beta_2=\frac{-A}{s_1\,.\,s_2},\quad \alpha_3=\frac{A}{s_1\,.\,s_2\,.\,s_3},\ \cdots\cdots\cdots$$

that is

$$\left.\begin{aligned} X_1(\sin\phi_1 - j\cos\phi_1) &= \frac{I_f\sqrt{2}}{s_1} \\ Y_2(\sin\theta_2 - j\cos\theta_2) &= \frac{-I_f\sqrt{2}}{s_1 . s_2} \\ X_3(\sin\phi_3 - j\cos\phi_3) &= \frac{I_f\sqrt{2}}{s_1 . s_2 . s_3} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots \end{aligned}\right\}*$$

where

$$\left\{\begin{aligned} s_1 &= \left(\frac{2b}{c} + j\frac{2a}{c\omega}\right) - \cfrac{1}{\left(\frac{2b_1}{c} + j\frac{2a_1}{2c\omega}\right) - \cfrac{1}{\left(\frac{2b}{c} + j\frac{2a}{3c\omega}\right) - \text{etc.}}} \\ s_2 &= \left(\frac{2b_1}{c} + j\frac{2a_1}{2c\omega}\right) - \cfrac{1}{\left(\frac{2b}{c} + j\frac{2a}{3c\omega}\right) - \cfrac{1}{\left(\frac{2b_1}{c} + j\frac{2a_1}{4c\omega}\right) - \text{etc.}}} \\ s_3 &= \left(\frac{2b}{c} + j\frac{2a}{3c\omega}\right) - \cfrac{1}{\left(\frac{2b_1}{c} + j\frac{2a_1}{4c\omega}\right) - \cfrac{1}{\left(\frac{2b}{c} + j\frac{2a}{5c\omega}\right) - \text{etc.}}} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right.$$

§ 2. **Solution of the fundamental equations,** continued.

The series of equations of α and β in the previous article seems at a glance impossible to solve, because the number of unknown quantities is greater than that of the equations by one; but if the functions of x and y are finite and otherwise satisfy the Dirichlet's condition, so that $\underset{n=\infty}{\text{Lt}}\ \alpha_{2n+1} = 0$ and $\underset{n=\infty}{\text{Lt}}\ \beta_{2n} = 0$, then the series of equations can be solved as will be described in the next few pages.

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* This coincides with what was obtained by Prof. Lyle (Philosophical Magazine 1909).

First $2n$ equations of the given series of equations are

$$\left.\begin{aligned}
&t_1\alpha_1+\beta_2=A\\
&\alpha_1+\tau_2\beta_2+\alpha_3=0\\
&\beta_2+t_3\alpha_3+\beta_4=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\alpha_{2n-3}+\tau_{2n-2}\beta_{2n-2}+\alpha_{2n-1}=0\\
&\beta_{2n-2}+t_{2n-1}\alpha_{2n-1}+\beta_{2n}=0\\
&\alpha_{2n-1}+\tau_{2n}\beta_{2n}+\alpha_{2n+1}=0
\end{aligned}\right\}\quad\ldots\ldots(1)$$

They give, when summed up,

$$(t_1+1)\,\alpha_1+(\tau_2+2)\,\beta_2+(t_3+2)\,\alpha_3+\cdots\cdots$$
$$\cdots\cdots+(t_{2n-1}+2)\,\alpha_{2n-1}+(\tau_{2n}+1)\,\beta_{2n}+\alpha_{2n+1}=A\ldots\ldots(2)$$

Now, take up a special case where $\alpha_1=0$ and denote β_2, α_3, β_4............ in that case by x_2, x_3, x_4............ respectively. Then we have the equations

$$\left.\begin{aligned}
&x_2=A\\
&\tau_2x_2+x_3=0\\
&x_2+t_3x_3+x_4=0\\
&x_3+\tau_4x_4+x_5=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&x_{2n-3}+\tau_{2n-2}x_{2n-2}+x_{2n-1}=0\\
&x_{2n-2}+t_{2n-1}x_{2n-1}+x_{2n}=0\\
&x_{2n-1}+\tau_{2n}x_{2n}+x_{2n+1}=0
\end{aligned}\right\}\quad\ldots\ldots(3)$$

which, when summed up, give

$$(\tau_2+2)\,x_2+(t_3+2)\,x_3+(\tau_4+2)\,x_4+\cdots\cdots$$
$$\cdots\cdots+(t_{2n-1}+2)\,x_{2n-1}+(\tau_{2n}+1)\,x_{2n}+x_{2n+1}=A\ldots\ldots(4)$$

Next, take up another special case where $\alpha_1=1$ and $A=0$ and denote β_2, α_3, β_4 .... in that case by y_2, y_3, y_4 .... respectively. Then we have the equations

$$\left.\begin{aligned}
&t_1+y_2=0\\
&1+\tau_2y_2+y_3=0\\
&y_2+t_3y_3+y_4=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&y_{2n-3}+\tau_{2n-2}y_{2n-2}+y_{2n-1}=0\\
&y_{2n-2}+t_{2n-1}y_{2n-1}+y_{2n}=0\\
&y_{2n-1}+\tau_{2n}y_{2n}+y_{2n+1}=0
\end{aligned}\right\}\quad\ldots\ldots(5)$$

which, when summed up, give

$$(t_1+1)+(\tau_2+2)y_2+(t_3+2)y_3+\dots\dots\dots\dots$$
$$\dots\dots\dots\dots+(t_{2n-1}+2)y_{2n-1}+(\tau_{2n}+1)y_{2n}+y_{2n+1}=0\dots\dots(6)$$

Now adding equation (6) multiplied by α_1 to equation (4) and then subtracting equation (2) we have

$$(\tau_2+2)(\alpha_1y_2+x_2-\beta_2)+(t_3+2)(\alpha_1y_3+x_3-\alpha_3)+\dots\dots\dots\dots$$
$$\dots\dots\dots\dots+(t_{2n-1}+2)(\alpha_1y_{2n-1}+x_{2n-1}-\alpha_{2n-1})+(\tau_{2n}+1)(\alpha_1y_{2n}+x_{2n}-\beta_{2n})$$
$$+(\alpha_1y_{2n+1}+x_{2n+1}-\alpha_{2n+1})=0$$

which, when $n=1$, becomes

$$(\tau_2+1)(\alpha_1y_2+x_2-\beta_2)+(\alpha_1y_3+x_3-\alpha_3)=0$$

and, when $n=2$, becomes

$$(\tau_2+2)(\alpha_1y_2+x_2-\beta_2)+(t_3+2)(\alpha_1y_3+x_3-\alpha_3)$$
$$+(\tau_4+1)(\alpha_1y_4+x_4-\beta_4)+(\alpha_1y_5+x_5-\alpha_5)=0$$

and so on.

Similarly, when first $(2n-1)$ equations of the given series of equations are taken up, we have

$$(\tau_2+2)(\alpha_1y_2+x_2-\beta_2)+(t_3+2)(\alpha_1y_3+x_3-\alpha_3)+\dots\dots\dots\dots$$
$$\dots\dots\dots\dots+(\tau_{2n-2}+2)(\alpha_1y_{2n-2}+x_{2n-2}-\beta_{2n-2})$$
$$+(t_{2n-1}+1)(\alpha_1y_{2n-1}+x_{2n-1}-\alpha_{2n-1})+(\alpha_1y_{2n}+x_{2n}-\beta_{2n})=0$$

which, when $n=1$, becomes

$$\alpha_1y_2+x_2-\beta_2=0$$

and, when $n=2$, becomes

$$(\tau_2+2)(\alpha_1y_2+x_2-\beta_2)+(t_3+1)(\alpha_1y_3+x_3-\alpha_3)+(\alpha_1y_4+x_4-\beta_4)=0$$

and so on.

Therefore, we have

$$\alpha_1y_2+x_2-\beta_2=0 \quad \alpha_1y_3+x_3-\alpha_3=0 \quad \alpha_1y_4+x_4-\beta_4=0 \ \dots\dots\dots\dots$$
$$\dots\dots\dots\dots \alpha_1y_{2n}+x_{2n}-\beta_{2n}=0 \quad \alpha_1y_{2n+1}+x_{2n+1}-\alpha_{2n+1}=0$$

so that

$$\alpha_1 = \frac{\beta_{2n}}{y_{2n}} - \frac{x_{2n}}{y_{2n}} \quad \text{or} \quad = \frac{\alpha_{2n+1}}{y_{2n+1}} - \frac{x_{2n+1}}{y_{2n+1}}$$

But $\underset{n=\infty}{\text{Lt}}\, \beta_{2n} = \underset{n=\infty}{\text{Lt}}\, \alpha_{2n+1} = 0$ as was said before; while $\underset{n=\infty}{\text{Lt}}\, y_{2n}$ and $\underset{n=\infty}{\text{Lt}}\, y_{2n+1}$ are not equal to zero, for the product $t \,.\, \tau$ is > 1.

Therefore

$$\alpha_1 = - \underset{n=\infty}{\text{Lt}} \frac{x_{2n}}{y_{2n}} \quad \text{or} \quad = - \underset{n=\infty}{\text{Lt}} \frac{x_{2n+1}}{y_{2n+1}}$$

But now relation (3) gives

$$\frac{-x_{2n+1}}{x_{2n}} = \tau_{2n} - \frac{1}{t_{2n-1}} - \frac{1}{\tau_{2n-2}} - \dots\dots\dots\dots - \frac{1}{t_3} - \frac{1}{\tau_2} = \frac{K(2, 2n)}{K(2, 2n-1)}$$

$$\frac{-x_{2n}}{x_{2n-1}} = t_{2n-1} - \frac{1}{\tau_{2n-2}} - \frac{1}{t_{2n-3}} - \dots\dots\dots\dots - \frac{1}{t_3} - \frac{1}{\tau_2} = \frac{K(2, 2n-1)}{K(2, 2n-2)}$$

....

$$\frac{-x_4}{x_3} = t_3 - \frac{1}{\tau_2} = \frac{K(2, 3)}{K(2, 2)}$$

$$\frac{-x_3}{x_2} = \tau_2 = K(2, 2)$$

$$x_2 = A$$

so that

$$(-1)^{2n-1} x_{2n+1} = A \,.\, K(2, 2n)$$

where K denotes a continuant such as

$$K(1, 2n) = \begin{vmatrix} t_1 & 1 & 0 \dots\dots\dots\dots & 0 & 0 \\ 1 & \tau_2 & 1 \dots\dots\dots\dots & 0 & 0 \\ 0 & 1 & t_3 \dots\dots\dots\dots & 0 & 0 \\ \dots & \dots & \dots\dots\dots\dots & \dots & \dots \\ 0 & 0 & 0 \dots\dots\dots\dots & t_{2n-1} & 1 \\ 0 & 0 & 0 \dots\dots\dots\dots & 1 & \tau_{2n} \end{vmatrix}$$

Next, relation (5) gives

$$\frac{-y_{2n+1}}{y_{2n}}=\tau_{2n}-\frac{1}{t_{2n-1}}-\frac{1}{\tau_{2n-2}}-\cdots\cdots\cdots\cdots-\frac{1}{\tau_2}-\frac{1}{t_1}=\frac{K(1,\,2n)}{K(1,\,2n-1)}$$

$$\frac{y_{2n}}{y_{2n-1}}=t_{2n-1}-\frac{1}{\tau_{2n-2}}-\frac{1}{t_{2n-3}}-\cdots\cdots\cdots\cdots-\frac{1}{\tau_2}-\frac{1}{t_1}=\frac{K(1,\,2n-1)}{K(1,\,2n-2)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{-y_3}{y_2}=\tau_2-\frac{1}{t_1}=\frac{K(1,\,2)}{K(1,\,1)}$$

$$-y_2=t_1=K(1,\,1)$$

so that

$$(-1)^{2n}y_{2n+1}=K(1,\,2n)$$

Therefore, we have

$$\frac{-x_{2n+1}}{y_{2n+1}}=A\frac{K(2,\,2n)}{K(1,\,2n)}=A\Big/\frac{K(2n,\,1)}{K(2n,\,2)}$$

accordingly

$$\alpha_1=\operatorname*{Lt}_{n=\infty}\left(A\Big/\frac{K(2n,\,1)}{K(2n,\,2)}\right)$$

$$=\operatorname*{Lt}_{n=\infty}\left[A\Big/\left(t_1-\frac{1}{\tau_2}-\frac{1}{t_3}-\cdots\cdots\cdots\cdots-\frac{1}{t_{2n-1}}-\frac{1}{\tau_{2n}}\right)\right]$$

$$=A\Big/\left(t_1-\frac{1}{\tau_2}-\frac{1}{t_3}-\cdots\cdots\cdots\cdots\text{ to }\infty\right)$$

and since

$$\frac{\beta_2}{\alpha_1}=\frac{A}{\alpha_1}-t_1=-\frac{1}{\tau_2}-\frac{1}{t_3}-\frac{1}{\tau_4}-\cdots\cdots\cdots\cdots\text{ to }\infty$$

we have

$$\beta_2=-\alpha_1\Big/\left(\tau_2-\frac{1}{t_3}-\frac{1}{\tau_4}-\cdots\cdots\cdots\cdots\text{ to }\infty\right)$$

and since

$$\frac{\alpha_3}{\beta_2}=-\frac{\alpha_1}{\beta_2}-\tau_2=-\frac{1}{t_3}-\frac{1}{\tau_4}-\frac{1}{t_5}-\cdots\cdots\cdots\cdots\text{ to }\infty$$

we have

$$\alpha_3=-\beta_2\Big/\left(t_3-\frac{1}{\tau_4}-\frac{1}{t_5}-\cdots\cdots\cdots\cdots\text{ to }\infty\right)$$

and so on,

which is what was obtained in the previous article.

Note that we arrive at the same result if we take up the relation

$$\alpha_1=-\operatorname*{Lt}_{n=\infty}\frac{x_{2n}}{y_{2n}}$$

§ 3. Amplitudes of the higher harmonics of the field and armature currents.

$$s_{2p} = \left(\frac{2b_1}{c} + j\frac{1}{2p}\cdot\frac{2a_1}{c\omega}\right) - \frac{1}{s_{2p+1}}$$

so that denoting s_{2p+1} by $\frac{b}{c}(g+jh)$ we have

$$s_{2p} = \frac{b_1}{c}\cdot\left[\left(2 - \frac{g}{\frac{bb_1}{c^2}(g^2+h^2)}\right) + j\left(\frac{1}{2p}\cdot\frac{2a_1}{b_1\omega} + \frac{h}{\frac{bb_1}{c^2}(g^2+h^2)}\right)\right]$$

$$= \frac{b_1}{c}\cdot\left[\left(2 - \frac{g}{\nu\nu_f(g^2+h^2)}\right) + j\left(\frac{1}{2p}\cdot\frac{2a_1}{b_1\omega} + \frac{h}{\nu\nu_f(g^2+h^2)}\right)\right]$$

where ν = armature leakage coefficient > 1

and ν_f = field leakage coefficient > 1

accordingly

$$s_{2p}s_{2p+1} = \frac{bb_1}{c^2}\cdot\left[\left(2 - \frac{g}{\nu\nu_f(g^2+h^2)}\right) + j\left(\frac{1}{2p}\cdot\frac{2a_1}{b_1\omega} + \frac{h}{\nu\nu_f(g^2+h^2)}\right)\right](g+jh)$$

$$= \nu\nu_f\cdot\left[\left(2 - \frac{g}{\nu\nu_f(g^2+h^2)}\right) + j\left(\frac{1}{2p}\cdot\frac{2a_1}{b_1\omega} + \frac{h}{\nu\nu_f(g^2+h^2)}\right)\right](g+jh)$$

of which the modulus is

$$= \nu\nu_f\sqrt{\left(2 - \frac{g}{\nu\nu_f(g^2+h^2)}\right)^2 + \left(\frac{1}{2p}\cdot\frac{2a_1}{b_1\omega} + \frac{h}{\nu\nu_f(g^2+h^2)}\right)^2}\sqrt{g^2+h^2}$$

which is > 1 when $\sqrt{g^2+h^2} > 1$ for $\frac{g}{\nu\nu_f(g^2+h^2)} < 1$

for $\frac{g}{\sqrt{g^2+h^2}} = \frac{1}{\sqrt{1+\left(\frac{h}{g}\right)^2}} < 1$ and $\frac{1}{\sqrt{g^2+h^2}} < 1$ when $\sqrt{g^2+h^2} > 1$

But now, when p is sufficiently large, we have

$$s_{2p+1}=\frac{2b}{c}-\cfrac{1}{\cfrac{2b_1}{c}-\cfrac{1}{s_{2p+1}}}$$

$$=\frac{b}{c}\left(1+\sqrt{1-\frac{c^2}{bb_1}}\right)$$

$$=\frac{b}{c}\left(1+\sqrt{1-\frac{1}{\nu\nu_f}}\right)$$

so that $g>1$ and $h=0$ when p is sufficiently large.

Therefore we can conclude that $[s_{2p}s_{2p+1}]>1$ and hence the higher harmonics of both the field and armature currents diminish in amplitude with the orders of the harmonics.

§ 4. Permanent short-circuit currents in the field and armature circuits.

Usually $\frac{a_1}{b_1\omega}$ is $\doteqdot 0$ and at short-circuit $\frac{a}{b\omega}$ is $=\frac{a_i}{b_i\omega}$ which is also usually $\doteqdot 0$. Therefore at short-circuit we have usually

$$s_1=s_3=s_5=\cdots\cdots\cdots$$

$$=\frac{2b_i}{c}-\cfrac{1}{\cfrac{2b_1}{c}-\cfrac{1}{\cfrac{2b_i}{c}-\cfrac{1}{\cfrac{2b_1}{c}-\cdots\cdots\cdots}}}\text{ to }\infty$$

$$=\frac{b_i}{c}(1+\sqrt{\sigma})$$

where
$$\sigma=1-\frac{c^2}{bb_1}=1-\frac{1}{\nu\nu_f}$$

and
$$s_2=s_4=s_6=\cdots\cdots\cdots$$

$$=\frac{2b_1}{c}-\cfrac{1}{\cfrac{2b_i}{c}-\cfrac{1}{\cfrac{2b_1}{c}-\cfrac{1}{\cfrac{2b_i}{c}-\cdots\cdots\cdots}}}\text{ to }\infty$$

$$=\frac{b_1}{c}(1+\sqrt{\sigma})$$

so that

$$X_1 \sin \phi_1 = \frac{I_f \sqrt{2}}{\frac{b_i}{c}} \cdot \frac{1}{k} \quad \text{where} \quad k = 1 + \sqrt{\sigma}$$

$$Y_2 \sin \theta_2 = \frac{-I_f \sqrt{2}}{\frac{b_i b_1}{c}} \cdot \frac{1}{k^2} = -I_f \sqrt{2} \cdot \frac{m}{k^2} \quad \text{where} \quad m = \frac{c^2}{b b_1} = \frac{1}{\nu \nu_f} = 1 - \sigma$$

$$X_3 \sin \phi_3 = \frac{I_f \sqrt{2}}{\frac{b_i}{c}} \cdot \frac{m}{k^3}$$

$$Y_4 \sin \theta_4 = -I_f \sqrt{2} \cdot \frac{m^2}{k^4}$$

etc.

and $$X_1 \cos \phi_1 = Y_2 \cos \theta_2 = X_3 \cos \phi_3 = Y_4 \cos \theta_4 = \ldots\ldots\ldots\ldots = 0$$

so that the permanent short-circuit current x_s in the armature is

$$x_s = \frac{-2I_f}{\frac{b_i}{c} k} \cdot (\cos \omega t + n \cos 3\omega t + n^2 \cos 5\omega t + \ldots\ldots\ldots\ldots \text{ to } \infty)$$

$$= \frac{-2I_f}{\frac{b_i}{c} k} \frac{(1-n) \cos \omega t}{1 - 2n \cos 2\omega t + n^2}$$

$$= \frac{-2I_f}{\frac{b_i}{c} k} \frac{(1-n) \cos \omega t}{(1+n)^2 - 4n \cos^2 \omega t}$$

where $$n = \frac{m}{k^2}$$

But $$1 - n = 1 - \frac{1-\sigma}{(1+\sqrt{\sigma})^2} = \frac{2\sqrt{\sigma}}{k} \quad \text{and} \quad 1 + n = \frac{2}{k}$$

Therefore

$$x_s = \frac{-2I_f}{\frac{b_i}{c}k} \cdot \frac{\frac{2\sqrt{\sigma}}{k}\cos \omega t}{\left(\frac{2}{k}\right)^2 - 4\frac{m}{k^2}\cos^2 \omega t}$$

$$= -\frac{cI_f\omega}{b_i\omega} \cdot \frac{\sqrt{\sigma}\cos \omega t}{1-(1-\sigma)\cos^2 \omega t}$$

which coincides with the result obtained by Mr Boucherot and traces the curves as shown in fig. 1.

The permanent short-circuit current y_s in the field winding, not including the exciting current I_f, is

$$y_s = 2I_f n\,(\cos 2\omega t + n\cos 4\omega t + n^2 \cos 6\omega t + \ldots\ldots\ldots\ldots \text{ to } \infty)$$

$$= 2I_f n\,\frac{\cos 2\omega t - n}{1 - 2n\cos 2\omega t + n^2}$$

$$= 2I_f n\,\frac{2\cos^2 \omega t - (1+n)}{(1+n)^2 - 4n\cos^2 \omega t}$$

$$= I_f \frac{m\cos^2 \omega t - \frac{m}{k}}{1 - m\cos^2 \omega t}$$

$$= I_f \frac{(1-\sigma)\cos^2 \omega t - (1-\sqrt{\sigma})}{1-(1-\sigma)\cos^2 \omega t}$$

accordingly

$$y_s + I_f = I_f \frac{\sqrt{\sigma}}{1-(1-\sigma)\cos^2 \omega t}$$

which also coincides with the result obtained by Mr Boucherot and traces the curves as shown in fig. 2.

§ 5. Maximum and effective values of the permanent short-circuit currents.

$\frac{dx_s}{d\theta} = 0$ where $\theta = \omega t$ gives $\sin\theta = 0$ or $1 + (1-\sigma)\cos^2\theta = 0$, the latter of which gives imaginary $\cos\theta$.

Therefore x_s is maximum when $\sin\theta = 0$ that is when $\theta = \pi$, 2π, 3π, etc., and the value of $(x_s)_{\max.}$ is

$$(x_s)_{\max.} = \pm \frac{cI_f\omega}{b_i\omega} \cdot \frac{1}{\sqrt{\sigma}}$$

$$= \pm \frac{\text{max. value of the E.M.F. induced in the armature}}{\text{total armature reactance} \times \text{square root of the dispersion coeff.}}$$

Next $\frac{dy_s}{d\theta} = 0$ gives $\sin\theta\cos\theta = 0$ that is $\sin\theta = 0$ or $\cos\theta = 0$, the former of which gives $\theta = 0$, π, 2π, etc., so that

$$(y_s)_{\text{positive max.}} = I_f \frac{(1-\sigma)-(1-\sqrt{\sigma})}{1-(1-\sigma)} = I_f \frac{1-\sqrt{\sigma}}{\sqrt{\sigma}}$$

and the latter gives $\theta = \frac{\pi}{2}$, $3\frac{\pi}{2}$, $5\frac{\pi}{2}$, etc., so that

$$(y_s)_{\text{negative max.}} = -I_f(1-\sqrt{\sigma}) \text{ which is } < I_f \text{ in magnitude}$$

Accordingly

$$(y_s + I_f)_{\max.} = I_f \cdot \frac{1}{\sqrt{\sigma}} \text{ and } (y_s + I_f)_{\min.} = I_f\sqrt{\sigma}$$

Next, since

$$x_s = \frac{-2I_f}{\frac{b_i}{c}k}(\cos\omega t + n\cos 3\omega t + n^2\cos 5\omega t + \dots\dots\dots\dots \text{ to } \infty)$$

its effective value $(x_s)_{\text{eff.}}$ is

$$(x_s)_{\text{eff.}} = \frac{2I_f}{\frac{b_i}{c}\cdot k} \cdot \sqrt{\frac{1}{2}(1 + n^2 + n^4 + \dots\dots\dots\dots)}$$

$$= \sqrt{2} \cdot \frac{c}{b_i} \cdot I_f \cdot \frac{1}{k} \cdot (1-n^2)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{cI_f\omega}{b_i\omega} \cdot \sigma^{-\frac{1}{4}}$$

and since

$$y_s = 2I_f n(\cos 2\omega t + n\cos 4\omega t + n^2\cos 6\omega t + \dots\dots\dots\dots \text{ to } \infty)$$

its effective value $(y_s)_{\text{eff.}}$ is

$$(y_s)_{\text{eff.}} = 2I_f \cdot \frac{m}{k^2} \cdot \sqrt{\frac{1}{2}(1 + n^2 + n^4 + \ldots\ldots\ldots\ldots)}$$

$$= \sqrt{2} \,.\, I_f \cdot \frac{m}{k^2}(1 - n^2)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot I_f \,.\, (1 - \sqrt{\sigma}) \,.\, \sigma^{-\frac{1}{4}}$$

so that

$$(y_s + I_f)_{\text{eff.}} = I_f \sqrt{1 + \frac{1}{2}(1 - \sqrt{\sigma})^2 \,.\, \sigma^{-\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{2}} \cdot I_f \,.\, \sqrt{1 + \sigma} \,.\, \sigma^{-\frac{1}{4}}$$

Note that $(x_s)_{\text{eff.}}$ can be checked as follows:—

$$(x_s)^2_{\text{eff.}} = \frac{4}{T}\int_0^{\frac{T}{4}} x^2 dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{-2I_f c}{b_i k}\right)^2 \left[\frac{(1 - n)\cos\theta}{1 - 2n\cos 2\theta + n^2}\right]^2 d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(\frac{-2I_f c}{b_i k}\right)^2 \frac{(1 - n)^2 \frac{1}{2}(1 + \cos\alpha)}{(1 - 2n\cos\alpha + n^2)} d\alpha \quad \text{where} \quad \alpha = 2\theta$$

$$= \frac{2}{\pi}\left(\frac{2I_f c}{b_i k}\right)^2 p^2 \int_{\tan 0}^{\tan\frac{\pi}{2}} \frac{dx}{(p^2 + q^2 x^2)^2}$$

where $p = 1 - n$, $q = 1 + n$ and $x = \tan\frac{\alpha}{2} = \tan\theta$

$$= \frac{1}{\pi}\left(\frac{2I_f c}{b_i k}\right)^2 \cdot \left[\frac{x}{p^2 + q^2 x^2} + \frac{1}{pq}\tan^{-1}\frac{q}{p}x\right]_{\tan 0}^{\tan\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{2I_f c}{b_i k}\right)^2 \cdot \frac{1}{pq} \cdot \frac{\pi}{2}$$

$$= 2\left(\frac{I_f c}{b_i k}\right)^2 \frac{1}{1 - n^2}$$

so that $(x_s)_{\text{eff.}} = \sqrt{2}\,\frac{I_f c}{b_i} \cdot \frac{1}{k} \cdot (1 - n^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} \cdot \frac{c I_f \omega}{b_i \omega} \,.\, \sigma^{-\frac{1}{4}}$

Next $(y_s)_{\text{eff.}}$ can be checked as follows:—

$$(y_s)^2_{\text{eff.}}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(2I_f n\frac{\cos 2\theta-n}{1-2n\cos 2\theta+n^2}\right)^2 d\theta$$

$$=(2I_f n)^2\frac{1}{\pi}\int_0^{\pi}\left(\frac{\cos\alpha-n}{1-2n\cos\alpha+n^2}\right)^2 d\alpha$$

$$=(2I_f n)^2\frac{1}{\pi}\int_0^{\tan\frac{\pi}{2}}\left(\frac{p-qx^2}{p^2+q^2x^2}\right)^2\frac{2}{1+x^2}dx$$

$$=\frac{2}{\pi}\frac{(2I_f n)^2}{(p-q)^2}\left[\frac{(q^2-p^2)x}{2(p^2+q^2x^2)}+\frac{p^2+q^2-4pq}{2pq}\tan^{-1}\frac{q}{p}x+\tan^{-1}x\right]_0^{\tan\frac{\pi}{2}}$$

$$=\frac{2}{\pi}\frac{(2I_f n)^2}{(p-q)^2}\left(\frac{p^2+q^2-4pq}{2pq}\cdot\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$=\frac{(2I_f n)^2}{2pq}$$

so that $(y_s)_{\text{eff.}}=2I_f\dfrac{1-\sigma}{k^2}\Big/\sqrt{\dfrac{8\sqrt{\sigma}}{k^2}}=\dfrac{1}{\sqrt{2}}\cdot I_f\dfrac{1-\sigma}{k}\cdot\sigma^{-\frac{1}{4}}$

$$=\frac{1}{\sqrt{2}}\cdot I_f(1-\sqrt{\sigma})\,\sigma^{-\frac{1}{4}}$$

§ 6. Ordinary treatment of the single phase generator.

In the ordinary treatment of the single phase generator, it is usual to consider that the average armature M.M.F., of which the magnitude is $\frac{1}{2}nI\sqrt{2}$ where nI is the effective armature ampere-turns, affects the original field as shown in fig. 3, while that part of the armature M.M.F. which is alternating in direction with regard to the original field is quite inactive.

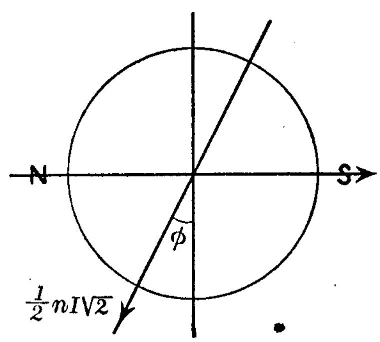


Fig. 3.

With this conception the E.M.F. induced in the armature due to the armature M.M.F. will be, as stated in the theory of the two and three phase generators, $=-\dfrac{1}{2\nu}b_i\dfrac{di}{dt}$ where b_i is the total armature inductance, ν the armature leakage coefficient and i the armature current.

Thus in the ordinary treatment the fundamental equation of the armature circuit will be

$$(a_e + a_i) + \left(b_e + b_l + \frac{1}{2\nu} b_i\right) \frac{di}{dt} = cI_f \omega \sin \omega t$$

where a_e and b_e are the load resistance and inductance and b_l the armature leakage inductance.

Now solving this equation, we have

$$I\sqrt{2} = \frac{cI_f\omega}{\sqrt{(a_e + a_i)^2 + \left(b_e + b_l + \frac{1}{2\nu} b_i\right)^2 \omega^2}}$$

$$= \frac{cI_f\omega}{\sqrt{(a_e + a_i)^2 + \left[b_e + b_i\left(1 - \frac{1}{2\nu}\right)\right]^2 \omega^2}}$$

for $\nu = \frac{b_i}{b_i - b_l}$ so that $b_l = b_i\left(1 - \frac{1}{\nu}\right)$

so that the effective value of the permanent short-circuit current is

$$(i_s)_{\text{eff.}} = \frac{cI_f\omega/\sqrt{2}}{\sqrt{a_i^2 + b_i^2\left(1 - \frac{1}{2\nu}\right)^2 \omega^2}} \doteqdot \frac{1}{\sqrt{2}} \cdot \frac{cI_f\omega}{b_i\omega} \cdot \frac{1}{1 - \frac{1}{2\nu}}$$

Comparing this $(i_s)_{\text{eff.}}$ with that $(x_s)_{\text{eff.}}$ obtained in the previous article we have the ratio $\sigma^{\frac{1}{4}} \Big/ \left(1 - \frac{1}{2\nu}\right)$ which, when $\nu \doteqdot \nu_f$ so that $\sigma = 1 - \frac{1}{\nu^2}$ that is $\frac{1}{\nu} = \sqrt{1 - \sigma}$, becomes as shown in the following table:

σ	0·1	0·2	0·3	0·4	0·46	0·5	0·6	1·0
$\sigma^{\frac{1}{4}} \Big/ \left(1 - \frac{1}{2\nu}\right)$	1·07	1·21	1·27	1·3	1·305	1·3	1·29	1·0

Thus the ordinary treatment is on the safe side with regard to the effective value of the permanent short-circuit current.

Comparing $(i_s)_{\max.}$ we have the ratio $\sigma^{\frac{1}{2}}\Big/\left(1-\frac{1}{2\nu}\right)$ which, when $\nu \doteqdot \nu_f$, becomes

σ	0·1	0·2	0·3	0·4	0·46	0·5	0·6	1·0
$\sigma^{\frac{1}{2}}\Big/\left(1-\frac{1}{2\nu}\right)$	·6	·81	·94	1·04	1·07	1·095	1·13	1·0

Thus with regard to the maximum value of the permanent short-circuit current, the ordinary treatment is on the safe side or not according as $\sigma^{\frac{1}{2}} \gtrless 1 - \frac{1}{2\nu}$, that is according as $\sigma \gtrless 0{\cdot}36$.

§ 7. Sudden short-circuit currents.

Below is given an approximate solution of the fundamental equations for sudden short-circuit currents.

If we consider y including the exciting current, then the fundamental equations may be written in the form

$$a_i x + b_i \frac{dx}{dt} + c \frac{d}{dt}(y \cos \omega t) = 0$$

$$a_1 y + b_1 \frac{dy}{dt} + c \frac{d}{dt}(x \cos \omega t) = a_1 I_f$$

that is

$$\frac{dx}{d\theta} + Ax + B \frac{d}{d\theta}(y \cos \theta) = 0 \quad \text{.........................(1)}$$

$$\frac{dy}{d\theta} + A_1 y + B_1 \frac{d}{d\theta}(x \cos \theta) = A_1 I_f \quad \text{....................(2)}$$

where $\quad A = \frac{a_i}{b_i \omega}, \quad A_1 = \frac{a_1}{b_1 \omega}, \quad B = \frac{c}{b}, \quad B_1 = \frac{c}{b_1}$ and $\theta = \omega t$

Neglecting for approximation the term containing A, we have from (1)

$$x = -By \cos \theta + k$$

so that equation (2) becomes

$$\frac{dy}{d\theta} + A_1 y - B_1 \frac{d}{d\theta}[(k - By \cos \theta) \cos \theta] = A_1 I_f$$

that is

$$(1-BB_1\cos^2\theta)\frac{dy}{d\theta}+(A_1+2BB_1\sin\theta\cos\theta)\,y=A_1I_f+B_1k\sin\theta$$

that is

$$\frac{dy}{d\theta}+\frac{A_1+2m\sin\theta\cos\theta}{1-m\cos^2\theta}\,y=\frac{A_1I_f+B_1k\sin\theta}{1-m\cos^2\theta}$$

where $$m=BB_1=\frac{c^2}{bb_1}=\frac{1}{\nu\nu_f}=1-\sigma$$

which solves to

$$y\epsilon^{\int P\,d\theta}=\int\frac{A_1I_f+B_1k\sin\theta}{1-m\cos^2\theta}\,.\,\epsilon^{\int P\,d\theta}\,.\,d\theta+k_1$$

where $$P=\frac{A_1+2m\sin\theta\cos\theta}{1-m\cos^2\theta}$$

Hence, neglecting the constants k and k_1, the principal integral is

$$y\epsilon^{\int P\,d\theta}=\int\frac{A_1I_f}{1-m\cos^2\theta}\cdot\epsilon^{\int P\,d\theta}\,.\,d\theta$$

But

$$\int P\,d\theta=\frac{A_1}{2}\int\left(\frac{d\theta}{1-\sqrt{m}\cos\theta}+\frac{d\theta}{1+\sqrt{m}\cos\theta}\right)+\sqrt{m}\int\frac{2\sqrt{m}\cos\theta\sin\theta}{1-m\cos^2\theta}\,.\,d\theta$$

$$=\qquad\qquad ,, \qquad\qquad +\int\left(\frac{\sqrt{m}\sin\theta\,d\theta}{1-\sqrt{m}\cos\theta}-\frac{\sqrt{m}\sin\theta\,d\theta}{1+\sqrt{m}\cos\theta}\right)$$

$$=\frac{A_1}{\sqrt{1-m}}\left[\tan^{-1}\left(\frac{\sqrt{1+\sqrt{m}}}{\sqrt{1-\sqrt{m}}}\tan\frac{\theta}{2}\right)+\tan^{-1}\left(\frac{\sqrt{1-\sqrt{m}}}{\sqrt{1+\sqrt{m}}}\tan\frac{\theta}{2}\right)\right]$$

$$+\log(1-\sqrt{m}\cos\theta)+\log(1+\sqrt{m}\cos\theta)$$

and $$\tan^{-1}\left(\frac{\sqrt{1+\sqrt{m}}}{\sqrt{1-\sqrt{m}}}\tan\frac{\theta}{2}\right)+\tan^{-1}\left(\frac{\sqrt{1-\sqrt{m}}}{\sqrt{1+\sqrt{m}}}\tan\frac{\theta}{2}\right)$$

$$=\tan^{-1}\left(\frac{2}{\sqrt{1-m}}\,\frac{\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\frac{1}{\sqrt{\sigma}}\tan\theta\right)$$

which is nearly equal to θ when θ is sufficiently large.

Figure 4 represents the curves of $\tan^{-1}\left(\frac{1}{\sqrt{\sigma}}\tan\theta\right)$ and its components

$$\tan^{-1}\left(\frac{\sqrt{1+\sqrt{m}}}{\sqrt{1-\sqrt{m}}}\tan\frac{\theta}{2}\right) \quad \text{and} \quad \tan^{-1}\left(\frac{\sqrt{1-\sqrt{m}}}{\sqrt{1+\sqrt{m}}}\tan\frac{\theta}{2}\right)$$

Therefore

$$\int P d\theta \doteqdot \frac{A_1}{\sqrt{1-m}}\theta + \log(1 - m\cos^2\theta)$$

so that

$$\epsilon^{\int P d\theta} \doteqdot (1 - m\cos^2\theta) \, . \, \epsilon^{\frac{A_1}{\sqrt{1-m}}\theta}$$

Hence we have

$$y(1 - m\cos^2\theta)\,\epsilon^{\frac{A_1}{\sqrt{1-m}}\theta} = \int A_1 I_f \epsilon^{\frac{A_1}{\sqrt{1-m}}\theta} \, . \, d\theta = I_f\sqrt{1-m} \, . \, \epsilon^{\frac{A_1}{\sqrt{1-m}}\theta}$$

so that

$$y = I_f \frac{\sqrt{1-m}}{1 - m\cos^2\theta} = I_f \frac{\sqrt{\sigma}}{1-(1-\sigma)\cos^2\theta}$$

and hence

$$x = -I_f \frac{B\sqrt{\sigma}\cos\theta}{1-(1-\sigma)\cos^2\theta} = -\frac{cI_f\omega}{b_i\omega} \cdot \frac{\sqrt{\sigma}\cos\theta}{1-(1-\sigma)\cos^2\theta}$$

which coincide with those expressions of the permanent short-circuit currents obtained in Art. 4.

Next, to determine the complementary functions we have the equations with the right-hand sides equal to zero, namely

$$\frac{dx}{d\theta} + Ax + B\frac{d}{d\theta}(y\cos\theta) = 0 \quad \text{......................(3)}$$

$$\frac{dy}{d\theta} + A_1 y + B_1\frac{d}{d\theta}(x\cos\theta) = 0 \quad \text{....................(4)}$$

which give, when the terms containing A and A_1 are neglected,

$$x = -By\cos\theta \quad \text{and} \quad y = -B_1 x \sin\theta$$

so that

$$x = \pm y\sqrt{\frac{B}{B_1}}$$

Note that here for finding particular integrals of equations (3) and (4) we drop the integration constants.

Now, put this relation between x and y in equation (4). Then we have

$$(1 \pm \sqrt{BB_1}\cos\theta)\frac{dy}{d\theta} + (A_1 \mp \sqrt{BB_1}\sin\theta)\,y = 0$$

which solves to

$$\log y = -\int\frac{A_1 \mp \sqrt{m}\sin\theta}{1 \pm \sqrt{m}\cos\theta}\,d\theta = -A_1\int\frac{d\theta}{1 \pm \sqrt{m}\cos\theta} \pm \int\frac{\sqrt{m}\sin\theta\, d\theta}{1 \pm \sqrt{m}\cos\theta}$$

$$= -\frac{2A_1}{\sqrt{1-m}}\tan^{-1}\left(\frac{\sqrt{1 \mp \sqrt{m}}}{\sqrt{1 \pm \sqrt{m}}}\tan\frac{\theta}{2}\right) - \log(1 \pm \sqrt{m}\cos\theta)$$

that is

$$y = \frac{1}{1 \pm \sqrt{m}\cos\theta}\cdot \epsilon^{-\frac{2A_1}{\sqrt{1-m}}\cdot\tan^{-1}\left(\frac{\sqrt{1 \mp \sqrt{m}}}{\sqrt{1 \pm \sqrt{m}}}\tan\frac{\theta}{2}\right)}$$

Therefore the complementary function of y is

$$\frac{C_1}{1+\sqrt{m}\cos\theta}\cdot\epsilon^{-\alpha_1(\theta)} + \frac{C_2}{1-\sqrt{m}\cos\theta}\cdot\epsilon^{-\alpha_2(\theta}$$

where C_1 and C_2 are arbitrary constants

and $$\alpha_1(\theta) = \frac{2A_1}{1-m}\tan^{-1}\left(\frac{\sqrt{1-\sqrt{m}}}{\sqrt{1+\sqrt{m}}}\tan\frac{\theta}{2}\right)$$

and $$\alpha_2(\theta) = \frac{2A_1}{1-m}\tan^{-1}\left(\frac{\sqrt{1+\sqrt{m}}}{\sqrt{1-\sqrt{m}}}\tan\frac{\theta}{2}\right)$$

and since $x = \pm y\sqrt{\frac{B}{B_1}}$ the complementary function of x is

$$\sqrt{\frac{b_1}{b_i}}\cdot\left(\frac{C_1}{1+\sqrt{m}\cos\theta}\cdot\epsilon^{-\alpha_1(\theta)} - \frac{C_2}{1-\sqrt{m}\cos\theta}\cdot\epsilon^{-\alpha_2(\theta)}\right)$$

If we put $y = \pm x\sqrt{\frac{B_1}{B}}$ in equation (3) then, similarly as above, the complementary function of x is

$$\frac{D_1}{1+\sqrt{m}\cos\theta}\cdot\epsilon^{-\beta_1(\theta)} + \frac{D_2}{1-\sqrt{m}\cos\theta}\cdot\epsilon^{-\beta_2(\theta)}$$

where D_1 and D_2 are arbitrary constants

and $$\beta_1(\theta) = \frac{2A}{\sqrt{1-m}}\tan^{-1}\left(\frac{\sqrt{1-\sqrt{m}}}{\sqrt{1+\sqrt{m}}}\tan\frac{\theta}{2}\right)$$

and $$\beta_2(\theta) = \frac{2A}{\sqrt{1-m}}\tan^{-1}\left(\frac{\sqrt{1+\sqrt{m}}}{\sqrt{1-\sqrt{m}}}\tan\frac{\theta}{2}\right)$$

and that of y is

$$\sqrt{\frac{b_i}{b_1}}\cdot\left(\frac{D_1}{1+\sqrt{m}\cos\theta}\cdot\epsilon^{-\beta_1(\theta)} - \frac{D_2}{1-\sqrt{m}\cos\theta}\cdot\epsilon^{-\beta_2(\theta)}\right)$$

Now, if we take up the first form of the complementary functions and put $\alpha_1(\theta) \div \alpha_2(\theta) \div \frac{A_1}{\sqrt{\sigma}}\theta$, then the complete solutions of the sudden short-circuit currents are

$$y = \frac{I_f}{2}\left(\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta} + \frac{\sqrt{\sigma}}{1-\sqrt{m}\cos\theta}\right)$$

$$+\left(\frac{C_1}{1+\sqrt{m}\cos\theta} + \frac{C_2}{1-\sqrt{m}\cos\theta}\right)\epsilon^{-\frac{A_1}{\sqrt{\sigma}}\theta}$$

$$x=\frac{I_f}{2}\cdot\sqrt{\frac{b_1}{b_i}}\cdot\left(\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta}-\frac{\sqrt{\sigma}}{1-\sqrt{m}\cos\theta}\right)$$

$$+\sqrt{\frac{b_1}{b_i}}\cdot\left(\frac{C_1}{1+\sqrt{m}\cos\theta}-\frac{C_2}{1-\sqrt{m}\cos\theta}\right)\epsilon^{-\frac{A_1}{\sqrt{\sigma}}\theta}$$

for

$$\frac{c}{b_i}\frac{1}{\sqrt{m}}=\sqrt{\frac{b_1}{b_i}}$$

Therefore if the initial condition for fixing the constants C_1 and C_2 be $x=0$ and $y=I_f$ at $\theta=\theta_0$, then we have

$$\frac{I_f}{2}\left(\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta_0}+\frac{\sqrt{\sigma}}{1-\sqrt{m}\cos\theta_0}\right)$$

$$+\left(\frac{C_1}{1+\sqrt{m}\cos\theta_0}+\frac{C_2}{1-\sqrt{m}\cos\theta_0}\right)\cdot\epsilon^{-\frac{A_1}{\sqrt{\sigma}}\theta_0}=I_f$$

$$\frac{I_f}{2}\left(\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta_0}-\frac{\sqrt{\sigma}}{1-\sqrt{m}\cos\theta_0}\right)$$

$$+\left(\frac{C_1}{1+\sqrt{m}\cos\theta_0}-\frac{C_2}{1-\sqrt{m}\cos\theta_0}\right)\cdot\epsilon^{-\frac{A_1}{\sqrt{\sigma}}\theta_0}=0$$

so that

$$I_f\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta_0}+\frac{2C_1}{1+\sqrt{m}\cos\theta_0}\cdot\epsilon^{-\frac{A_1}{\sqrt{\sigma}}\theta_0}=I$$

that is

$$C_1=\frac{1}{2}I_f(1-\sqrt{\sigma}+\sqrt{m}\cos\theta_0)\,.\,\epsilon^{\frac{A_1}{\sqrt{\sigma}}\theta_0}$$

and similarly

$$C_2=\frac{1}{2}I_f(1-\sqrt{\sigma}-\sqrt{m}\cos\theta_0)\,.\,\epsilon^{\frac{A_1}{\sqrt{\sigma}}\theta_0}$$

Accordingly the complete solutions become

$$y = I_f \frac{\sqrt{\sigma}}{1 - m\cos^2\theta} + \frac{1}{2} I_f \left(\frac{1 - \sqrt{\sigma} + \sqrt{m}\cos\theta_0}{1 + \sqrt{m}\cos\theta} + \frac{1 - \sqrt{\sigma} - \sqrt{m}\cos\theta_0}{1 - \sqrt{m}\cos\theta} \right) \times \epsilon^{-\frac{A_1}{\sqrt{\sigma}}(\theta - \theta_0)}$$

$$= I_f \frac{\sqrt{\sigma}}{1 - m\cos^2\theta} + I_f \frac{1 - \sqrt{\sigma} - m\cos\theta_0\cos\theta}{1 - m\cos^2\theta} \epsilon^{-\frac{A_1}{\sqrt{\sigma}}(\theta - \theta_0)}$$

and

$$x = -I_f \cdot \sqrt{\frac{b_1}{b_i}} \cdot \frac{\sqrt{m\sigma}\cos\theta}{1 - m\cos^2\theta} + I_f \cdot \sqrt{\frac{b_1}{b_i}} \cdot \frac{\sqrt{m}\cos\theta_0 - (1 - \sqrt{\sigma})\sqrt{m}\cos\theta}{1 - m\cos^2\theta} \times \epsilon^{-\frac{A_1}{\sqrt{\sigma}}(\theta - \theta_0)}$$

$$= -I_f \cdot \frac{c}{b_i} \cdot \frac{\sqrt{\sigma}\cos\theta}{1 - m\cos^2\theta} - I_f \cdot \frac{c}{b_i} \cdot \frac{(1 - \sqrt{\sigma})\cos\theta - \cos\theta_0}{1 - m\cos^2\theta} \cdot \epsilon^{-\frac{A_1}{\sqrt{\sigma}}(\theta - \theta_0)}$$

which, when $A_1 \doteqdot 0$, become

$$y = I_f \cdot \frac{1 - m\cos\theta_0\cos\theta}{1 - m\cos^2\theta} \quad \text{and} \quad x = -I_f \frac{c}{b_i} \cdot \frac{\cos\theta - \cos\theta_0}{1 - m\cos^2\theta}$$

If we do not put $\alpha_1(\theta) = \alpha_2(\theta)$ in the above solution, then we have

$$x = \frac{I_f}{2} \cdot \sqrt{\frac{b_1}{b_i}} \cdot \left(\frac{\sqrt{\sigma}}{1 + \sqrt{m}\cos\theta} - \frac{\sqrt{\sigma}}{1 - \sqrt{m}\cos\theta} \right) + \sqrt{\frac{b_1}{b_i}} \cdot \left(\frac{C_1}{1 + \sqrt{m}\cos\theta} \epsilon^{-\alpha_1(\theta)} - \frac{C_2}{1 - \sqrt{m}\cos\theta} \epsilon^{-\alpha_2(\theta)} \right)$$

and

$$y = \frac{I_f}{2} \cdot \left(\frac{\sqrt{\sigma}}{1 + \sqrt{m}\cos\theta} + \frac{\sqrt{\sigma}}{1 - \sqrt{m}\cos\theta} \right) + \left(\frac{C_1}{1 + \sqrt{m}\cos\theta} \epsilon^{-\alpha_1(\theta)} + \frac{C_2}{1 - \sqrt{m}\cos\theta} \epsilon^{-\alpha_2(\theta)} \right)$$

which are also obtained by putting $\frac{a}{b}=\frac{a_1}{b_1}$ in the fundamental equations as Mr Biermann* did. Putting $\frac{a}{b}=\frac{a_1}{b_1}$ in the fundamental equations we arrive at these results without any neglect. This assumption is however not allowable in general.

If we take up the second form of the complementary functions and put $\beta_1(\theta) \doteqdot \beta_2(\theta) \doteqdot \frac{A}{\sqrt{\sigma}}\,\theta$, then the complete solutions become

$$x=\frac{I_f}{2}\cdot\sqrt{\frac{b_1}{b_i}}\left(\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta}-\frac{\sqrt{\sigma}}{1-\sqrt{m}\cos\theta}\right)$$
$$+\left(\frac{D_1}{1+\sqrt{m}\cos\theta}+\frac{D_2}{1-\sqrt{m}\cos\theta}\right)\cdot\epsilon^{-\frac{A}{\sqrt{\sigma}}\theta}$$

$$y=\frac{I_f}{2}\cdot\left(\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta}+\frac{\sqrt{\sigma}}{1-\sqrt{m}\cos\theta}\right)$$
$$+\sqrt{\frac{b_i}{b_1}}\left(\frac{D_1}{1+\sqrt{m}\cos\theta}-\frac{D_2}{1-\sqrt{m}\cos\theta}\right)\cdot\epsilon^{-\frac{A}{\sqrt{\sigma}}\theta}$$

so that with the same initial condition as before we have

$$\frac{I_f}{2}\left(\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta_0}-\frac{\sqrt{\sigma}}{1-\sqrt{m}\cos\theta_0}\right)$$
$$+\sqrt{\frac{b_i}{b_1}}\cdot\left(\frac{D_1}{1+\sqrt{m}\cos\theta_0}+\frac{D_2}{1-\sqrt{m}\cos\theta_0}\right)\cdot\epsilon^{-\frac{A}{\sqrt{\sigma}}\theta}=0$$

$$\frac{I_f}{2}\left(\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta_0}+\frac{\sqrt{\sigma}}{1-\sqrt{m}\cos\theta_0}\right)$$
$$+\sqrt{\frac{b_i}{b_1}}\cdot\left(\frac{D_1}{1+\sqrt{m}\cos\theta_0}-\frac{D_2}{1-\sqrt{m}\cos\theta_0}\right)\cdot\epsilon^{-\frac{A}{\sqrt{\sigma}}\theta}\doteqdot I_f$$

so that

$$I_f\cdot\frac{\sqrt{\sigma}}{1+\sqrt{m}\cos\theta_0}+\sqrt{\frac{b_i}{b_1}}\cdot\frac{2D_1}{1+\sqrt{m}\cos\theta_0}=I_f$$

* E.T.Z. 1915. p. 579.

that is

$$D_1 = \frac{1}{2} \cdot I_f \cdot \sqrt{\frac{b_i}{b_1}} \cdot (1 - \sqrt{\sigma} + \sqrt{m} \cos \theta_0) \cdot \epsilon^{\frac{A}{\sqrt{\sigma}} \theta_0}$$

and similarly

$$D_2 = -\frac{1}{2} \cdot I_f \cdot \sqrt{\frac{b_i}{b_1}} \cdot (1 - \sqrt{\sigma} - \sqrt{m} \cos \theta_0) \cdot \epsilon^{\frac{A}{\sqrt{\sigma}} \theta_0}$$

Accordingly the complete solutions become, similarly as before,

$$x = -I_f \cdot \frac{c}{b_i} \cdot \frac{\sqrt{\sigma} \cos \theta}{1 - m \cos^2 \theta} - I_f \cdot \frac{c}{b_i} \cdot \frac{(1 - \sqrt{\sigma}) \cos \theta - \cos \theta_0}{1 - m \cos^2 \theta} \cdot \epsilon^{-\frac{A}{\sqrt{\sigma}} (\theta - \theta_0)}$$

$$y = I_f \frac{\sqrt{\sigma}}{1 - m \cos^2 \theta} + I_f \frac{1 - \sqrt{\sigma} - m \cos \theta_0 \cos \theta}{1 - m \cos^2 \theta} \cdot \epsilon^{-\frac{A}{\sqrt{\sigma}} (\theta - \theta_0)}$$

which are nothing other than those found before with A_1 changed to A and so give, when A is put $\doteqdot 0$,

$$y = I_f \frac{1 - m \cos \theta_0 \cos \theta}{1 - m \cos^2 \theta} \quad \text{and} \quad x = -I_f \cdot \frac{c}{b_i} \cdot \frac{\cos \theta - \cos \theta_0}{1 - m \cos^2 \theta}$$

which are the same as obtained before by taking up the first form of complementary functions.

§ 8. Maximum sudden short-circuit currents.

We saw in the previous article that when $A \doteqdot A_1 \doteqdot 0$ the sudden short-circuit currents are

$$x = -I_f \cdot \frac{c}{b_i} \cdot \frac{\cos \theta - \cos \theta_0}{1 - m \cos^2 \theta} \quad \text{and} \quad y = I_f \frac{1 - m \cos \theta_0 \cos \theta}{1 - m \cos^2 \theta}$$

Now to find the maxima and minima of these,

$$\frac{\partial x}{\partial \theta_0} = 0 \text{ gives } \sin \theta_0 = 0 \text{ and } \frac{\partial x}{\partial \theta} = 0 \text{ gives } \sin \theta = 0$$

or $\quad m \cos^2 \theta - 2m \cos \theta_0 \cos \theta + 1 = 0$ that is $\cos \theta = \cos \theta_0 \pm \sqrt{\cos^2 \theta_0 - \frac{1}{m}}$

which is imaginary.

Therefore

$$x_{\text{max.}} = -I_f \frac{c}{b_i} \cdot \frac{\cos\theta \pm 1}{1 - m\cos^2\theta}$$

and

$$x_{\text{max. max.}} = \pm 2 \cdot \frac{cI_f\omega}{b_i\omega} \cdot \frac{1}{\sigma}$$

Next $\frac{\partial y}{\partial \theta_0} = 0$ gives $\sin\theta_0 = 0$ and $\frac{\partial y}{\partial \theta} = 0$ gives $\sin\theta = 0$ or

$$m^2 \cos\theta_0 \cos^2\theta - 2m\cos\theta + m\cos\theta_0 = 0$$

that is

$$\cos\theta = \frac{1 \pm \sqrt{1 - m\cos^2\theta_0}}{m\cos\theta_0}$$

of which only the negative sign is permissible.

Therefore

$$y_{\text{max.}} \text{ (including the exciting current } I_f) = I_f \frac{1 - m\cos\theta}{1 - m\cos^2\theta}$$

and

$$y_{\text{max. max.}} = I_f \frac{1+m}{1-m} \quad \text{or} \quad I_f \quad \text{or} \quad I_f \frac{1 - (1 - \sqrt{1-m})}{m - (1 - \sqrt{1-m})^2} m$$

that is

$$y_{\text{max max.}} = I_f\left(\frac{2}{\sigma} - 1\right) \quad \text{or} \quad I_f \quad \text{or} \quad I_f \frac{m\sqrt{1-m}}{2\sqrt{1-m} - 2(1-m)}$$

$$= I_f\left(\frac{2}{\sigma} - 1\right) \quad \text{or} \quad I_f \quad \text{or} \quad \frac{1}{2} I_f (1 + \sqrt{\sigma})$$

which shows that the maximum value of y is $I_f\left(\frac{2}{\sigma} - 1\right)$ and the minimum value is $\frac{1}{2} I_f (1 + \sqrt{\sigma})$.

These maximum values of x and y as found above coincide with those obtained by Mr Boucherot. Mr Biermann also arrived at the same result by putting $\frac{a}{b}=\frac{a_1}{b_1}$ and $\alpha_1(\theta) \doteqdot \alpha_2(\theta) \doteqdot \frac{A}{\sqrt{\sigma}}\theta \doteqdot \frac{A_1}{\sqrt{\sigma}}\theta$. Mr Berg's solution, as described before in the three phase generator, gives too big a figure of the maximum sudden short-circuit current.

Curves showing $x_{\text{max.}} = I_f \frac{c}{b_i} \cdot \frac{1-\cos\theta}{1-m\cos^2\theta}$ are given in figures 5 and 6 and those showing $y_{\text{max.}} = I_f \frac{1-m\cos\theta}{1-m\cos^2\theta}$ in figures 7 and 8.

Putting $\theta_0 = 0$ in the complete solution of x and y obtained in the previous article, we have

$$(x_s)_{\text{max.}} \doteqdot -I_f \cdot \frac{c}{b_i} \cdot \frac{\sqrt{\sigma}\cos\theta}{1-m\cos^2\theta} + I_f \cdot \frac{1-(1-\sqrt{\sigma})\cos\theta}{1-m\cos^2\theta} \cdot \epsilon^{-\frac{A_1}{\sqrt{\sigma}}(\theta-\theta_0)}$$

and

$$(y_s)_{\text{max.}} \doteqdot I_f \cdot \frac{\sqrt{\sigma}}{1-m\cos^2\theta} + I_f \cdot \frac{1-\sqrt{\sigma}-(1-\sigma)\cos\theta}{1-m\cos^2\theta} \cdot \epsilon^{-\frac{A_1}{\sqrt{\sigma}}(\theta-\theta_0)}$$

which show approximately the manner in which the instantaneous values of the maximum sudden short-circuit currents change with time starting at $\theta=\theta_0$. Curves showing this $(x_s)_{\text{max.}}$ are given in figures 9 and 10 and those showing $(y_s)_{\text{max.}}$ in figures 11 and 12. In all these figures A is taken $= 3/100$.

§ 9. Electromotive forces induced in open phases at short-circuit of one phase when the armature is wound in two or three phases.

As shown in Art. 4, the permanent short-circuit currents are

$$(x_s)_{\text{permanent}} = -\frac{c}{b} \cdot I_f \frac{\sqrt{\sigma}\cos\theta}{1-m\cos^2\theta}$$

$$(y_s + I_f)_{\text{permanent}} = I_f \frac{\sqrt{\sigma}}{1-m\cos^2\theta}$$

where $\theta = \omega t$ and $m = 1-\sigma$.

Therefore if the armature be wound in three phases, then the E.M.F.s induced in the open phases at steady short-circuit of any one phase are

$$= -c\omega \frac{d}{d\theta}\left[I_f \frac{\sqrt{\sigma}\cos\left(\theta \mp \frac{2\pi}{3}\right)}{1 - m\cos^2\theta}\right] - \frac{-1}{2\nu} b_i\omega \frac{d}{d\theta}\left(\frac{c}{b_i} I_f \frac{\sqrt{\sigma}\cos\theta}{1 - m\cos^2\theta}\right)$$

$$= -cI_f\omega\sqrt{\sigma}\,.\frac{d}{d\theta}\cdot\left[\frac{\cos\left(\theta \mp \frac{2\pi}{3}\right) + \frac{1}{2\nu}\cos\theta}{1 - m\cos^2\theta}\right]$$

$$\doteqdot -cI_f\omega\sqrt{\sigma}\,.\frac{d}{d\theta}\cdot\left[\frac{\cos\left(\theta \mp \frac{2\pi}{3}\right) + \frac{1}{2}\cos\theta}{1 - m\cos^2\theta}\right]$$

$$= \mp\frac{\sqrt{3}}{2} cI_f\omega\sqrt{\sigma}\frac{d}{d\theta}\left(\frac{\sin\theta}{1 - m\cos^2\theta}\right)$$

$$= \mp\frac{\sqrt{3}}{2} cI_f\omega\sqrt{\sigma}\,.\frac{\sigma - m\sin^2\theta}{(1 - m\cos^2\theta)^2}\cdot\cos\theta$$

which trace the curves as shown in figure 13, and the maximum instantaneous values of which appear at $\theta = 0$ or π and have the value

$$\pm\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{\sigma}}\cdot cI_f\omega$$

Next as shown in Art. 7, the sudden short-circuit currents are, when $A \doteqdot A_1 \doteqdot 0$,

$$(x_s)_{\text{sudden}} = -\frac{c}{b_i}\cdot I_f\frac{\cos\theta - \cos\theta_0}{1 - m\cos^2\theta} \quad \text{where } \theta_0 = \omega t_0$$

$$(y_s + I_f)_{\text{sudden}} = I_f\frac{1 - m\cos\theta_0\cos\theta}{1 - m\cos^2\theta}$$

so that if the armature be wound in three phases, then the E.M.F.s induced in the open phases at sudden short-circuit are, similarly as above,

$$\doteqdot - cI_f\omega \frac{d}{d\theta}\cdot\left[\frac{1-m\cos\theta_0\cos\theta}{1-m\cos^2\theta}\cos\left(\theta\mp\frac{2\pi}{3}\right)+\frac{1}{2}\,\frac{\cos\theta-\cos\theta_0}{1-m\cos^2\theta}\right]$$

$$= - cI_f\omega\frac{d}{d\theta}\cdot\left[\frac{\cos\left(\theta\mp\frac{2\pi}{3}\right)+\frac{1}{2}\cos\theta}{1-m\cos^2\theta}\right]$$

$$+ cI_f\omega\cos\theta_0\,.\,\frac{d}{d\theta}\left[\frac{m\cos\theta\,.\,\cos\left(\theta\mp\frac{2\pi}{3}\right)+\frac{1}{2}}{1-m\cos^2\theta}\right]$$

$$=\mp\frac{\sqrt{3}}{2}cI_f\omega\,\frac{\sigma-m\sin^2\theta}{(1-m\cos^2\theta)^2}\cos\theta$$

$$+\frac{1}{2}cI_f\omega\cos\theta_0\,.\,\frac{d}{d\theta}\left(\frac{1-m\cos^2\theta\pm\sqrt{3}\,m\cos\theta\sin\theta}{1-m\cos^2\theta}\right)$$

$$=\mp\frac{\sqrt{3}}{2}cI_f\omega\left[\frac{\sigma-m\sin^2\theta}{(1-m\cos^2\theta)^2}\cos\theta-m\cos\theta_0\,.\,\frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{1-m\cos^2\theta}\right)\right]$$

But
$$\frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{1-m\cos^2\theta}\right)=\frac{1-m-(2-m)\sin^2\theta}{(1-m\cos^2\theta)^2}$$

Therefore the E.M.F.s induced in the open phases are

$$=\mp\frac{\sqrt{3}}{2}cI_f\omega\,\frac{(\sigma-m\sin^2\theta)\cos\theta-m\left[\sigma-(1+\sigma)\sin^2\theta\right]\cos\theta_0}{(\sigma+m\sin^2\theta)^2}$$

which are maximum when $\sin\theta_0=0$ and the maximum E.M.F.s are

$$=\mp\frac{\sqrt{3}}{2}cI_f\omega\,\frac{(\sigma-m\sin^2\theta)\cos\theta-m\left[\sigma-(1+\sigma)\sin^2\theta\right]}{(\sigma+m\sin^2\theta)^2}$$

Figure 14 shows the curves of

$$-\frac{\sqrt{3}}{2}\cdot\frac{(\sigma-m\sin^2\theta)\cos\theta-m\left[\sigma-(1+\sigma)\sin^2\theta\right]}{(\sigma+m\sin^2\theta)^2}\quad\text{when }\sigma=0{\cdot}4\text{ and }0{\cdot}1$$

The maximum values of these curves take place at $\theta=\pi$ and the maximum value is $\frac{\sqrt{3}}{2}\cdot\frac{1+m}{\sigma}$ that is $\frac{\sqrt{3}}{2}\cdot\frac{2-\sigma}{\sigma}$.

If we take another sign of the above expression of the E.M.F. induced, then we have curves the same in character and magnitude as those shown in figure 14, and so we can conclude that the maximum values of the E.M.F.s induced in the open phases are

$$\pm \frac{\sqrt{3}}{2} \cdot \frac{2-\sigma}{\sigma} \cdot cI_f \omega$$

Next if the armature be wound in two phases, then the E.M.F.s induced in the open phase at steady short-circuit of one phase are

$$= - c\omega \frac{d}{d\theta} \cdot \left[I_f \frac{\sqrt{\sigma} \cos\left(\theta \mp \frac{\pi}{2}\right)}{1 - m\cos^2\theta} \right]$$

$$= \mp c\omega \frac{d}{d\theta} \left(I_f \frac{\sqrt{\sigma} \sin\theta}{1 - m\cos^2\theta} \right)$$

$$= \mp cI_f \omega \sqrt{\sigma} \frac{\sigma - m\sin^2\theta}{(1 - m\cos^2\theta)^2} \cos\theta$$

which trace the same curves as shown in figure 13 and the maximum instantaneous values are

$$\pm \frac{1}{\sqrt{\sigma}} cI_f \omega$$

which coincide with the result obtained by Mr Boucherot.

Next, the E.M.F.s induced in the two phase generator at sudden short-circuit of one phase are

$$= - c\omega \frac{d}{d\theta} \left[I_f \frac{1 - m\cos\theta_0 \cos\theta}{1 - m\cos^2\theta} \cdot \cos\left(\theta \mp \frac{\pi}{2}\right) \right]$$

$$= - cI_f \omega \cdot \frac{d}{d\theta} \left(\frac{1 - m\cos\theta_0\cos\theta}{1 - m\cos^2\theta} \cdot \sin\theta \right)$$

$$= \mp cI_f \omega \cdot \frac{(\sigma - m\sin^2\theta)\cos\theta - m\left[\sigma - (1+\sigma)\sin^2\theta\right]\cos\theta_0}{(\sigma + m\sin^2\theta)^2}$$

which are the same as those for the three phase generator except that the numerical coefficient $\frac{\sqrt{3}}{2}$ is replaced by 1.

The maximum instantaneous values of these induced E.M.F.s are obviously

$$\pm \frac{2-\sigma}{\sigma} \cdot cI_f \omega$$

which do not coincide with those results obtained by Mr Boucherot. He gives $\pm \frac{1}{\sigma} cI_f\omega$ in place of these maximum instantaneous values. Also he does not give the expressions for the three phase generator, which we have treated in the beginning of this article.

§ 10. Permanent and sudden short-circuit between two terminals of interconnected three and two phase generators.

In the previous article we have considered sudden and permanent short-circuit of one phase, of two and three phase generators. In this article we will consider the case when sudden and permanent short-circuit takes place between two terminals of the interconnected three and two phase generators.

First, consider the three phases in star connection and as before denote by b and c the total self inductance and the maximum mutual inductance of one phase. Now referring to figure 15, if short-circuit takes place between the terminals B and C while the phase A is open, then, since current flows from B to C or from C to B, we have to place $\sqrt{3}\,c$ in place of c and $3b_i$ in place of b_i in calculating the permanent and sudden short-circuit currents from the formulae for the single phase generator. As to the calculation of the E.M.F. induced in the open phase A when short-circuit takes place between the terminals B and C, the same expressions as given for the two phase generator in the previous article will do without any change because the resultant field produced by the two phases B and C carrying current in the direction B to C or C to B is perpendicular to and hence independent of the phase A. Note that, in the expressions of the short-circuit current and E.M.F.s induced in the open phases, the origin of time is that instant when the axis of the coil A is at the neutral point of the original field produced by the exciting current I_f. Also note that the value of σ increases owing to the increase of the armature leakage flux.

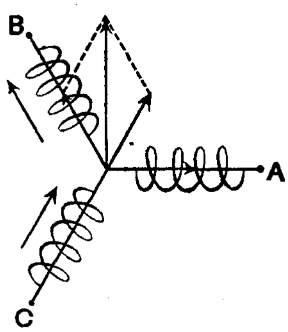


Fig. 15.

Next, considering the three phase generator in delta connection and short-circuited between any two terminals A and B as shown in figure 16, there are two circuits shorted, the one consisting of two phases AC and CB in series, and the other consisting of the phase AB only. Neglecting the ohmic resistance, the two circuits carry the currents in the same phase; and assuming the magnetic leakage of the two circuits equal, the two circuits have the same total self inductance; so that the total short-circuit current is nearly double that when only one phase AB is short-circuited. Here note that, since the magnetic leakage of the circuit consisting of two phases AC and CB is greater than that of the phase AB only, the total short-circuit current will be less than double that when only one phase AB is short-circuited.

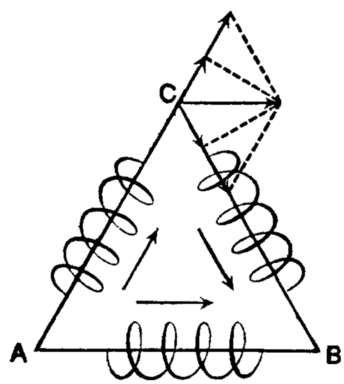


Fig. 16.

Next, if the two phase generator inter-connected be short-circuited between the out-side wires, then we have to put $\sqrt{2}\,c$ in place of c and $2b$ in place of b in calculating the permanent and sudden short-circuit currents from the formula given for the single phase generator.

In conclusion the authors wish to express their thanks to Mr T. Otake for his suggestions in completing proofs.

Figure. 1.

Armature current at permanent short-circuit.

$$x_s = \frac{-cI_f\omega}{b_i\omega} \cdot \frac{\sqrt{\sigma}\cos\omega t}{1-(1-\sigma)\cos^2\omega t}$$

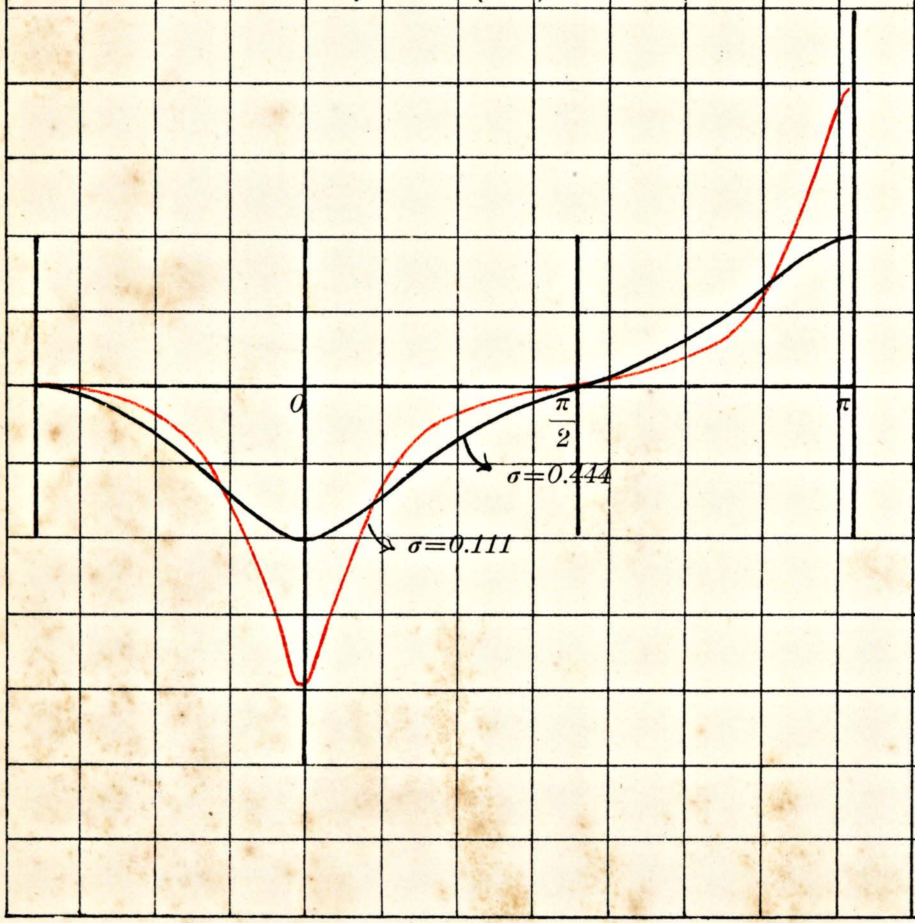
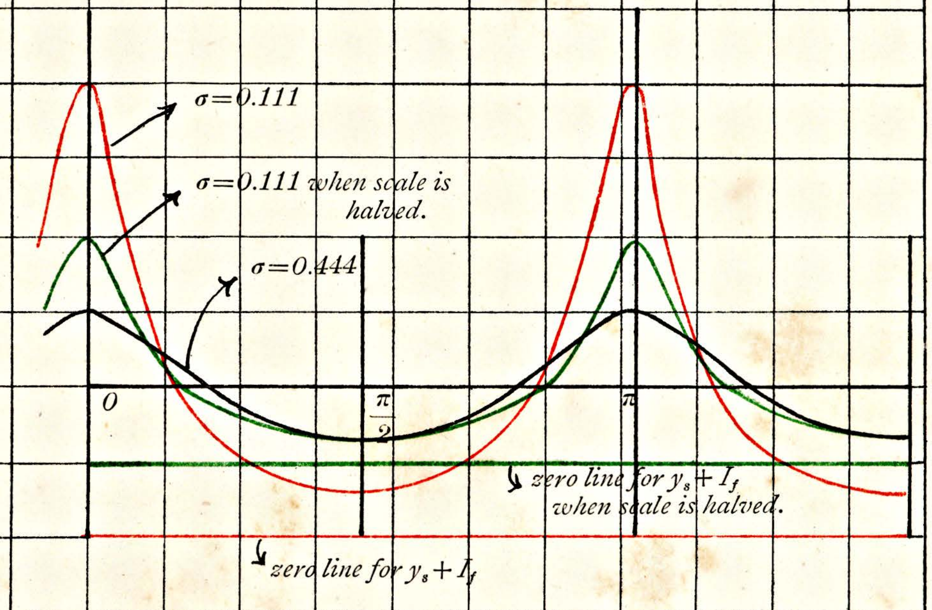


Figure. 2.

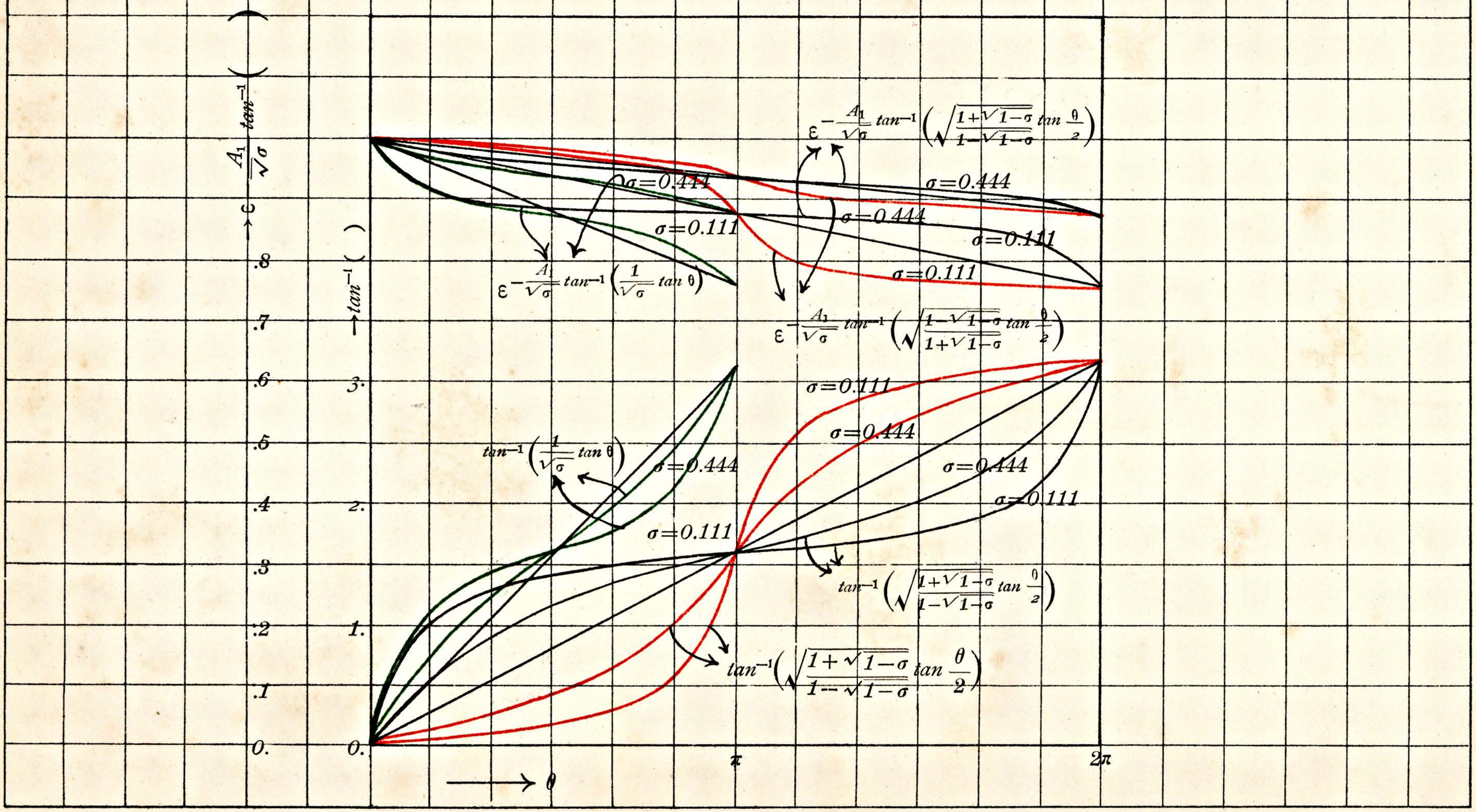
Field current at permanent short-circuit.

$y_s + I_f$



$$y_s + I_f = I_f \frac{\sqrt{\sigma}}{1-(1-\sigma)\cos^2\omega t}$$

Figure. 4.



$\rightarrow \varepsilon^{-\frac{A_1}{\sqrt{\sigma}} tan^{-1}}(\ )$
$\rightarrow tan^{-1}(\ )$
$\varepsilon^{-\frac{A_1}{\sqrt{\sigma}} tan^{-1}\left(\sqrt{\frac{1+\sqrt{1-\sigma}}{1-\sqrt{1-\sigma}}} tan\frac{\theta}{2}\right)}$
$\varepsilon^{-\frac{A_1}{\sqrt{\sigma}} tan^{-1}\left(\frac{1}{\sqrt{\sigma}} tan\,\theta\right)}$
$\varepsilon^{-\frac{A_1}{\sqrt{\sigma}} tan^{-1}\left(\sqrt{\frac{1-\sqrt{1-\sigma}}{1+\sqrt{1-\sigma}}} tan\frac{\theta}{2}\right)}$
$\sigma=0.444$
$\sigma=0.111$
$tan^{-1}\left(\frac{1}{\sqrt{\sigma}} tan\,\theta\right)$
$tan^{-1}\left(\sqrt{\frac{1+\sqrt{1-\sigma}}{1-\sqrt{1-\sigma}}} tan\frac{\theta}{2}\right)$
.8
.7
.6
.5
.4
.3
.2
.1
0.
3.
2.
1.
0.
π
2π
$\longrightarrow \theta$

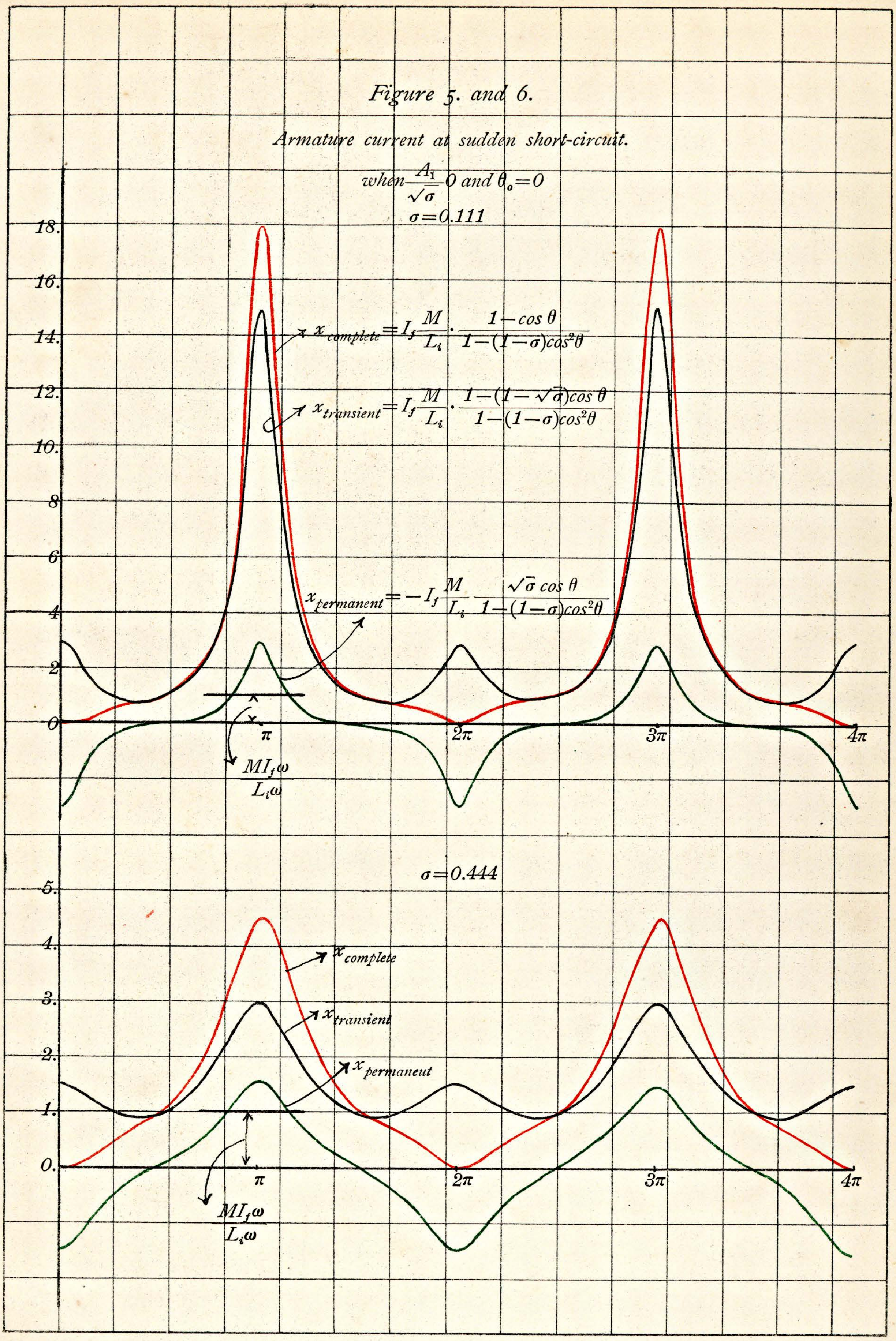


Figure 5. and 6.

Armature current at sudden short-circuit.

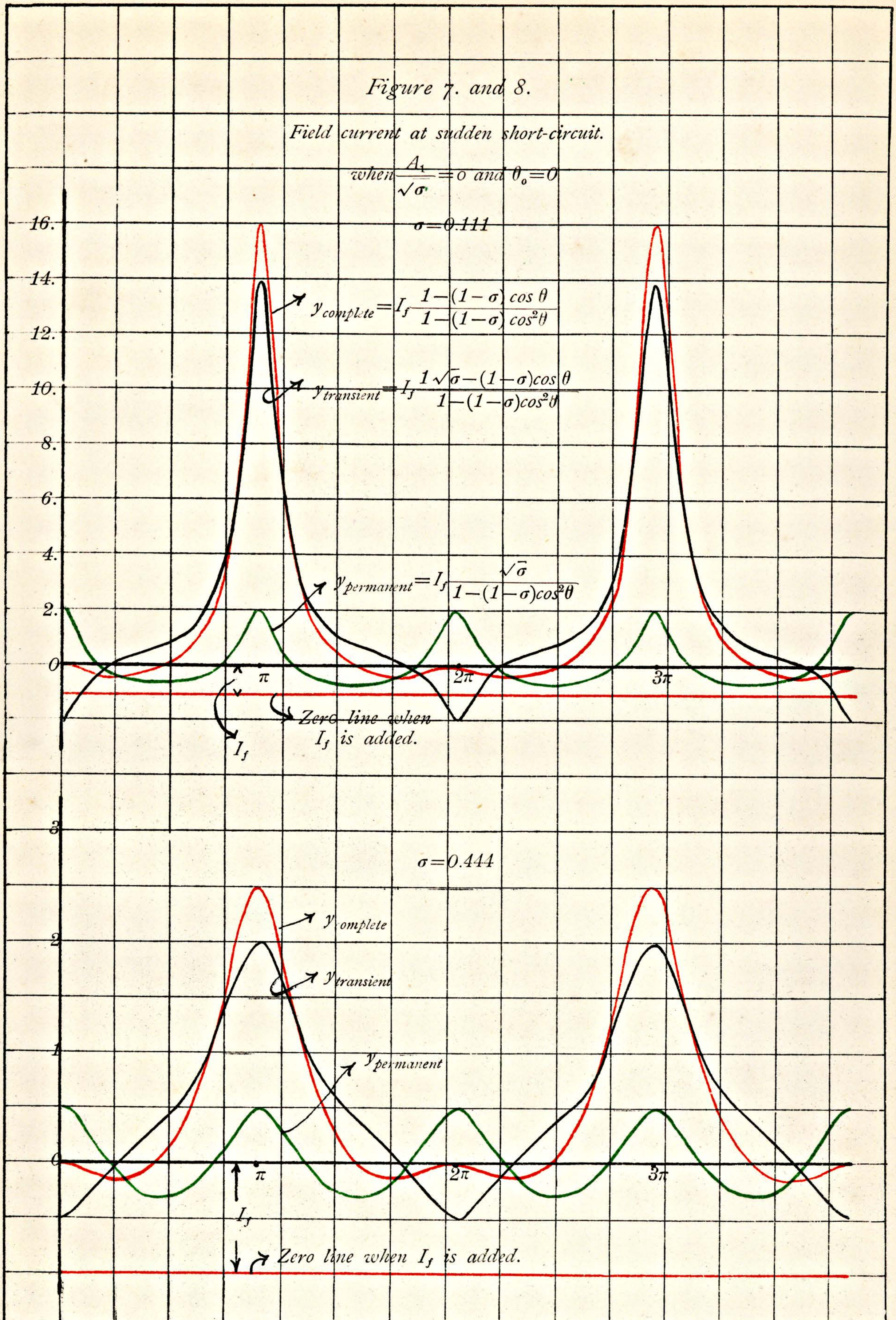


Figure 7. and 8.

Field current at sudden short-circuit.

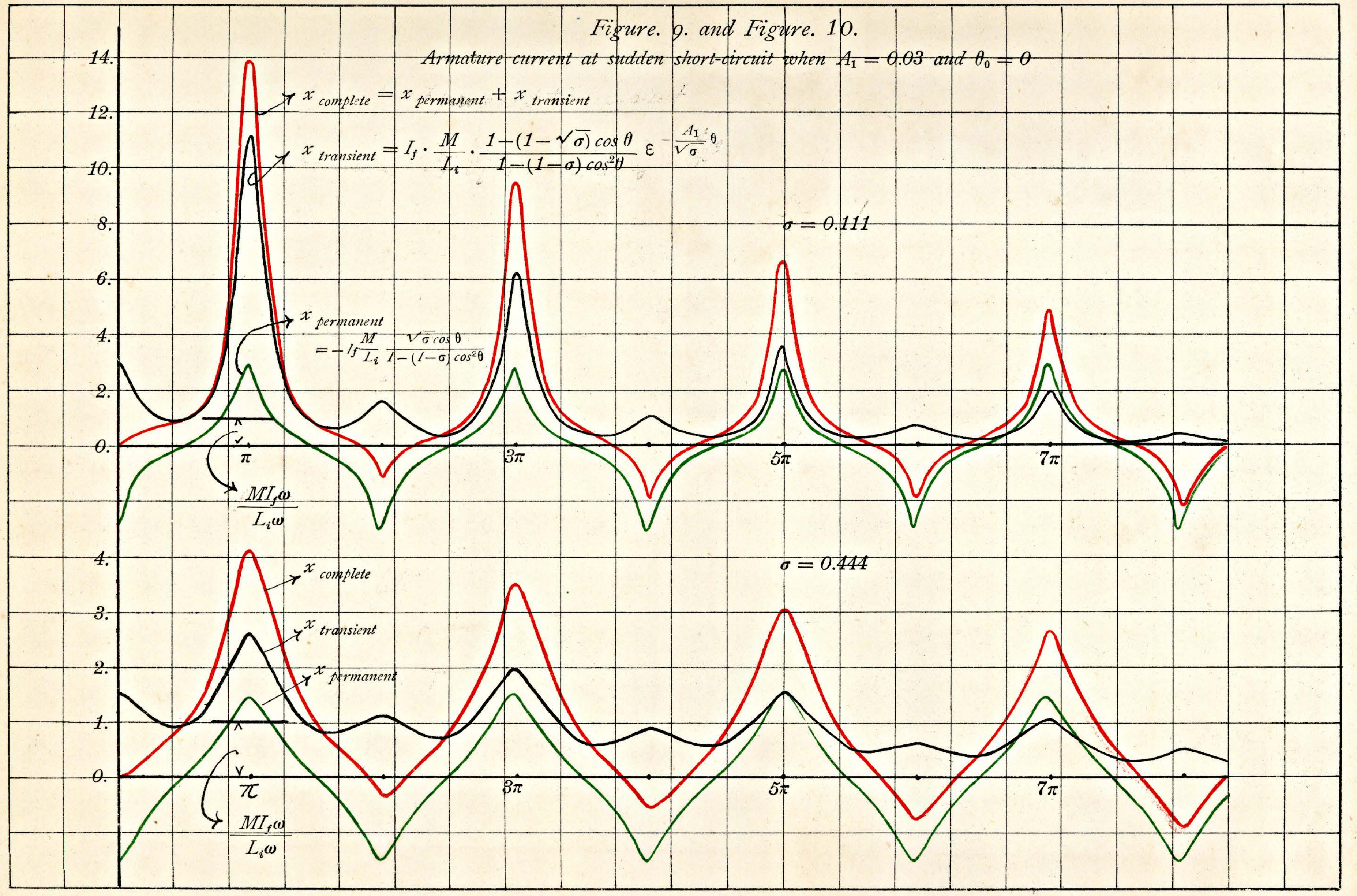


Figure. 9. and Figure. 10.
Armature current at sudden short-circuit when $A_1 = 0.03$ and $\theta_0 = 0$
$x_{complete} = x_{permanent} + x_{transient}$
$x_{transient} = I_f \cdot \frac{M}{L_i} \cdot \frac{1-(1-\sqrt{\sigma})\cos\theta}{1-(1-\sigma)\cos^2\theta}\,\varepsilon^{-\frac{A_1}{\sqrt{\sigma}}\theta}$
$x_{permanent} = -I_f \frac{M}{L_i} \frac{\sqrt{\sigma}\cos\theta}{1-(1-\sigma)\cos^2\theta}$
$\sigma = 0.111$
$\sigma = 0.444$
$\frac{MI_f\omega}{L_i\omega}$
$x_{complete}$
$x_{transient}$
$x_{permanent}$
14. 12. 10. 8. 6. 4. 2. 0.
4. 3. 2. 1. 0.
π 3π 5π 7π

Figure. 11. and 12.

Figure. 13.

E. m. f. induced in the open phase of the three phase generator at steady short circuit of any one phase.

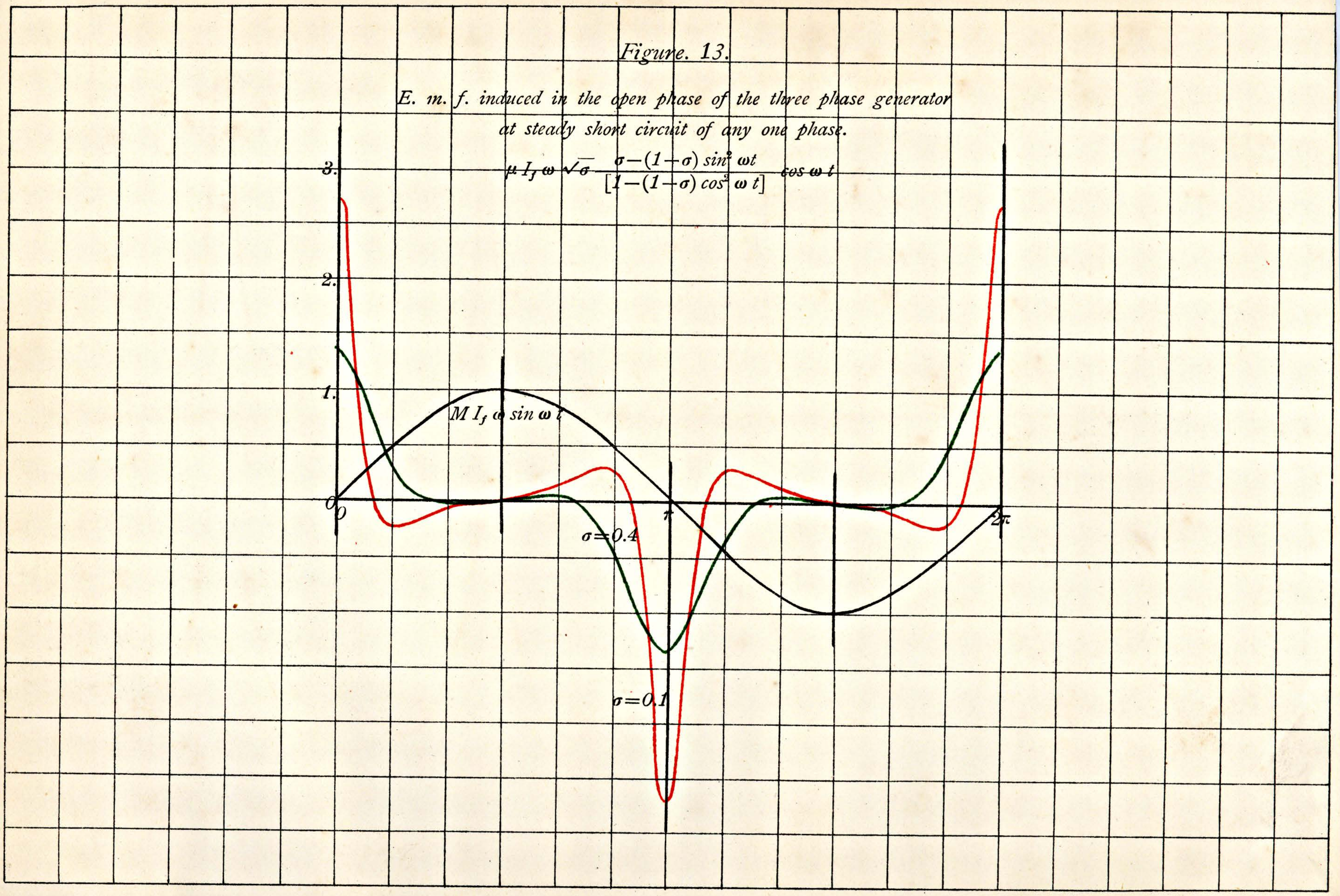


Figure. 14.

E. m. f. induced in the open phase of three phase generator at sudden short-circuit of any one phase.

$$-\frac{\sqrt{3}}{2}\cdot\frac{(\sigma-m\sin^2\theta)\cos\theta-m\left[\sigma-(1+\sigma)\sin^2\theta\right]}{(\sigma+m\sin^2\theta)^2} \quad \text{where } m=1-\sigma$$