A Contribution to the Theory of Thermal Stress in a Long Hollow Cylinder.

By

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The problem of thermal stress in a long hollow cylinder, in which heat is transmitted at a uniform rate from the inner to the outer surface or in the reverse direction, was solved among others by Lorenz.[1] In the solution he assumed the coefficient of thermal expansion and the shearing modulus of elasticity as constant, but it is experimentally known that they are not constant, the former increasing and the latter diminishing as temperature rises. Among the recent experimenters on the elastic modulus may be mentioned K. Iokibe and S. Sakai[2] and T. Kikuta.[3]

As far as the writer is aware, there has been published no solution of thermal stress considering the coefficient and the modulus as variable. To fill up the gap he has studied the problem, taking them as functions of temperature and accomplished a solution, which will be recorded hereinafter.

The following notation will be employed :—

r=radial distance of any point in the cylindrical wall,

z=axial distance of the same point from the coordinate cross plane,

Δr=change of r,

ζ=change of z,

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1) H. Lorenz, „Technische Elastizitätslehre".

2) The 45th Rep. of Iron and Steel Research Inst., Sendai.

3) The 50th Rep. of Iron and Steel Research Inst., Sendai.

$\varepsilon_r,\ \varepsilon_t,\ \varepsilon_z$=radial, tangential and axial strains,

$\sigma_r,\ \sigma_t,\ \sigma_z$=corresponding stresses,

G =shearing modulus of elasticity,

α =cofficient of thermal expansion,

m =Poisson's constant,

T =absolute temperature at radius r,

T_m= mean temperature in the wall.

The radial, tangential and axial stresses in terms of the corresponding strains are

$$\left.\begin{aligned}\sigma_r &= 2G\left(\varepsilon_r+\frac{\varepsilon_r+\varepsilon_t+\varepsilon_z}{m-2}\right),\\ \sigma_t &= 2G\left(\varepsilon_t+\frac{\varepsilon_r+\varepsilon_t+\varepsilon_z}{m-2}\right),\\ \sigma_2 &= 2G\left(\varepsilon_z+\frac{\varepsilon_r+\varepsilon_t+\varepsilon_z}{m-2}\right),\end{aligned}\right\}\quad\cdots\cdots\cdots(1)$$

while we have the relations

$$\left.\begin{aligned}\varepsilon_r+\alpha\,(T-T_m) &= \frac{\partial\Delta r}{\partial r},\\ \varepsilon_t+\alpha\,(T-T_m) &= \frac{\Delta r}{r},\\ \varepsilon_z+\alpha\,(T-T_m) &= \frac{\partial\zeta}{\partial z}.\end{aligned}\right\}\quad\cdots\cdots\cdots(2)$$

Substituting (2) in (1) we get

$$\left.\begin{aligned}\sigma_r&=2G\left[\frac{\partial\Delta r}{\partial r}+\frac{1}{m-2}\left(\frac{\partial\Delta r}{\partial r}+\frac{\Delta r}{r}+\frac{\partial\zeta}{\partial z}\right)-\frac{m+1}{m-2}\alpha(T-T_m)\right],\\ \sigma_t&=2G\left[\frac{\Delta r}{r}+\frac{1}{m-2}\left(\frac{\partial\Delta r}{\partial r}+\frac{\Delta r}{r}+\frac{\partial\zeta}{\partial z}\right)-\frac{m+1}{m-1}\alpha(T-T_m)\right],\\ \sigma_z&=2G\left[\frac{\partial\zeta}{\partial z}+\frac{1}{m-1}\left(\frac{\partial\Delta r}{\partial r}+\frac{\Delta r}{r}+\frac{\partial\zeta}{\partial z}\right)-\frac{m+1}{m-1}\alpha(T-T_m)\right].\end{aligned}\right\}(3)$$

The condition of equilibrium of stresses requires,

$$\left.\begin{aligned}&\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_t}{r}=0,\\&\frac{\partial\sigma_z}{\partial z}=0.\end{aligned}\right\}\qquad\qquad(4)$$

In a long cylinder, in which the temperature distribution is alike in all cross sections, T is a function of r alone, i.e.

$$\frac{\partial T}{\partial z}=0,\qquad\qquad(5)$$

while α and G are in turn functions of T alone. Therefore they must also be functions of r alone, i.e.

$$\frac{\partial\alpha}{\partial z}=0,\qquad\frac{\partial G}{\partial z}=0.\qquad\qquad(6)$$

Also $$\frac{\partial\zeta}{\partial r}=0,\qquad\frac{\partial\Delta r}{\partial z}=0,\qquad\text{and}\qquad\frac{\partial}{\partial z}\frac{1}{r}\frac{\partial(r\Delta r)}{\partial r}=0.\quad\ldots\ (7)$$

Substituting the values of (3) in (4) and putting as in (5), (6) and (7) we have

$$\left.\begin{aligned}&\frac{dG}{dr}\left[(m-2)\frac{d\Delta r}{dr}+\left(\frac{d\Delta r}{dr}+\frac{\Delta r}{r}+\frac{d\zeta}{dz}\right)-(m+1)\,\alpha(T-T_m)\right]\\&\quad+G\left\{(m-1)\frac{d}{dr}\left[\frac{1}{r}\frac{d(r\Delta r)}{dr}\right]-(m+1)\left[(T-T_m)\frac{d\alpha}{dr}+\alpha\frac{dT}{dr}\right]\right\}=0,\\&\frac{d^2\zeta}{dz^2}=0.\end{aligned}\right\}\ (8)$$

If α and G be assumed constant,

$$\left.\begin{aligned}&\frac{d}{dr}\left[\frac{1}{r}\frac{d(r\Delta r)}{dr}\right]=\frac{m+1}{m-1}\,\alpha\frac{dT}{dr},\\&\frac{d^2\zeta}{dz^2}=0.\end{aligned}\right\}\qquad\qquad(9)$$

Whence we get

$$\left.\begin{aligned} \Delta r &= \frac{m+1}{m-1}\alpha r T' + Br + \frac{C}{r}, \\ \zeta &= Az, \end{aligned}\right\} \qquad (10)$$

where $T' = \frac{1}{r^2}\int Trdr$ and A, B, C are the integration constants.

Equations (3) then become

$$\left.\begin{aligned} \frac{\sigma_r}{2G} &= \frac{A}{m-2} + \frac{mB}{m-2} + \frac{m+1}{m-2}\alpha T_m - \frac{C}{r^2} - \frac{m+1}{m-1}\alpha T', \\ \frac{\sigma_t}{2G} &= \frac{A}{m-2} + \frac{mB}{m-2} + \frac{m+1}{m-2}\alpha T_m + \frac{C}{r^2} + \frac{m+1}{m-1}\alpha(T'-T), \\ \frac{\sigma_z}{2G} &= \frac{m-1}{m-2}A + \frac{2B}{m-2} + \frac{m+1}{m-2}\alpha T_m - \frac{m+1}{m-1}\alpha T, \end{aligned}\right\} (11)_a$$

or if we put

$$K = \frac{A}{m-2} + \frac{mB}{m-2} + \frac{m+1}{m-2}\alpha T_m,$$

$$H = \frac{m-1}{m-2}A + \frac{2B}{m-2} + \frac{m+1}{m-2}\alpha T_m,$$

we have

$$\left.\begin{aligned} \frac{\sigma_r}{2G} &= K - \frac{C}{r^2} - \frac{m+1}{m-1}\alpha T', \\ \frac{\sigma_t}{2G} &= K + \frac{C}{r^2} + \frac{m+1}{m-1}\alpha(T'-T), \\ \frac{\sigma_z}{2G} &= H - \frac{m+1}{m-1}\alpha T. \end{aligned}\right\} \qquad (11)_b$$

Considering α and G as functions of temperature and accordingly as those of radial distance r the solution of the first of the differential equations (8) will not be accomplished. Functions, which correspond fairly well with the experimental results and at the same time make the equation integrable are unlikely to be found. Thus the problem is left unsolved.

The writer has succeeded in a solution, perhaps hitherto unknown, in which α and G may be of any forms whatever as empirical functions of temperature.

A cylindrical wall is divided into a number of coaxial thin cylindrical layers 1, 2, ... n as shown in Fig. 1 and in any one layer α and G are assumed constant. Then for each layer the equations as $(11)_b$ will be built up. Let for the layers 1, 2, ...n, α , α_2, ...α_n be the coefficients α's; G_1, G_2, ...G_n the moduli G's; K_1, K_2, ...K_n the constants K's; C_1, C_2, ...C_n the constants C's; H_1, H_2, ...H_n the constants H's. Then for any layer i

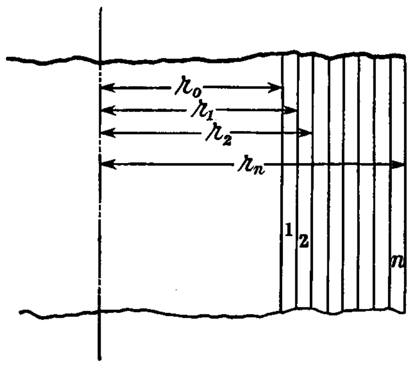

Fig. 1.

$$\left.\begin{aligned}
\frac{\sigma_r}{2G_i} &= K_i - \frac{C_i}{r^2} - \frac{m+1}{m-1}\alpha_i T',\\
\frac{\sigma_r}{2G_i} &= K_i + \frac{C_i}{r^2} + \frac{m+1}{m-1}\alpha_i (T'-T),\\
\frac{\sigma_z}{2G_i} &= H_i - \frac{m+1}{m-1}\alpha_i T,
\end{aligned}\right\} \qquad \text{.............................} \quad (12)$$

i standing for any one of 1, 2, ...n.

At the junction of any two adjacent layers the radial stress belonging to the one layer will be equal to that belonging to the other layer. For the tangential and axial stresses the same could not be said. The stresses belonging to the one layer may at the junction be different in magnitude from those belonging to the other layer. Nevertheless we put them equal but avoid any error arising therefrom by making $n=\infty$ as will be done later.

The radial and tangential stresses at r_1 are

$$\sigma_{r1}=\begin{cases} 2G_1\left(K_1-\dfrac{C_1}{r_1^2}-\dfrac{m+1}{m-1}a_1T_1'\right), & \text{.........(13)}_1 \\ 2G_2\left(K_2-\dfrac{C_2}{r_1^2}-\dfrac{m+1}{m-1}a_2T_1'\right), & \text{.........(13)}_2 \end{cases}$$

$$\sigma_{t1}=\begin{cases} 2G_1\left[K_1+\dfrac{C_1}{r_1^2}+\dfrac{m+1}{m-1}a_1(T_1'-T_1)\right], & \text{.........(14)}_1 \\ 2G_2\left[K_2+\dfrac{C_2}{r_1^2}+\dfrac{m+1}{m-1}a_2(T_1'-T_1)\right]. & \text{.........(14)}_2 \end{cases}$$

Equating $(14)_1-(13)_1$ to $(14)_2-(13)_2$

$$G_2C_2=G_1C_1+\frac{1}{2}\frac{m+1}{m-1}r_1^2(a_1G_1-a_2G_2)(2T_1'-T_1).$$

Similarly

$$G_3C_3=G_2C_2+\frac{1}{2}\frac{m+1}{m-1}r_2^2(a_2G_2-a_3G_3)(2T_2'-T_2)$$

$$=G_1C_1+\frac{1}{2}\frac{m+1}{m-1}\left[r_1^2(a_1G_1-a_2G_2)(2T_1'-T_1)+r_2^2(a_2G_2-a_3G_3)(2T_2'-T_2)\right],$$

$$G_iC_i=G_1C_1+\frac{1}{2}\frac{m+1}{m-1}\left[r_1^2(a_1G_1-a_2G_2)(2T_1'-T_1)+r_2^2(a_2G_2-a_3G_3)(2T_2'-T_2)+\right.$$
$$\left.\cdots\cdots+r_{i-1}^2(a_{i-1}G_{i-1}-a_iG_i)(2T_{i-1}'-T_{i-1})\right],$$

$$G_nC_n=G_1C_1+\frac{1}{2}\frac{m+1}{m-1}\left[r_1^2(a_1G_1-a_2G_2)(2T_1'-T_1)+r_2^2(a_2G_2-a_3G_3)(2T_2'-T_2)+\right.$$
$$\left.\cdots\cdots+r_{n-1}^2(a_{n-1}G_{n-1}-a_nG_n)\,(2T_{n-1}'-T_{n-1})\right],$$

or if the series in the square brackets be denoted by

$$-\sum_1^{i-1}\left[r^2\Delta aG\,(2T'-T)\right] \quad \text{and} \quad -\sum_1^{n-1}\left[r^2\Delta aG\,(2T'-T)\right]$$

respectively, we have

$$G_iC_i = G_1C_1 - \frac{1}{2}\,\frac{m+1}{m-1}\sum_1^{i-1}\Big[r^2\Delta aG(2T'-T)\Big], \quad \text{.................}(15)_i$$

$$G_nC_n = G_1C_1 - \frac{1}{2}\,\frac{m+1}{m-1}\sum_1^{n-1}\Big[r^2\Delta aG(2T'-T)\Big], \quad \text{.................}(15)_n$$

Next equating $(13)_1+(14)_1$ to $(13)_2+(14)_2$

$$G_2K_2 = G_1K_1 - \frac{1}{2}\,\frac{m+1}{m-1}(a_1G_1 - a_2G_2)T_1.$$

Similarly

$$G_3K_3 = G_2K_2 - \frac{1}{2}\,\frac{m+1}{m-1}(a_2G_2 - a_3G_3)T_2$$

$$= G_1K_1 - \frac{1}{2}\,\frac{m+1}{m-1}\Big[(a_1G_1 - a_2G_2)T_1 + (a_2G_2 - a_3G_3)T_2\Big],$$

$$G_iK_i = G_1K_1 - \frac{1}{2}\,\frac{m+1}{m-1}\Big[(a_1G_1 - a_2G_2)T_1 + (a_2G_2 - a_3G_3)T_2 + \cdots\cdots\cdots \cdots\cdots\cdots + (a_{i-1}G_{i-1} - a_iG_i)T_{i-1}\Big],$$

$$G_nK_n = G_1K_1 - \frac{1}{2}\,\frac{m+1}{m-1}\Big[(a_1G_1 - a_2G_2)T_1 + (a_2G_2 - a_3G_3)T_2 + \cdots\cdots\cdots \cdots\cdots\cdots + (a_{n-1}G_{n-1} - a_nG_n)T_{n-1}\Big],$$

or if the series in the square brackets be denoted by

$$-\sum_1^{i-1}\Big[\Delta aG\,T\Big] \quad \text{and} \quad -\sum_1^{n-1}\Big[\Delta aG\,T\Big]$$

respectively, we have

$$G_iK_i = G_1K_1 + \frac{1}{2}\,\frac{m+1}{m-1}\sum_1^{i-1}\Big[\Delta aG\,T\Big], \quad \text{..............................} (16)_i$$

$$G_nK_n = G_1K_1 + \frac{1}{2}\,\frac{m+1}{m-1}\sum_1^{n-1}\Big[\Delta aG\,T\Big]. \quad \text{..............................} (16)_n$$

The constants C_1, C_n, K_1, K_n are determined by the boundary condition that $\sigma_r=0$ at r_0 and r_n, with the aid of $(15)_n$ and $(16)_n$ and from C_1, K_1 the constants C_i, K_i are found. By (12) we have

$$G_1K_1-\frac{1}{r_0^2}G_1C_1-\frac{m+1}{m-1}a_1G_1T_0'=0, \qquad (17)_1$$

$$G_nK_n-\frac{1}{r_n^2}G_nG_n-\frac{m+1}{m-1}a_nG_nT_n'=0. \qquad (17)_n$$

Substituting $(15)_n$ and $(16)_n$ in $(17)_n$

$$G_1K_1+\frac{1}{2}\frac{m+1}{m-1}\sum_1^{n-1}\Big[\Delta aG\,T\Big]-\frac{1}{r_n^2}\left\{G_1C_1-\frac{1}{2}\frac{m+1}{m-1}\sum_1^{n-1}\Big[r^2\Delta aG\,(2T'-T)\Big]\right\}$$
$$-\frac{m+1}{m-1}a_nG_nT'_n=0. \qquad (18)$$

Subtracting $(17)_1$ from (18)

$$\left(\frac{1}{r_0^2}-\frac{1}{r_n^2}\right)G_1C_1+\frac{m+1}{m-1}(a_1G_1T'_0-a_nG_nT_n')$$
$$+\frac{1}{2}\frac{m+1}{m-1}\sum_1^{n-1}\Big[\Delta aG\,T\Big]$$
$$+\frac{1}{2}\frac{m+1}{m-1}\frac{1}{r_n^2}\sum_1^{n-1}\Big[r^2\Delta aG\,(2T'-T)\Big]=0,$$

whence

$$G_1C_1=\frac{m+1}{m-1}\frac{r_0^2}{r_n^2-r_0^2}\left\{-r_n^2(a_1G_1T'_0-a_nG_nT'_n)-\frac{r_n^2}{2}\sum_1^{n-1}\Big[\Delta aG\,T\Big]\right.$$
$$\left.-\frac{1}{2}\sum_1^{n-1}\Big[r^2\Delta aG\,(2T'-T)\right\}, \qquad (19)_1$$

and then by $(15)_i$

$$G_iC_i=\frac{m+1}{m-1}\frac{r_0^2}{r_n^2-r_0^2}\left\{-r_n^2(a_1G_1T_0'-a_nG_nT_n')-\frac{r_n^2}{2}\sum_1^{n-1}\Big[\varDelta aG\,T\Big]\right.$$

$$-\frac{1}{2}\sum_1^{n-1}\Big[r^2\varDelta aG(2T'-T)\Big]$$

$$\left.-\frac{1}{2}\left(\frac{r_n^2}{r_0^2}-1\right)\sum_1^{i-1}\Big[r^2\varDelta aG\,(2T'-T)\Big]\right\}.\ (19)_i$$

With the value of G_1C_1 in $(19)_1$ we have from $(17)_1$

$$G_1K_1=\frac{m+1}{m-1}\frac{1}{r_n^2-r_0^2}\left\{-r_0^2a_1G_1T_0'+r_n^2a_nG_nT_n'-\frac{r_n^2}{2}\sum_1^{n-1}\Big[\varDelta aG\,T\Big]\right.$$

$$\left.-\frac{1}{2}\sum_1^{n-1}\Big[r^2\varDelta aG\,(2T'-T)\Big]\right\}\cdots\cdots\cdots\cdots(20)_1$$

and then by $(16)_i$

$$G_iK_i=\frac{m+1}{m-1}\frac{1}{r_n^2-r_0^2}\left\{-r_0^2\,a_1G_1T_0'+r_n^2\,a_nG_nT_n'-\frac{r_n^2}{2}\sum_1^{n-1}\Big[\varDelta aG\,T\Big]\right.$$

$$+\frac{1}{2}\left(r_n^2-r_0^2\right)\sum_1^{i-1}\Big[\varDelta aG\,T\Big]$$

$$\left.-\frac{1}{2}\sum_1^{n-1}\Big[r^2\varDelta aG\,(2T'-T)\Big]\right\}.\ \cdots\cdots\cdots\cdots\ (20)_i$$

Further by the condition that the resultant of the axial stresses must vanish

$$2\pi\int_{r_0}^{r_n}\sigma_z\,r\,dr=0.$$

The integral is the sum of the integrals for the layers 1, 2, ...n. We have with the value of σ_z in (12)

$$0=G_1H_1(r_1^2-r_0^2)+G_2H_2(r_2^2-r_1^2)+\dots\dots\dots+G_nH_n(r_n^2-r_{n-1}^2)$$

$$-2\frac{m+1}{m-1}\left[\alpha_1G_1\int_{r_0}^{r_1}Trdr+\alpha_2G_2\int_{r_1}^{r_2}Trdr+\dots\dots\dots+\alpha_nG_n\int_{r_{n-1}}^{r_n}Trdr\right].$$

But the last term is

$$-2\frac{m+1}{m-1}\Big[-r_0^2\alpha_1G_1T'_0+r_1^2(\alpha_1G_1-\alpha_2G_2)T'_1+r_2^2(\alpha_2G_2-\alpha_3G_3)T'_2+$$

$$\dots\dots\dots\dots+r_{n-1}^2(\alpha_{n-1}G_{n-1}-\alpha_nG_n)T'_{n-1}+r_n^2\alpha_nG_nT'_n\Big]$$

or

$$-2\frac{m+1}{m-1}\left\{-r_0^2\alpha_1G_1T'_0+r_n^2\alpha_nG_nT_n'-\sum_1^{n-1}\Big[r^2\Delta\alpha G\ T'\Big]\right\}.$$

Therefore

$$0=G_1H_1(r_1^2-r_0^2)+G_2H_2(r_2^2-r_1^2)+\dots\dots\dots+G_nH_n(r_n^2-r_{n-1}^2)$$

$$-2\frac{m+1}{m-1}\left\{-r_0^2\alpha_1G_1T'_0+r_n^2\alpha_nG_nT'_n-\sum_1^{n-1}\Big[r^2\Delta\alpha G\ T'\Big]\right\}.\quad\dots\ (21)$$

On the other hand the axial stress at r_1 is

$$\frac{\sigma_z}{2}=\begin{cases}G_1H_1-\dfrac{m+1}{m-1}\alpha_1G_1T_1,\\[2ex] G_2H_2-\dfrac{m+1}{m+1}\alpha_2G_2T_1,\end{cases}$$

whence $\quad G_2H_2=G_1H_1-\dfrac{m+1}{m-1}(\alpha_1G_1-\alpha_2G_2)T_1.$

Similarly $\quad G_3H_3=G_2H_2-\dfrac{m+1}{m-1}(\alpha_2G_2-\alpha_3G_3)T_2$

$$=G_1H_1-\frac{m+1}{m-1}\Big[(\alpha_1G_1-\alpha_2G_2)T_1+(\alpha_2G_2-\alpha_3G_3)T_2\Big],$$

$$G_iH_i = G_1H_1 - \frac{m+1}{m-1}\Big[(a_1G_1 - a_2G_2)T_1 + (a_2G_2 - a_3G_3)T_2 +$$

$$\cdots\cdots + (a_{i-1}G_{i-1} - a_iG_i)T_{i-1}\Big]$$

$$= G_1H_1 + \frac{m+1}{m-1}\sum_1^{i-1}\Big[\varDelta aG\ T\Big], \qquad \cdots\cdots \quad (22)_i$$

$$G_nH_n = G_1H_1 - \frac{m+1}{m+1}\Big[(a_1G_1 - a_2G_2)T_1 + (a_2G_2 - a_3G_3)T_2 +$$

$$\cdots\cdots(a_{n-1}G_{n-1} - a_nG_n)T_{n-1}\Big]$$

$$= G_1H_1 + \frac{m+1}{m+1}\sum_1^{n-1}\Big[\varDelta aG\ T\Big]. \qquad \cdots\cdots \quad (22)_n$$

Putting all these in (21)

$$0 = G_1H_1[(r_1^2 - r_0^2) + (r_2^2 - r_1^2) + \cdots\cdots + (r_n^2 - r_{n-1}^2)]$$

$$-\frac{m+1}{m-1}\Big[(r_n^2 - r_1^2)(a_1G_1 - a_2G_2)T_1 + (r_n^2 - r_2^2)(a_2G_2 - a_3G_3)T_2 +$$

$$\cdots\cdots + (r_n^2 - r_{n-1}^2)(a_{n-1}G_{n-1} - a_nG_n)T_{n-1}\Big]$$

$$-2\frac{m+1}{m-1}\Big\{-r_0^2a_1G_1T'_0 + r_n^2a_nG_nT'_n - \sum_1^{n-1}\Big[r^2\varDelta a\,G\,T'\Big]\Big\}$$

$$= G_1H_1(r_n^2 - r_0^2) - \frac{m+1}{m-1}\Big\{-2\,r_0^2a_1G_1T'_0 + 2\,r_n^2a_nG_nT'_n$$

$$-r_n^2\sum_1^{n-1}\Big[\varDelta aG\ T\Big]$$

$$-\sum_1^{n-1}\Big[r^2\varDelta aG\,(2T' - T)\Big]\Big\},$$

whence

$$G_1H_1=\frac{m+1}{m-1}\frac{1}{r_n^2-r_0^2}\left\{-2r_0^2\alpha_1G_1T'_0+2r_n^2\alpha_nG_nT'_n\right.$$

$$-r_n^2\sum_1^{n-1}\left[\varDelta\alpha G\,T\right]$$

$$\left.-\sum_1^{n-1}\left[r^2\varDelta\alpha G\,(2T'-T)\right]\right\},\;\ldots\ldots\ldots\ldots(23)_1$$

and by $(22)_i$

$$G_iH_i=\frac{m+1}{m-1}\frac{1}{r_n^2-r_0^2}\left\{-2r_0^2\alpha_1G_1T'_0+2r_n^2\alpha_nG_nT'_n-r_n^2\sum_1^{n-1}\left[\varDelta\alpha G\,T\right]\right.$$

$$+(r_n^2-r_0^2)\sum_1^{i-1}\left[\varDelta\alpha G\,T\right]$$

$$\left.-\sum_1^{n-1}\left[r^2\varDelta\alpha G(2T'-T)\right]\right\}.\;\ldots\ldots\ldots\ldots(23)_i$$

Introducing the values of $(19)_i$, $(20)_i$ and $(23)_i$ into (12) we get the component stresses at a point in the layer i.

So far the number of layers has been considered finite but it may as well be taken as infinite in order to arrive at the exact solution. Then α_i, G_i, C_i, K_i and H_i in (12), which *were the stepwise changing constants now become the coutinuous functions of* r. They will be written henceforth dropping the suffix i. Also will be written

$$\alpha_0 \text{ for } \alpha_1, \qquad G_0 \text{ for } G_1,$$

$$\int_{\alpha_0G_0}^{\alpha G}Td\alpha G \text{ for } \sum_1^{i-1}\left[\varDelta\alpha G\,T\right], \qquad \int_{\alpha_0G_0}^{\alpha_nG_n}T\,d\alpha G \text{ for } \sum_1^{n-1}\left[\varDelta\alpha G\,T\right],$$

$$\int_{\alpha_0G_0}^{\alpha G}r^2(2T'-T)d\alpha G \quad\text{for}\quad \sum_1^{i-1}\left[r^2\varDelta\alpha G\,(2T'-T)\right],$$

$$\int_{\alpha_0G_0}^{\alpha_nG_n}r^2(2T'-T)d\alpha G \quad\text{for}\quad \sum_1^{n-1}\left[r^2\varDelta\alpha G(2T'-T)\right].$$

Then by (12) with $(19)_i$, $(20)_i$ and $(23)_i$

$$\left.\begin{aligned}
\sigma_r &= 2GK-\frac{2GC}{r^2}-2\frac{m+1}{m-1}aGT',\\
\sigma_t &= 2GK+\frac{2GC}{r^2}+2\frac{m+1}{m-1}aG\,(T'-T),\\
\sigma_z &= 2GH-\frac{m+1}{m-1}aGT,
\end{aligned}\right\} \qquad\qquad (24)$$

with

$$2GK=\frac{m+1}{m-1}\,\frac{1}{r_n^2-r_0^2}\Big\{-2\,r_0^2a_0G_0T_0'+2\,r_n^2a_nG_nT_n'$$

$$-r_0^2\int_{\alpha_0G_0}^{\alpha G}TdaG-r_n^2\int_{\alpha G}^{\alpha_nG_n}TdaG$$

$$-\int_{\alpha_0G_0}^{\alpha_nG_n}r^2(2T'-T)\,daG\Big\},$$

$$2GC=\frac{m+1}{m-1}\,\frac{r_0^2}{r_n^2-r_0^2}\Big\{-2\,r_n^2(a_0G_0T_0'-a_nG_nT_n')-r_n^2\int_{\alpha_0G_0}^{\alpha_nG_n}TdaG$$

$$-\frac{r_n^2}{r_0^2}\int_{\alpha_0G_0}^{\alpha G}r^2(2T'-T)daG-\int_{\alpha G}^{\alpha_nG_n}r^2(2T'-T)daG\Big\},$$

$$2GH=2\frac{m+1}{m-1}\,\frac{1}{r_n^2-r_0^2}\Big\{-2\,r_0^2a_0G_0T_0'+2\,r_n^2a_nG_nT_n'-r_0^2\int_{\alpha_0G_0}^{\alpha G}TdaG$$

$$-r_n^2\int_{\alpha G}^{\alpha_nG_n}TdaG-\int_{\alpha_0G_0}^{\alpha_nG_n}r^2(2T'-T)daG\Big\}.$$

As $GH=2GK$, we have

$$\sigma_r+\sigma_t=\sigma_z.$$

Let us now try a verification, whether the values of the component stresses here obtained satisfy the fundamental equations (4). From (24)

$$\frac{d\sigma_r}{dr}=\frac{m+1}{m-1}T\frac{d\alpha G}{dr}+\frac{m+1}{m-1}\left(2T'-T\right)\frac{d\alpha G}{dr}+\frac{4GC}{r^3}$$

$$-2\frac{m+1}{m-1}\left(\alpha G\frac{dT'}{dr}+T'\frac{d\alpha G}{dr}\right)$$

$$=\frac{4GC}{r^3}-2\frac{m+1}{m-1}\alpha G\frac{dT'}{dr}.$$

But

$$\frac{dT'}{dr}=\frac{d}{dr}\left(\frac{1}{r^2}\int Tr\,dr\right)=-\frac{2}{r^3}\int Trdr+\frac{T}{r}=-\frac{2T'-T}{r}.$$

Therefore

$$\frac{d\sigma_r}{dr}=\frac{4GC}{r^3}+2\frac{m+1}{m-1}\alpha G\frac{2T'-T}{r}.$$

And

$$\frac{\sigma_r-\sigma_t}{r}=-\frac{4GC}{r^3}-2\frac{m+1}{m-1}\alpha G\frac{2T'-T}{r}.$$

With these values the first of the equations (4) is satisfied to the proof of correctness of the solution.

To find the component stresses at the inner surface of the hollow cylinder we put in (24)

$$\alpha=\alpha_0,\qquad G=G_0,\qquad r=r_0,\qquad T'=T'_0,\qquad T=T_0.$$

They are

$$\sigma_{r0}=0,$$

$$\sigma_{t0}=\sigma_{z0}=2\frac{m+1}{m-1}\,\frac{1}{r_n^2-r_0^2}\Bigg\{-2\,r_0^2\alpha_0G_0T'_0+2\,r_n^2\alpha_nG_nT'_n$$

$$-(r_n^2-r_0^2)\alpha_0G_0T_0-r_n^2\int_{\alpha_0G_0}^{\alpha_nG_n}T\,d\alpha G$$

$$-\int_{\alpha_0G_0}^{\alpha_nG_n}r^2(2T'-T)d\alpha G\Bigg\}.\quad\ldots\ldots(26)_0$$

And to find those at the outer surface we put in the same equations

$$a=a_n, \qquad G=G_n, \qquad r=r_n, \qquad T'=T'_n, \qquad T=T_n.$$

They are

$$\sigma_{rn}=0,$$

$$\sigma_{tn}=\sigma_{zn}=2\frac{m+1}{m-1}\frac{1}{r_n^2-r_0^2}\Big\{-2\,r_0^2a_0G_0T'_0+2\,r_n^2a_nG_nT'_n - (r_n^2-r_0^2)a_nG_nT_n-r_0^2\int_{a_0G_0}^{a_nG_n}TdaG - \int_{a_0G_0}^{a_nG_n}r^2(2T'-T)daG\Big\}. \quad\text{.........}(26)_n$$

The law of temperature distribution was determined by Lorenz from the fact that the quantity of heat passing in unit time through the cylindrical plate of the area $2\pi rl$ and the thickness dr is constant, l being the length of the cylinder. It is

$$T=\frac{T_n-T_0}{\log r_n-\log r_0}\log r+\frac{T_0\log r_n-T_n\log r_0}{\log r_n-\log r_0} \quad\text{..................}\ (27)_a$$

or

$$T-T_0=(T_n-T_0)\frac{\log r-\log r_0}{\log r_n-\log r_0}, \quad\text{....................................}\ (27)_b$$

and

$$2T'=\frac{2}{r^2}\int Trdr=\frac{T_n-T_0}{\log r_n-\log r_0}\left(\log r-\frac{1}{2}\right)+\frac{T_0\log r_n-T_n\log r_0}{\log r_n-\log r_0} \quad\text{...........................}\ (28)$$

or

$$2T'-T=-\frac{1}{2}\frac{T_n-T_0}{\log r_n-\log r_0}=\text{a const.}=D \text{ say.} \quad\text{..............}\ (29)$$

Putting in $(26)_0$ and $(26)_n$

$$2T'-T=D, \qquad 2T'_0=D+T_0, \qquad 2T'_n=D+T_n,$$

we get

$$\sigma_{t0}=\sigma_{z0}=2\frac{m+1}{m-1}\frac{1}{r_n^2-r_0^2}\left\{r_n^2\left(\alpha_n G_n T_n-\alpha_0 G_0 T_0-\int_{\alpha_0 G_0}^{\alpha_n G_n} T d\alpha G\right)\right.$$
$$\left.+D\left(r_n^2\alpha_n G_n-r_0^2\alpha_0 G_0-\int_{\alpha_0 G_0}^{\alpha_n G_n} r^2 d\alpha G\right)\right\}, \quad (30)_0$$

$$\sigma_{tn}=\sigma_{zn}=2\frac{m+1}{m-1}\frac{1}{r_n^2-r_0^2}\left\{r_0^2\left(\alpha_n G_n T_n-\alpha_0 G_0 T_0-\int_{\alpha_0 G_0}^{\alpha_n G_n} T d\alpha G\right)\right.$$
$$\left.+D\left(r_n^2\alpha_n T_n-r_0^2\alpha_0 T_0-\int_{\alpha_0 G_0}^{\alpha_n G_n} r^2 d\alpha G\right)\right\}. \quad (30)_n$$

$\int_{\alpha_0 G_0}^{\alpha_n G_n} T d\alpha G$ and $\int_{\alpha_0 G_0}^{\alpha_n \alpha_n} r^2 d\alpha G$ can easily be found graphically.

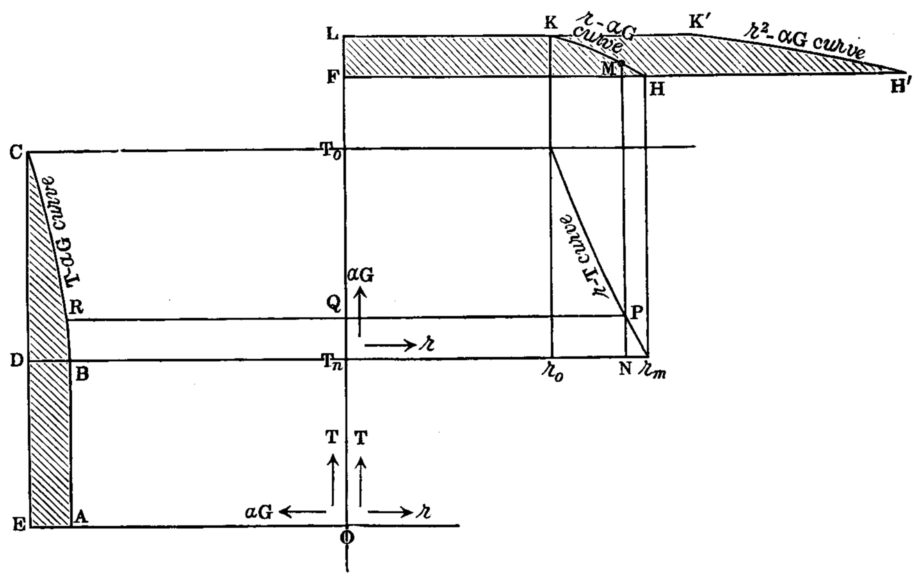

Fig. 2.

Draw T-αG curve from the experimental data and r-T curve by (27) as in Fig. 2. Then the area $ABCDE$ represents $\pm\int_{\alpha_0 G_0}^{\alpha_n G_n} T d\alpha G$, the upper or lower sign being taken according as αG increases or decreases from $\alpha_0 G_9$ to $\alpha_n G_n$. To show the variation of αG in dependence on r draw ordinate MN at any radial distance, from the intersection P draw a straight line parallel to the base line and take $MN=RQ$. Then the points M's similarly plotted will trace r-αG curve. Next draw r^2-αG curve by taking the horizontal ordinates equal to the squares of those of r-αG curve. Then the area $FH'K'L$ represents $\pm\int_{\alpha_0 G_0}^{\alpha_n G_n} r^2 d\alpha G$, the upper or lower sign being taken as said above. Drawing r^2-αG curve may be inconvenient in some cases, then draw $\frac{r^2}{r_0^2}$-αG curve instead of it and multiply the estimated area by r_0^2.

In the foregoing Poisson's constant m was considered constant but it is like α and G subject to a variation as temperature rises and according to Cl. Schäfer approaches to 2 at the melting point of the material, although the experimental data in this concern hitherto published are but scanty. This variableness can also be taken into account. In equations (12) α always accompanies the factor $\frac{m+1}{m-1}$. Therefore considering m as variable and denoting $\frac{m+1}{m-1}$ by μ we may write the expressions of $2GK$, $2GC$ and $2GH$ belonging to equations (24) as

$$2GK=\frac{1}{r_n^2-r_0^2}\left\{-2r_0^2\mu_0\alpha_0G_0T'_0+2r_n^2\mu_n\alpha_nG_nT'_n-r_0^2\int_{\mu_0\alpha_0G_0}^{\mu\alpha G}Td(\mu\alpha G)\right.$$

$$\left.-r_n^2\int_{\mu\alpha G}^{\mu_n\alpha_nG_n}Td(\mu\alpha G)-\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}r^2(2T'-T)d(\mu\alpha G)\right\},$$

$$2GC=\frac{r_0^2}{r_n^2-r_0^2}\left\{-2\,r_n^2(\mu_0\alpha_0G_0T'_0-\mu_n\alpha_nG_nT'_n)-r_n^2\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}Td(\mu\alpha G)\right.$$

$$-\frac{r_n^2}{r_0^2}\int_{\mu_0\alpha_0G_0}^{\mu\alpha G}r^2(2T'-T)\,d(\mu\alpha G)$$

$$\left.-\int_{\mu\alpha G}^{\mu_n\alpha_nG_n}r^2(2T'-T\,)\,d(\mu\alpha G)\right\},$$

$$2GH=\frac{2}{r_n^2-r_0^2}\left\{-2\,r_0^2\mu_0\alpha_0G_0T_0'+2\,r_n^2\mu_n\alpha_nG_nT_n'-r_0^2\int_{\mu_0\alpha_0G_0}^{\mu\alpha G}Td(\mu\alpha G)\right.$$
$$\left.-r_n^2\int_{\mu\alpha G}^{\mu_n\alpha_nG_n}Td(\mu\alpha G)-\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}r^2(2T_0-T)d(\mu\alpha G)\right\},$$

equations $(26)_0$ and $(26)_n$ as

$$\sigma_{t0}=\sigma_{z0}=\frac{2}{r_n^2-r_0^2}\left\{-2\,r_0^2\mu_0\alpha_0G_0T_0'+2\,r_n^2\mu_n\alpha_nG_nT_n'-(r_n^2-r_0^2)\mu_0\alpha_0G_0T_0\right.$$
$$\left.-r_n^2\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}Td(\mu\alpha G)-\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}r^2(2T'-T)d(\mu\alpha G)\right\},\qquad\ldots\ldots(31)_0$$

$$\sigma_{tn}=\sigma_{zn}=\frac{2}{r_n^2-r_0^2}\left\{-2\,r_0^2\mu_0\alpha_0G_0T_0'+2\,r_n^2\mu_n\alpha_nG_nT_n'-(r_n^2-r_0^2)\mu_n\alpha_nG_nT_n\right.$$
$$\left.-r_0^2\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}Td(\mu\alpha G)-\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}r^2(2T'-T)d(\mu\alpha G)\right\},\qquad\ldots\ldots(31)_n$$

and equations $(30)_0$ and $(30)_n$ as

$$\sigma_{t0}=\sigma_{z0}=\frac{2}{r_n^2-r_0^2}\left\{r_n^2\left[\mu_n\alpha_nG_nT_n-\mu_0\alpha_0G_0T_0-\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}Td(\mu\alpha G)\right]\right.$$
$$\left.+D\left[r_n^2\mu_n\alpha_nG_n-r_0^2\mu_0\alpha_0G_0-\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}r^2d(\mu\alpha G)\right]\right\},\qquad\ldots\ldots(32)_0$$

$$\sigma_{tn}=\sigma_{zn}=\frac{2}{r_n^2-r_0^2}\left\{r_0^2\left[\mu_n\alpha_nG_nT_n-\mu_0\alpha_0G_0T_0-\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}Td(\mu\alpha G)\right]\right.$$
$$\left.+D\left[r_n^2\mu_n\alpha_nG_n-r_0^2\mu_0\alpha_0G_0-\int_{\mu_0\alpha_0G_0}^{\mu_n\alpha_nG_n}r^2d(\mu\alpha G)\right]\right\}.\qquad\ldots\ldots(32)_n$$

Summary.

The object of tho present paper is to find the thermal stress in a long hollow cylinder, in which heat is transmitted at a uniform rate from the inner to the outer surface or in the reverse direction, considering the coefficient of thermal expansion and the shearing modulus of elasticity as functions of temperature. The writer has succeeded in a solution, perhaps hitherto unknown, in which the coefficient and the modulus may be of any forms whatever as empirical functions of temperature. Dividing the cylindrical wall into a number of coaxial thin cylindrical layers and assuming the coefficient and the modulus in any one layer as constant the radial, tangential and axial stresses at any point of any layer are found. At last the number of layers is taken as infinite in order to arrive at the exact solution. The tangential and axial stresses at the inner and outer surfaces are reduced to

$$\sigma_{t0}=\sigma_{z0}=2\frac{m+1}{m-1}\,\frac{1}{r_n^2-r_0^2}\left\{r_n^2\left(\alpha_n G_n T_n-\alpha_0 G_0 T_0-\int_{\alpha_0 G_0}^{\alpha_n G_n} T\,d\alpha G\right)\right.$$
$$\left.+\,D\left(r_n^2\alpha_n G_n-r_0^2\alpha_0 G_0-\int_{\alpha_0 G_0}^{\alpha_n G_n} r^2\,d\alpha G\right)\right\},$$

$$\sigma_{tn}=\sigma_{zn}=2\frac{m+1}{m-1}\,\frac{1}{r_n^2-r_0^2}\left\{r_0^2\left(\alpha_n G_n T_n-\alpha_0 G_0 T_0-\int_{\alpha_0 G_0}^{\alpha_n G_n} T\,d\alpha G\right)\right.$$
$$\left.+\,D\left(r_n^2\alpha_n G_n-r_0^2\alpha_0 G_0-\int_{\alpha_0 G_0}^{\alpha_n G_n} r^2\,d\alpha G\right)\right\}.$$

For the determination of the definite integrals a graphical method may be conveniently used.

If, besides the coefficient and the modulus, Poisson's constant be considered variable the stresses become

$$\sigma_{t0}=\sigma_{z0}=\frac{2}{r_n^2-r_0^2}\left\{r_n^2\left[\mu_n\alpha_n G_n T_n-\mu_0\alpha_0 G_0 T_0-\int_{\mu_0\alpha_0 G_0}^{\mu_n\alpha_n G_n} T d(\mu\alpha G)\right]\right.$$

$$\left.+D\left[r_n^2\mu_n\alpha_n G_n-r_0^2\mu_0\alpha_0 G_0-\int_{\mu_0\alpha_0 G_0}^{\mu_n\alpha_n G_n} r^2 d(\mu\alpha G)\right]\right\},$$

$$\sigma_{tn}=\sigma_{zn}=\frac{2}{r_n^2-r_0^2}\left\{r_0^2\left[\mu_n\alpha_n G_n T_n-\mu_0\alpha_0 G_0 T_0-\int_{\mu_0\alpha_0 G_0}^{\mu_n\alpha_n G_n} T d(\mu\alpha G)\right]\right.$$

$$\left.+D\left[r_n^2\mu_n\alpha_n G_n-r_0^2\mu_0\alpha_0 G_0-\int_{\mu_0\alpha_0 G_0}^{\mu_n\alpha_n G_n} r^2 d(\mu\alpha G)\right]\right\},$$

where

$$\mu=\frac{m+1}{m-1}.$$