Bending Strength of Curved Rods.

BY

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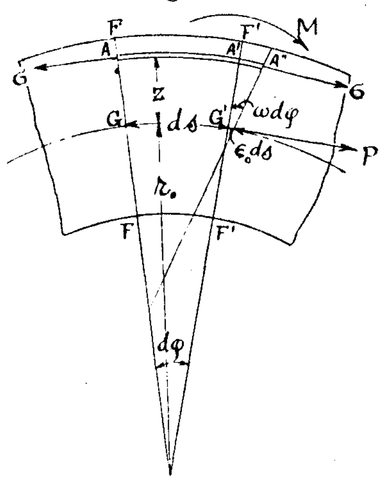
ON THE ORDINARY THEORY.

The ordinary theory of bending strength of curved rods was established on the following base condition:—In Fig. 1 if FF and $F'F'$ two cross-sections infinitely near to each other, then when the longitudinal stress σ causes the fibre element AA' to elongate or to shorten to the length AA'' the point A'' shall lie on a plane, which being the plane of the cross-section $F'F'$ in the deformed state.

Fig. 1.



If ω be the specific variation of angle between the two cross-sections and ϵ_0 the strain of the element ds on the centre line, the expressions of σ, ω and ϵ_0 as found as the results of this theory are

$$\sigma = \frac{P}{F} + \frac{M}{Fr_0} + \frac{M}{Fr_0\,x}\;\frac{z}{r_0+z},$$

$$\omega = \frac{1}{EF}\left(P + \frac{M}{r_0} + \frac{M}{r_0 x}\right),$$

$$\epsilon_0 = \frac{1}{EF}\left(P + \frac{M}{r_0}\right).$$

where E = modulus of elasticity,
F = cross-sectional area,
P = normal component of the resultant force acting at G',
M = resultant moment acting round the axis through G' and perpendicular to the plane of centre line,

$$x = \text{a consiant} = -\frac{1}{F}\int \frac{z}{r_0+z}\,dF.$$

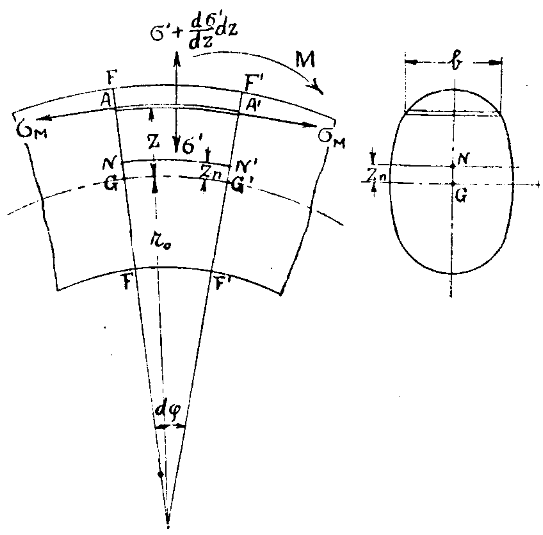
This theory is known very early. It appears in Grashof's Elasticität und Festigkeit published in 1878. Theory of curved rods given in the majority of text books on the strength of materials published thereafter is materially the same as that given in the work of Grashof. So it is likely that the theory has been regarded to be strict up to the present.

According to the present author, however, the theory is not free from some looseness and accordingly Equs. (1), (2) and (3) are not exact, especially when the rodius of curvature r_0 decreases relatively to the height of cross-section.

The assertion here made will soon be recognized in tracing the following development of the author's theory.

THE AUTHOR'S THEORY.

Fig. 2.



In Fig. 2 let GG' be the centre line and NN' the neutral fibre or the fibre, which neither elongates nor shortens, being not identical with the fibre of no longitudinal stress.

First consider the case, where M acts alone. The values of σ, ω and ϵ_0 concerning to M alone will in the following be distinguished by the suffix M.

The fibre element

AA' is subject at its ends to the longitudinal stresses σ_M, which including an angle between themselves form a resultant directed toward the centre of curvature. This resultant requires another force in counteraction for the retention of equilibrium, which latter must be the force consisting of the radial stresses σ' and $\sigma' + \frac{d\sigma'}{dz}\, dz$ shown in Fig. 2, that is

$$2\,\sigma_M \left(b + \frac{db}{2}\right) dz \sin \frac{d\varphi}{2} = \frac{d\,(b\,\sigma'\,r)}{dz}\, dz\, d\varphi,$$

Writing $\frac{d\varphi}{2}$ for $\sin \frac{d\varphi}{2}$ and neglecting $\frac{db}{2}$ against b we get

$$b\,\sigma_M = \frac{d\,(b\,\sigma'\,r)}{dz}. \tag{4}$$

By the condition that the cross-section $F'F'$ must remain as a plane after deformation we have

$$\frac{1}{E}\left(\sigma_M - \frac{\sigma'}{m}\right)\left(r + \frac{dr}{2}\right) d\varphi = \omega_M\, d\varphi\,(z - z_n),$$

m being Poisson's constant and $\omega_M\, d\varphi$ the variation of $d\varphi$. The ordinary theory leaves the effect of the radial stress σ' out of consideration.

Neglecting $\frac{dr}{2}$ against r we have

$$\left.\begin{aligned} \sigma_M\, r - \frac{\sigma'\, r}{m} &= E\,\omega_M\,(z - z_n) \\ \text{or} \qquad b\,\sigma_M\, r - \frac{b\,\sigma'\, r}{m} &= E\,\omega_M\, b\,(z - z_n). \end{aligned}\right\} \tag{5}$$

Differentiating

$$\frac{d\,(b\,\sigma_M\, r)}{dz} - \frac{1}{m}\,\frac{d\,(b\,\sigma'\, r)}{dz} = E\,\omega_M \left[\frac{db}{dz}\,(z - z_n) + b\right], \tag{5'}$$

or inserting the value of (4)

$$\frac{d\,(b\,\sigma_M\, r)}{dz} - \frac{1}{m}\, b\,\sigma_M = E\,\omega_M \left[\frac{db}{dz}\,(z - z_n) + b\right]$$

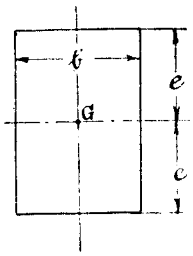
or

$$r\,\frac{d\,(b\,\sigma_M)}{dz} + \frac{m-1}{m}\, b\,\sigma_M - E\,\omega_M \left[\frac{db}{dz}\,(z - z_n) + b\right] = 0. \tag{6}$$

By solving the differential equation (6) the expression of σ_M is obtained. As this equation contains b and $\frac{db}{dz}$, the expression of σ_M will be of different types for different forms of cross-section.

In the following the rectangular, trapezoidal and circular sections will be considered in succession.

1. RECTANGLE.



In rectangle

$$b = \text{const.}, \quad \frac{db}{dz} = 0.$$

Then from (6) as $\frac{dr}{dz} = 1$

$$\frac{d\sigma_M}{dr} + \frac{m-1}{m}\frac{\sigma_M}{r} - E\,\omega_M \frac{1}{r} = 0,$$

and the solution is

$$\sigma_M = e^{-\int \frac{m-1}{m}\frac{dr}{r}} \left[\int \frac{E\,\omega_M}{r} e^{\int \frac{m-1}{m}\frac{dr}{r}} + C \right]$$

$$= r^{-\frac{m-1}{m}} \left[\frac{m}{m-1} E\,\omega_M e^{\frac{m-1}{m}} + C \right]$$

$$= \frac{m}{m-1} E\,\omega_M + \frac{C}{r^{\frac{m-1}{m}}}. \tag{7}$$

The constants ω_M and C are found by putting the whole stress equal to zero and the whole moment of stress equal to M, *i. e.* by the condition

$$0 = \int \sigma_M \, dF,$$

$$M = \int \sigma_M \, z \, dF.$$

Putting in these the value of (7) we get

$$0 = \frac{m}{m-1} E \omega_M \int dF + C \int \frac{dF}{r^{\frac{m-1}{m}}},$$

$$M = \frac{m}{m-1} E \omega_M \int z\, dF + C \int \frac{z dF}{r^{\frac{m-1}{m}}},$$

the integration being to be carried out for the whole cross-section. But $\int dF = F$ and $\int z\, dF = 0$. Therefore with

$$\alpha = \int \frac{dF}{r^{\frac{m-1}{m}}} \text{ and } \beta = \int \frac{z\, dF}{r^{\frac{m-1}{m}}}$$

$$0 = \frac{m}{m-1} EF \omega_M + C \alpha,$$

$$M = C \beta,$$

whence

$$\left. \begin{aligned} C &= \frac{M}{\beta}, \\ \omega_M &= -\frac{m-1}{m} \frac{C\alpha}{EF} = -\frac{m-1}{m} \frac{M}{EF} \frac{\alpha}{\beta}. \end{aligned} \right\} \quad (8)_M$$

With these values (7) reduces to

$$\omega_M = \frac{M}{\beta} \left(-\frac{\alpha}{F} + \frac{1}{r^{\frac{m-1}{m}}} \right). \quad (9)_M$$

This is the stress caused by M alone, to which the stress due to P, $\sigma_P = \frac{P}{F}$ is superposed. The resultant stress is

$$\sigma = \frac{P}{F} + \frac{M}{\beta} \left(-\frac{\alpha}{F} + \frac{1}{r^{\frac{m-1}{m}}} \right). \quad (9)$$

Strictly speaking σ_P like σ_M accompanies the radial stress too and accordingly σ_P will not be uniform throughout the cross section. For the equilibrium of forces acting on the fibre element AA' Fig. 3 we have

$$b\, \sigma_P = \frac{d(b\, \sigma' r)}{dz} + \frac{b}{F} \frac{dR}{d\varphi} f(z), \quad (10)$$

$f(z)$ being the function, according which the shearing stress distributes.

Fig. 3.

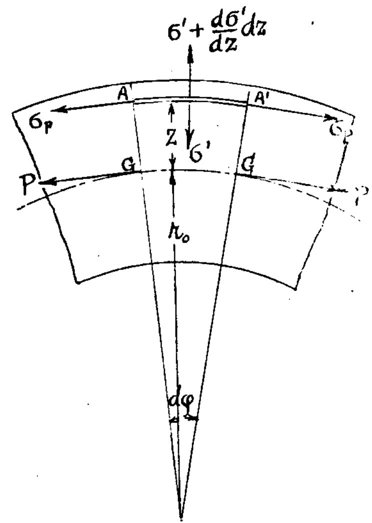
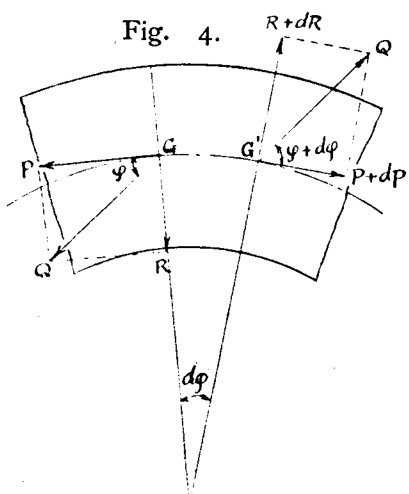


Fig. 4.



Referring to Fig. 4 if there act no external force between G and G', the resultant forces acting at these points will be parallel and of equal magnitude. Hence

$$P = Q\cos\varphi,$$
$$P+dP = Q\cos(\varphi+d\varphi)$$
$$= Q(\cos\varphi - \sin\varphi\, d\varphi),$$
$$R = Q\sin\varphi,$$
$$R+dR = Q\sin(\varphi+d\varphi)$$
$$= Q(\sin\varphi+\cos\varphi\, d\varphi).$$

The resultant of P and $P+dP$ neglecting dP, is

$$2P\sin\frac{d\varphi}{2} = P\,d\varphi = Q\cos\varphi\, d\varphi$$

directed toward the centre of curvature, while the increase of R is

$$dR = Q\cos\varphi\, d\varphi$$

acting in the opposite direction.

Thus the resultant of σ_P in the two cross-sections is counterbalanced by the increase of shearing force and further

$$\frac{dR}{d\varphi} = P.$$

Putting this in (10)

$$b\,\sigma_P = \frac{d(b\,\sigma'\,r)}{dz} + \frac{b}{F}\,Pf(z). \tag{11}$$

The equation similar to (5′) obtained from the condition of no distorsion of the cross-section $F'F'$ is

$$\frac{d(b\,\sigma_P\,r)}{dz} - \frac{1}{m}\,\frac{d(b\,\sigma'\,r)}{dz} = E\omega_P\left[\frac{db}{dz}(z-z'_n)+b\right],$$

where ω_P and z'_n are of the same meanings as ω_M and z_n but concerning to P alone.

Introducing to this the value of $\dfrac{d(b\sigma' r)}{dz}$ in (11)

$$\frac{d(b\,\sigma_P\,r)}{dz} - \frac{b}{m}\left[\sigma_P - \frac{P}{F}f(z)\right] = E\,\omega_P\left[\frac{db}{dz}(z - z'_n) + b\right].$$

The solution of this differential equation determines the distribution of σ_P. But if now, for the simplicity's sake, we assume dR to distribute uniformly throughout the cross-section, then for the fibre element AA' $\sigma_P = \dfrac{P}{F}$ acting along the fibre at A and A' will hold equilibrium with $\dfrac{dR}{F}$ acting across the fibre at A', so that the consideration of σ' might be dispensed with.

The error of max. σ arising from the above assumption should be very small as σ_P is almost always a small fraction less than one-tenth of max. σ_M.

If σ_P is uniform

$$r\,\omega_P\,d\varphi = \frac{P}{EF}\,r\,d\varphi \qquad \text{or} \qquad \omega_P = \frac{P}{EF}$$

and
$$\omega = \omega_P + \omega_M = \frac{1}{EF}\left(P - \frac{m-1}{m}\,\frac{\alpha}{\beta}\,M\right)$$

in case P and M act simultaneously.

If r_1 be the radius of the outer extreme fibre and r_2 that of the inner extreme fibre, the value of α and β are

$$\alpha = \int \frac{dF}{r^{\frac{m-1}{m}}} = b\int_{r_2}^{r_1} \frac{dr}{r^{\frac{m-1}{m}}} = bm\left(r_1^{\frac{1}{m}} - r_2^{\frac{1}{m}}\right), \qquad (12)$$

$$\beta = \int \frac{z\,dF}{r^{\frac{m-1}{m}}} = b\int_{r_2}^{r_1} \frac{z\,dz}{r^{\frac{m-1}{m}}}$$

$$= bm\left[z\,(r_0+z)^{\frac{1}{m}} - \frac{m}{m+1}\,(r_0+z)^{\frac{m+1}{m}}\right]_{-e}^{c}$$

$$= bm\left[(r_0+z)^{\frac{1}{m}}\left(\frac{z}{m+1} - \frac{m}{m+1}\,r_0\right)\right]_{-e}^{c}$$

$$= -b\,\frac{m}{m+1}\left[(m r_0 - e)\, r_1^{\frac{1}{m}} - (m r_0 + e)\, r_2^{\frac{1}{m}}\right]. \tag{13}$$

To find the distribution of σ' caused by σ_M in dependence on r we have by (4)

$$\sigma_M = \frac{d(\sigma' r)}{dz} = \frac{d(\sigma' r)}{dr},$$

whence
$$\sigma' r = \int \sigma_M\, dr + \text{const.} = \frac{M}{\beta}\left[\int\left(-\frac{\alpha}{F} + \frac{1}{r^{\frac{m-1}{m}}}\right) dr + K\right]$$

$$= \frac{M}{\beta}\left(-\frac{\alpha}{F}\, r + m r^{\frac{1}{m}} + K\right).$$

But for $r = r_1$ $\sigma' r = 0$. Therefore

$$0 = -\frac{\alpha}{F}\, r_1 + m r_1^{\frac{1}{m}} + K \quad \text{or} \quad K = \frac{\alpha}{F}\, r_1 - m r_1^{\frac{1}{m}}$$

and accordingly

$$\sigma' r = \frac{M}{\beta}\left[\frac{\alpha}{F}(r_1 - r) - m\left(r_1^{\frac{1}{m}} - r^{\frac{1}{m}}\right)\right]. \tag{14}$$

For the inner extreme fibre

$$\sigma' r_2 = \frac{M}{\beta}\left[\frac{\alpha}{F}(r_1 - r_2) - m\left(r_1^{\frac{1}{m}} - r_2^{\frac{1}{m}}\right)\right],$$

which with the value of α in (12) becomes zero, as must be the case.

The value of z_n is found from (5). Substitute therein $(9)_M$ and (14) and put $z = 0$ and $r = r_0$.

$$\frac{M}{\beta}\left(-\frac{\alpha}{F} + \frac{1}{r^{\frac{m-1}{m}}}\right) r_0 - \frac{1}{m}\,\frac{M}{\beta}\left[\frac{\alpha}{F}(r_1 - r_0) - m\left(r_1^{\frac{1}{m}} - r_0^{\frac{1}{m}}\right)\right]$$
$$= -E\,\omega_M z_n,$$

whence
$$z_n = -\frac{m}{m-1}\left[r_0 + \frac{e}{m} - \frac{F}{\alpha}\, r_1^{\frac{1}{m}}\right]. \tag{15}$$

Finally the strain of $G\,G'$ on the centre line is

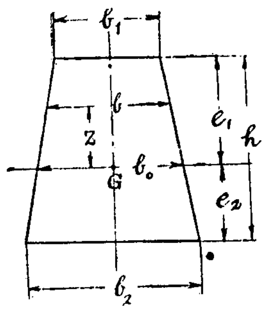
$$\epsilon_{0M} = \frac{-z_n\,\omega_M\, d\varphi}{r_0\, d\varphi} = -\frac{\omega_M z_n}{r_0}$$

$$= -\frac{M}{EF}\,\frac{\alpha}{\beta}\left[1 + \frac{1}{m}\,\frac{e}{r_0} + \frac{F}{\alpha r_0}\, r_1^{\frac{1}{m}}\right]. \tag{16}_M$$

In case P and M act simultaneously

$$\epsilon_0 = \epsilon_{0P} + \epsilon_{0M} = \frac{P}{EF} - \frac{M}{EF}\,\frac{\alpha}{\beta}\left[1 + \frac{1}{m}\,\frac{e}{r_0} + \frac{F}{\alpha r_0}\, r_1^{-\frac{1}{m}}\right]. \quad (16)$$

2. TRAPEZOID.



In trapezoid

$$b = b_0 - \mu z \text{ and } \frac{db}{dz} = -\mu$$

with $\mu = \dfrac{b_2 - b_1}{h}$.

We have by (6)

$$\frac{d(b\,\sigma_M)}{dz} - \frac{m-1}{m}\,\frac{b\,\sigma_M}{r} - E\,\omega_M\,\frac{-2\,\mu z + \mu z_n + b_0}{r} = 0,$$

the solution of which is

$$b\,\sigma_M = E\,\omega_M\,(r_0+z)^{-\frac{m-1}{m}}\left\{\int\left[-2\,\mu z + \mu z_n + b_0\right](r_0+z)^{-\frac{1}{m}}\,dz + K\right\}$$

$$= E\,\omega_M\,(r_0+z)^{-\frac{m-1}{m}}\left\{-2\,\mu\frac{m}{m-1}\,(r_0+z)^{\frac{m-1}{m}}\left(\frac{m-1}{2\,m-1}\,z - \frac{m}{2\,m-1}r_0\right)\right.$$

$$\left. + (\mu z_n + b_0)\,\frac{m}{m-1}\,(r_0+z)^{\frac{m-1}{m}} + K\right\}$$

$$= E\,\omega_M\,\frac{m}{m-1}\,\mu\left\{A + z_n - 2\,\frac{m-1}{2\,m-1}\,z + \frac{K'}{(r_0+z)^{\frac{m-1}{m}}}\right\}, \quad (17)$$

with

$$A = \frac{2\,m}{2\,m-1}\,r_0 + \frac{b_0}{\mu}. \quad (18)$$

By (4)

$$b\,\sigma'\,r = \int b\,\sigma_M\,dz + \text{const.},$$

which with the value of $b\,\sigma_M$ in (17) becomes

$$b\,\sigma'\,r = E\,\omega_M\,\frac{m}{m-1}\,\mu\left\{(A+z_n)\,z - \frac{m-1}{2\,m-1}\,z^2 + K'm\,(r_0+z)^{\frac{1}{m}} + H\right\}.$$

But for $z = e_1$ and $z = -e_2$ $b\,\sigma' r = 0$. These conditions determine K' and H, viz.

$$0 = (A+z_n)\,e_1 - \frac{m-1}{2\,m-1}\,e_1^2 + K'\,m\,r_1^{\frac{1}{m}} + H,$$

$$0 = -(A+z_n)\,e_2 - \frac{m-1}{2\,m-1}\,e_2^2 + K'\,m\,r_2^{\frac{1}{m}} + H,$$

whence we get

$$\left.\begin{aligned} K'\,m &= \frac{\frac{m-1}{2\,m-1}\,(e_1-e_2)-(A+z_n)}{r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}}}\,h, \\ H &= \frac{(A+z_n)\,(e_1\,r_2^{\frac{1}{m}}+e_2\,r_1^{\frac{1}{m}}) - \frac{m-1}{2\,m-1}\,(e_1^2\,r_2^{\frac{1}{m}}-e_2^2\,r_1^{\frac{1}{m}})}{r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}}}. \end{aligned}\right\} \tag{19}$$

To find z_n insert in (5) the values of $b\,\sigma_M$ and $b\,\sigma' r$ found above and put $z = 0$, $r = r_0$ and $b = b_0$.

$$E\,\omega_M \frac{m}{m-1}\,\mu \left\{A+z_n+\frac{K'}{r_0^{\frac{m-1}{m}}}\right\} r_0 - E\,\omega_M \frac{m}{m-1}\,\mu\left\{K'\,r_0^{\frac{1}{m}}+\frac{H}{m}\right\}$$

$$= -E\,\omega_M\,b_0\,z_n,$$

or

$$\frac{m}{m-1}\,\mu\left[(A+z_n)\,r_0 - \frac{H}{m}\right] = -b_0\,z_n.$$

Introducing to this the value of H in (19) and solving for z_n

$$z_n = -\frac{A\left[m\,r_0\,(r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}})-(e_1\,r_2^{\frac{1}{m}}+e_2\,r_1^{\frac{1}{m}})\right]+\frac{m-1}{2\,m-1}\,(e_1^2\,r_2^{\frac{1}{m}}-e_2^2\,r_1^{\frac{1}{m}})}{\left[(m-1)\,\frac{b_0}{\mu}+m\,r_0\right](r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}})-(e_1\,r_2^{\frac{1}{m}}+e_2\,r_1^{\frac{1}{m}})}. \tag{20}$$

Finally ω_M is found by the following condition.

$$M = \int \sigma_M\,z\,dF = \int_{-e_2}^{e_1} b\,\sigma_M\,z\,dz.$$

Putting therein the value of $b\,\sigma_M$ in (17)

$$M = E\,\omega_M \frac{m}{m-1}\,\mu \int_{-e_2}^{e_1}\left[(A+z_n)\,z - 2\,\frac{m-1}{2\,m-1}\,z^2 + K'\frac{z}{(r_0+z)^{\frac{m-1}{m}}}\right]dz$$

$$= E\,\omega_M \frac{m}{m-1}\,\mu\left\{\frac{1}{2}(A+z_n)\,h\,(e_1-e_2)-\frac{2}{3}\,\frac{m-1}{2m-1}\,h\,(e_1^2-e_1e_2+e_2^2)\right.$$

$$\left.+K'\frac{m}{m+1}\left[e_1 r_1^{\frac{1}{m}}+e_2 r_2^{\frac{1}{m}}-m\,r_0\left(r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}}\right)\right]\right\}.$$

Substituting to this the value of K' in (19) and solving for ω_M

$$\omega_M=\frac{m-1}{m}\,\frac{M}{E\mu h}\div\left\{(A+z_n)\,\frac{e_1-e_2}{2}-\frac{2}{3}\,\frac{m-1}{2m-1}(e_1^2-e_1e_2+e_2^2)\right.$$

$$\left.+\frac{1}{m+1}\left[\frac{m-1}{2m-1}(e_1-e_2)-(A+z_n)\right]\left[\frac{e_1 r_1^{\frac{1}{m}}+e_2 r_2^{\frac{1}{m}}}{r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}}}-m\,r_0\right]\right\}. \quad (21)_M$$

To recapitulate the expression of σ_M is

$$\sigma_M=\frac{m}{m-1}\,\mu\,E\,\omega_M\frac{1}{b}\left[A+z_n-2\,\frac{m-1}{2m-1}\,z+\frac{K'}{r^{\frac{m-1}{m}}}\right] \quad (22)_M$$

with

$$\frac{m}{m-1}\,\mu\,E\,\omega_M=\frac{M}{h}\div\left\{(A+z_n)\frac{e_1-e_2}{2}-\frac{2}{3}\,\frac{m-1}{2m-1}(e_1^2-e_1e_2+e_2^2)\right.$$

$$\left.+\frac{1}{m+1}\left[\frac{m-1}{2m-1}(e_1-e_2)-(A+z_n)\right]\left[\frac{e_1 r_1^{\frac{1}{m}}+e_2 r_2^{\frac{1}{m}}}{r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}}}-m\,r_0\right]\right\},$$

$$K'=\frac{\frac{m-1}{2m-1}(e_1-e_2)-(A+z_n)}{r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}}}\,\frac{h}{m},$$

where

$$z_n=-\frac{A\left[m\,r_0\left(r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}}\right)-\left(e_1 r_2^{\frac{1}{m}}+e_2 r_1^{\frac{1}{m}}\right)\right]+\frac{m-1}{2m-1}\left(e_1^2 r_2^{\frac{1}{m}}-e_2^2 r_1^{\frac{1}{m}}\right)}{\left[(m-1)\frac{b_0}{\mu}+m\,r_0\right]\left(r_1^{\frac{1}{m}}-r_2^{\frac{1}{m}}\right)-\left(e_1 r_2^{\frac{1}{m}}-e_2 r_1^{\frac{1}{m}}\right)}.$$

The strain of GG' on the centre line is

$$\epsilon_{0M}=-\frac{\omega_M z_n}{r_0}. \quad (23)_M$$

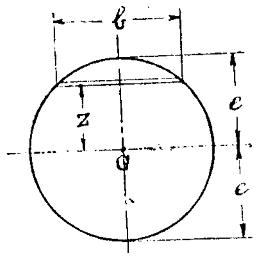
In case P and M act simultaneously

$$\sigma=\frac{P}{F}+\sigma_M, \quad (22)$$

$$\omega=\frac{P}{EF}+\omega_M, \quad (21)$$

$$\epsilon_0 = \frac{P}{EF} - \frac{\omega_M z_n}{r_0}. \tag{23}$$

3. CIRCLE.



In circle

$$b = 2\,(e^2 - z^2)^{\frac{1}{2}}$$

and $$\frac{db}{dz} = -2\,z\,(e^2 - z^2)^{-\frac{1}{2}}.$$

The case of circle is perplexing. Inserting in (6) the above values of b and $\frac{db}{dz}$ and solving as before we obtain the expression of $b\,\sigma_M$ of the following form:

$$b\,\sigma_M = 2\,E\,\omega_M\,r_0^{-1}\left(1+\frac{z}{r_0}\right)^{-\frac{m-1}{m}}\left\{\left[a + a_1 z + a_2 z^2 + \ldots\ldots\right]\sqrt{e^2 - z^2}\right.$$
$$\left. + c \arcsin\frac{z}{e} + K r_0^{\frac{1}{m}}\right\}.$$

For $z = e$ must $b\,\sigma_M = 0$ and accordingly the constants c and K are expected to vanish. But the proof that $c = 0$ is not easily attained. Hence we put as follows:

$$\sigma_M = E\,\omega_M\,(a + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots\ldots\ldots), \tag{24}$$

where a, a_1, a_2 etc. are unknown constants to be determined later. (24) is no other than Maclaurin's series and evidently the greater number of terms is taken, the more accurate is the result. Here the terms up to the term of z^4 will be taken.

By (4) we have

$$b\,\sigma'\,r = \int b\,\sigma_M\,dz + \text{const.}$$

$$= 2\,E\,\omega_M\left[\int (a + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4)\sqrt{e^2 - z^2}\,dz + K\right].$$

But

$$\int \sqrt{e^2-z^2}\,dz = \frac{e^2}{2}\arcsin\frac{z}{e}+\frac{z}{2}\sqrt{e^2-z^2},$$

$$\int z\sqrt{e^2-z^2}\,dz = \left(-\frac{e^2}{3}+\frac{z^2}{3}\right)\sqrt{e^2-z^2},$$

$$\int z^2\sqrt{e^2-z^2}\,dz = \frac{e^4}{8}\arcsin\frac{z}{e}+\left(-\frac{e^2}{8}z+\frac{z^3}{4}\right)\sqrt{e^2-z^2},$$

$$\int z^3\sqrt{e^2-z^2}\,dz = \left(-\frac{2e^4}{15}-\frac{e^2}{15}z^2+\frac{z^4}{5}\right)\sqrt{e^2-z^2},$$

$$\int z^4\sqrt{e^2-z^2}\,dz = \frac{e^6}{16}\arcsin\frac{z}{e}+\left(-\frac{e^4}{16}z-\frac{e^2}{24}z^3+\frac{z^5}{6}\right)\sqrt{e^2-z^2}.$$

Therefore

$$b\,\sigma' r = 2\,E\,\omega_M\Bigg\{\left(\frac{e^2}{2}a+\frac{e^4}{8}a_2+\frac{e^6}{16}a_4\right)\arcsin\frac{z}{e}$$

$$+\Bigg[\left(-\frac{e^2}{3}a_1-\frac{2e^4}{15}a_3\right)+\left(\frac{a}{2}-\frac{e^2}{8}a_2-\frac{e^4}{16}a_4\right)z$$

$$+\left(\frac{a_1}{3}-\frac{e^2}{15}a_3\right)z^2+\left(\frac{a_2}{4}-\frac{e^2}{24}\right)z^3$$

$$+\frac{a_3}{5}z^4+\frac{a_4}{6}z^5\Bigg]\sqrt{e^2-z^2}+K\Bigg\}.$$

But for $z=\pm e$ must $\sigma'=0$. Therefore

$$K=0,$$

$$\frac{e^2}{2}a+\frac{e^2}{8}a_2+\frac{e^6}{16}a_4=0$$

or

$$8\,a+2\,e^2\,a_2+e^4\,a_4=0. \tag{a}$$

Then

$$\sigma' r = -E\,\omega_M\Bigg[\left(\frac{e^2}{3}a_1+\frac{2e^4}{15}a_3\right)+\left(-\frac{a}{2}+\frac{e^2}{8}a_2+\frac{e^4}{16}a_4\right)z$$

$$+\left(-\frac{a_1}{3}+\frac{e^2}{15}a_3\right)z^2+\left(-\frac{a_2}{4}+\frac{e^2}{24}a_4\right)z^3-\frac{a_3}{5}z^4\Bigg], \tag{25}$$

taking up to the term of z^4.

By (24)

$$\sigma_M r = E\,\omega_M\,(r_0+z)\,(a+a_1 z+a_2 z^2+a_3 2^3+a_4 z^4)$$

$$= E\,\omega_M \Big[a\,r_0+(a_1 r_0+a)\,z+(a_2 r_0+a_1)\,z^2$$

$$+(a_3 r_0+a_2)\,z^3+(a_4 r_0+a_3)\,z^4 \Big], \qquad (26)$$

taking up to the term of z^4.

Putting in (5) the above values of $\sigma' r$ and $\sigma_M r$

$$\sigma_M r-\frac{\sigma' r}{m}-E\,\omega_M\,(z-z_n)$$

$$= E\,\omega_M \Big\{ a\,r_0+(a_1 r_0+a)\,z+(a_2 r_0+a_1)\,z^2+(a_3 r_0+a_2)\,z^3+(a_4 r_0+a_3)\,z^4$$

$$+\frac{1}{m}\Big[\Big(\frac{e^2}{3}a_1+\frac{2e^4}{15}a_3\Big)+\Big(-\frac{a}{2}+\frac{e^2}{8}a_2+\frac{e^4}{16}a_4\Big)z$$

$$+\Big(-\frac{a_1}{3}+\frac{e^2}{15}a_3\Big)z^2+\Big(-\frac{a_2}{4}+\frac{e^2}{24}a_4\Big)z^3-\frac{a_3}{5}z^4\Big]\Big\}$$

$$-E\,\omega_M\,(z-z_n) = 0.$$

As this equation must hold good for all values of z, the coefficients of z^0, z^1, z^2 etc. have to vanish by themselves. That is

$$a\,r_0+\frac{1}{m}\Big(\frac{e^2}{3}a_1+\frac{2e^4}{15}a_3\Big)+z_n = 0, \qquad \text{(b)}$$

$$a_1 r_0+a+\frac{1}{m}\Big(-\frac{a}{2}+\frac{e^2}{8}a_2+\frac{e^4}{16}a_4\Big)-1 = 0, \qquad \text{(c)}$$

$$a_2 r_0+a_1+\frac{1}{m}\Big(-\frac{a_1}{3}+\frac{e^2}{15}a_3\Big) = 0, \qquad \text{(d)}$$

$$a_3 r_0+a_2+\frac{1}{m}\Big(-\frac{a_2}{4}+\frac{e^2}{24}a_4\Big) = 0, \qquad \text{(e)}$$

$$a_4 r_0+a_3-\frac{1}{m}\,\frac{a_3}{5} = 0. \qquad \text{(f)}$$

A further condition remaining is

$$M = \int_{-e}^{e} b\,\sigma_M\,z\,dz$$

or with the values of b and σ_M

$$M = 2\,E\,\omega_M \int_{-e}^{e} (a\,z + a_1\,z^2 + a_2\,z^3 + a_3\,z^4 + a_4\,z^5)\sqrt{e^2 - z^2}\,dz.$$

But the integrals including the odd power of z are zero. Therefore

$$M = 2\,E\,\omega_M \left(\frac{e^4}{8}\,a_1 + \frac{e^6}{16}\,a_3\right)\left(\arcsin\frac{z}{e}\right)_{-e}^{e}$$

or

$$2\,e^4\,a_1 + e^6\,a_3 - \frac{8\,M}{E\,\omega_M\,\pi} = 0. \qquad (g)$$

From Equs. (a), (b), (c), (d), (e), (f) and (g) the unknown quantities are found as follows:

$$a = \frac{1}{C}\Big[(-180\,m^4 + 60\,m^3)\,r_0^2\,e^2 + (-90\,m^4 + 78\,m^3 - 22\,m^2 + 2\,m)\,e^4\Big],$$

$$a_1 = \frac{1}{C}\Big[-720\,m^4\,r_0^3 + (78\,m^3 - 18\,m^2)\,r_0\,e^2\Big],$$

$$a_2 = \frac{1}{C}\Big[(720\,m^4 - 240\,m^3)\,r_0^2 + (-30\,m^3 + 16\,m^2 - 2\,m)\,e^2\Big],$$

$$a_3 = \frac{1}{C}\Big[(-720\,m^4 + 420\,m^3 - 60\,m^2)\,r_0\Big],$$

$$a_4 = \frac{1}{C}\Big[720\,m^4 - 564\,m^3 + 144\,m^2 - 12\,m\Big],$$

$$z_n = -\frac{1}{C}\Big[(-180\,m^4 - 180\,m^3)\,r_0^3\,e^2 + (-90\,m^4 - 18\,m^3 + 60\,m^2 - 12\,m)\,r_0\,e^4\Big],$$

$$\omega_M = \frac{8\,M}{E\,\pi}\;\frac{C}{-1440\,m^4\,r_0^3\,e^4 + (-720\,m^4 + 576\,m^3 - 96\,m^2)\,r_0\,e^6},$$

where

$$C = -720\,m^4\,r_0^4 + (-180\,m^4 + 318\,m^3 - 78\,m^2)\,r_0^2\,e^2$$
$$+ (-90\,m^4 + 168\,m^3 - 100\,m^2 + 24\,\mathrm{m} - 2)\,e^4.$$

These values are much simplified by putting $m = \frac{10}{3}$, with which and with $\frac{1}{n} = \frac{e}{r_0}$ and $\zeta = \frac{z}{e}$

$$\left.\begin{aligned} a &= \frac{e^4}{C'}(-162\,n^2-69), \\ a_1 &= \frac{e^3}{C'}(-720\,n^3+22), \\ a_2 &= \frac{e^2}{C'}(648\,n^2-8), \\ a_3 &= \frac{e}{C'}(-599\,n), \\ a_4 &= \frac{563}{C'}, \end{aligned}\right\} \tag{27}$$

with

$$C' = e^4(-720\,n^4-92\,n^2-48), \tag{28}$$

$$z_n = -\frac{n(234\,n^2+90)}{720\,n^4+92\,n^2+48}\,e, \tag{29}$$

$$\omega_M = \frac{M}{E\pi e^7}\,\frac{C'}{n(-180\,n^2-68)} = \frac{M}{E\pi e^3}\,\frac{720\,n^4+92\,n^2+48}{n(180\,n^2+68)}. \tag{30$_M$}$$

With these values the expression of σ_M reduces to

$$\sigma_M = \frac{M}{\pi e^3}\,\frac{1}{n(180\,n^2+68)}\Big\{(162\,n^2+69)+(720\,n^3-22)\,\zeta$$
$$-(648\,n^2-8)\,\zeta^2+599\,n\,\zeta^3-563\,\zeta^4\Big\}. \tag{31$_M$}$$

For $\zeta = 1$

$$(\sigma_M)_1 = \frac{M}{Fe}\,\frac{1}{n(180\,n^2+68)}\Big[720\,n^3-486\,n^2+599\,n-508\Big]$$

and for $\zeta = -1$ (32)$_M$

$$(\sigma_M)_2 = -\frac{M}{Fe}\,\frac{1}{n(180\,n^2+68)}\Big[720\,n^3+486\,n^2+599\,n+464\Big].$$

Further the strain of GG' is

$$\epsilon_{0M} = -\frac{\omega_M z_n}{r_0} = \frac{M}{E\pi e^3 n}\,\frac{234\,n^2+90}{180\,n^2+68}. \tag{33$_M$}$$

In case P and M act simultaneously

$$\sigma = \frac{P}{F}+\sigma_M, \tag{31}$$

$$\left.\begin{aligned}(\sigma)_1 &= \frac{P}{F} + (\sigma_M)_1, \\ (\sigma)_2 &= \frac{P}{F} + (\sigma_M)_2,\end{aligned}\right\} \quad (32)$$

$$\omega = \frac{P}{EF} + \omega_M, \quad (30)$$

$$\epsilon_0 = \frac{P}{EF} + \epsilon_{0M}. \quad (33)$$

A remark may here be necessary about the distribution of σ'. In sections other than rectangle, especially in circle σ' may possibly be not uniform along a chord parallel to the bending axis. Its value in the present investigation has to be taken as the mean for all points on the chord.

NUMERICAL RESULTS FROM THE ORDINARY AND THE AUTHOR'S THEORIES.

Taking a few examples the values of σ_M were calculated by the ordinary as well as by the author's formulas and the results were made to Tables 1 to 3 for the purpose of comparison.

TABLE 1. Rectangle.

$\frac{e}{r_0}$	$\frac{z}{e}$	$\sigma_M / \frac{M}{Fe}$ by ordinary formula	$\sigma_M / \frac{M}{Fe}$ by new formula	Error of magnitude of max. stress. %
$\frac{1}{6}$	1	2.689	2.731	−1.54
	$\frac{1}{2}$	1.525	1.521	
	0	0.167	0.127	
	$-\frac{1}{2}$	−1.438	−1.447	
	−1	−3.364	−3.303	+1.84
$\frac{1}{3}$	1	2.431	2.511	−3.19
	$\frac{1}{2}$	1.532	1.535	
	0	0.333	0.285	
	$-\frac{1}{2}$	−1.345	−1.377	
	−1	−3.863	−3.715	+3.98
$\frac{1}{2}$	1	2.190	2.293	−4.49
	$\frac{1}{2}$	1.514	1.520	
	0	0.500	0.429	
	$-\frac{1}{2}$	−1.190	−1.240	
	−1	−4.571	−4.281	+6.75

TABLE 2. Trapezoid ($b_2 = 2\,b_1$).

$\frac{e_2}{r_0}$	z	$\sigma_M / \frac{M}{Fe_2}$ by ordinary formula	by new formula	Error of magnitude of max. stress. %
$\frac{1}{6}$	e_1	2.713	2.800	−3.11
	$\frac{e_1}{2}$	1.560	1.576	
	0	0.167	0.143	
	$-\frac{e_2}{2}$	−1.176	−1.202	
	$-e_2$	−2.787	−2.780	+0.25
$\frac{1}{3}$	e_1	2.430	2.517	−3.46
	$\frac{e_1}{2}$	1.563	1.562	
	0	0.333	0.287	
	$-\frac{e_2}{2}$	−1.093	−1.117	
	$-e_2$	−3.232	−3.112	−3.86
$\frac{1}{2}$	e_1	2.176	2.291	−5.02
	$\frac{e_1}{2}$	1.537	1.544	
	0	0.500	0.432	
	$-\frac{e_2}{2}$	0.952	−1.004	
	$-e_2$	3.857	−3.616	−6.66

TABLE 3. Circle.

$\frac{e}{r_0}$	$\frac{z}{e}$	$\sigma_M / \frac{M}{Fe}$ by ordinary formula	by new formula	Error of magnitude of max. stress. %
$\frac{1}{6}$	1	3.547	3.591	−1.23
	$\frac{1}{2}$	1.987	1.991	
	0	0.167	0.150	
	$-\frac{1}{2}$	−1.984	−1.989	
	−1	−4.567	−4.507	+1.33
$\frac{1}{3}$	1	3.166	3.228	−1.92
	$\frac{1}{2}$	1.951	1.969	
	0	0.333	0.301	
	$-\frac{1}{2}$	−1.933	1.954	
	−1	−5.332	5.147	+3.59

For circle the convergency of the expression of σ_M becomes slow as the value of $\frac{e}{r_0}$ increases. Taking in the expression the terms up to the term of z^4 the result will not be sufficiently accurate for cases where $\frac{e}{r_0}$ is considerably greater than $\frac{1}{3}$. Hence in the examples the case of $\frac{e}{r_0}=\frac{1}{2}$ was omitted.

To show what like is the distribution of σ_M according to the author's theory in comparison with that according to the ordinary theory Figs. 5 to 7 were drawn. Fig. 5 is for rectangle and $\frac{e}{r_0}=\frac{1}{2}$, Fig. 6 for trapezoid and $\frac{e_2}{r_0}=\frac{1}{2}$ and Fig. 7 for circle and $\frac{e}{r_0}=\frac{1}{3}$.

In conclusion the author expresses his hearty thanks to Mr. K. Tabushi, who assisted him in undertaking a part of deduction of formulas as well as the numerical calculations.

Fig. 5.

Rectangular Section, $\frac{e}{r_0}=\frac{1}{2}$

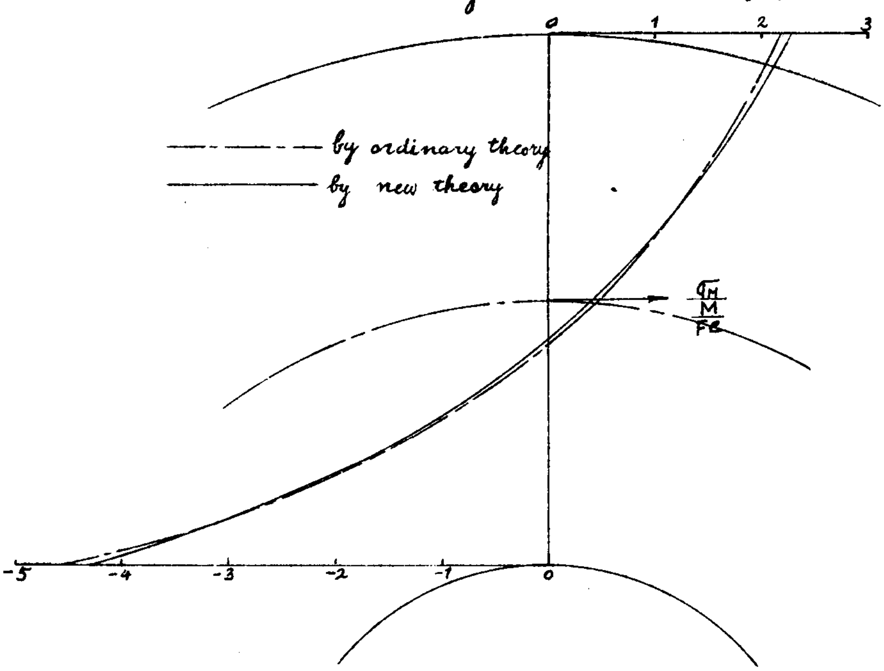


Fig. 6.

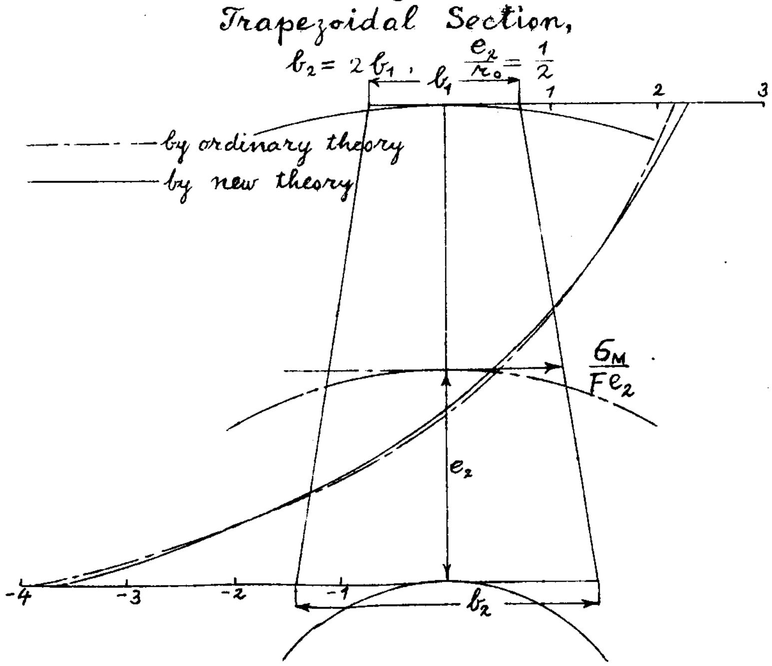
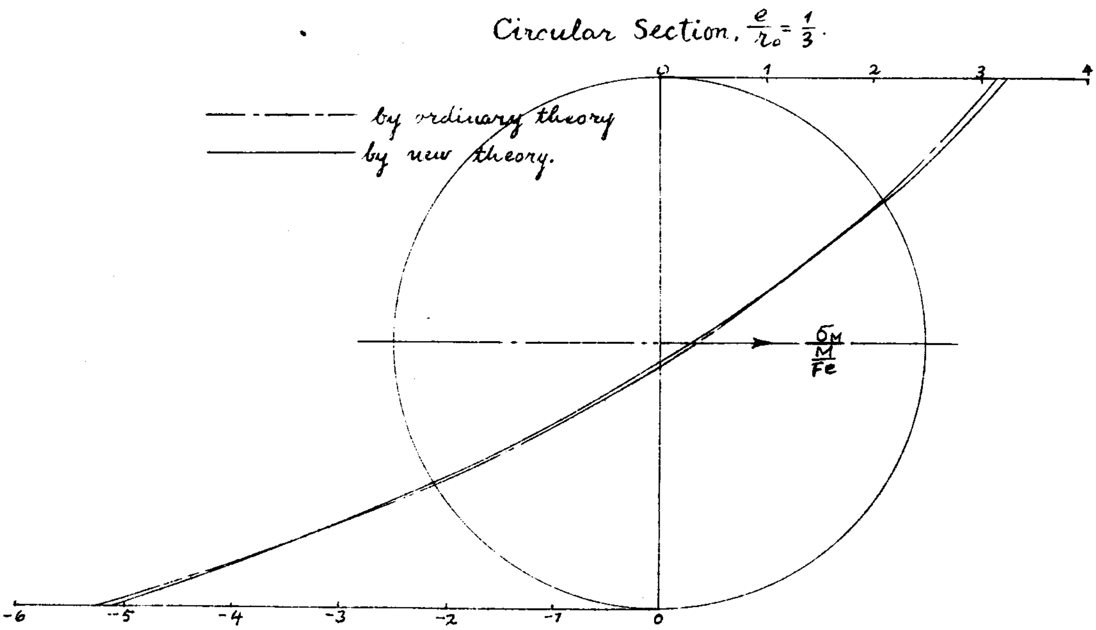

Trapezoidal Section,
$b_2 = 2b_1$, $\frac{e_2}{r_0} = \frac{1}{2}$
b_1
1
2
3
by ordinary theory
by new theory
$\frac{\sigma_M}{Fe_2}$
e_2
-4
-3
-2
-1
b_2

Fig. 7.


Circular Section, $\frac{e}{r_0} = \frac{1}{3}$.
0
1
2
3
4
by ordinary theory
by new theory.
$\frac{\sigma_M}{\frac{M}{Fe}}$
-6
-5
-4
-3
-2
-1
0