Analytical Investigation of Electrical Transient Phenomena in Transmission Line Circuit.

By

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INTRODUCTION.

The general methods of investigating transient phenomena in a transmission line terminated by impedances at both ends, have been studied persistently by many authorities, yet with unsuccessful results, though some special cases have been dealt with successfully, and many brilliant results have been obtained. For instance, surge propagation phenomena along an infinitely long line with four distributed constants, resistance R, inductance L, leakance G, and electrostatic capacity C per unit length of the line, have been discussed [1] by several authors. But when the terminal condition of the line come into play, or which is the same thing, when the line is of finite length, we are at a loss how to solve the question generally. When the dissipation constants of the line, R and G, are zero, what is called "D'Alumbert's method", by means of which Hô obtained many important results, is convenient. But this method is only effective when reflected waves are excluded. And in the case of L and G being neglected, the results already obtained for heat conduction phenomena are directly applicable. Wagner[3] proposes a method of discussing the transient phenomena in a line of finite length, with impedances at its terminals, and finds the solutions in infinite series, but his method is not general, and only a

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1. The so-called Riemann's method is usually applied. We find this method in Riemann—Weber's "Die Partiellen Differentialgleichungen Bd. 2", or in Goursat's "Cours d'Analyse mathematique Tome 3".

2. 鳳秀太郎；波動振動及避雷.

3. K. W. Wagner, Electromagnetische Ausgleichsvorgänge in Freileitungen und Kabeln, or J. Biermanns, Arch. f. Elek. 1916, p. 211.

few cases are dealt with thereby. Though Heaviside's expansion theorem proves itself more general, yet it is also inconvenient in the study of these kinds of phenomena, and numerical calculation of results often becomes impossible. Indeed, when the constants R, L, G and C as well as the terminal conditions of the line are taken into consideration, we shall find it so difficult and so complicated to treat transient phenomena therein that there is hardly any other method of solution than that proposed by Heaviside, known as "Operational Calculus."[1] But this method is too difficult for us to comprehend.

In course of an analytical study of Heaviside's operational calculus, the author tried to understand the mathematical meaning of the calculus and to modify its original form as given by Heaviside so as to make it more convenient for practical use. For this purpose, the author investigated transient phenomena in a finitely long transmission line terminated by impedances, taking four constants R, L, G and C into consideration and obtained some results which seem more generally applicable than those derived from any of the other methods mentioned above. The author has succeeded in solving some problems left untouched by Heaviside. He wishes to add that the present paper serves to a certain extent, to give the analytical explanation of the operational calculus, and at the same time, that the present results may serve to solve other physical problems such as the conduction of heat, the propagation of sound and of electromagnetic waves in space etc. also.

The present paper deals with transient phenomena in a line with the e.m.f. $E\varepsilon^{-p_0 t}$ applied at one terminal since $t=0$, assuming that the initial potential and current distributions are zero along the whole circuit.

1. FUNDAMENTAL FORMULAS.

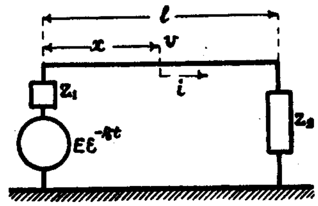
The author's results are as follows :—

The potential v and the current i at the instant t and at the point x on the transmission line shown in fig. 1, due to the e.m.f. $E\varepsilon^{-p_0 t}$ applied since

1. O. Heaviside, Electromagnetic Theory Vol. 2 & 3.

$t=0$, at which time the current and potential in the circuit were identically zero, are given by the following formulas :—

Fig. 1



$$\left.\begin{aligned}
v&=0 \qquad \text{for } t<x/g,\\
&=v_0+v_1+v_2+v_3+\cdots\cdots\cdots+v_{2m}\\
&\qquad \text{for } \frac{2(m+1)l-x}{g}>t>\frac{2ml+x}{g},\\
&=v_0+v_1+v_2+v_3+\cdots\cdots\cdots+v_{2m+1}\\
&\qquad \text{for } \frac{2(m+1)l+x}{g}>t>\frac{2(m+1)l-x}{g},\\
i&=0 \qquad \text{for } t<x/g,\\
&=i_0+i_1+i_2+i_3+\cdots\cdots\cdots+i_{2m}\\
&\qquad \text{for } \frac{2(m+1)l-x}{g}>t>\frac{2ml+x}{g},\\
&=i_0+i_1+i_2+i_3+\cdots\cdots\cdots+i_{2m+1}.\\
&\qquad \text{for } \frac{2(m+1)l+x}{g}>t>\frac{2(m+1)l-x}{g},
\end{aligned}\right\} \cdots\cdots\cdots\cdots(1)$$

where

$$\left.\begin{aligned}
v_{2m}&=\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{-q(2ml+x)+pt}}{p+p_0}f_1 f^m\,dp,\\
v_{2m+1}&=-\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{-q\{2(m+1)l-x\}+pt}}{(p+p_0)}f_1 f_2 f^m\,dp,\\
i_{2m}&=\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{-q(2ml+x)+pt}}{(p+p_0)z}f_1 f^m\,dp,\\
i_{2m+1}&=\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{-q\{2(m+1)l-x\}+pt}}{(p+p_0)z}f_1 f_2 f^m\,dp,
\end{aligned}\right\} \cdots\cdots\cdots\cdots(2)$$

where the path of integration K is any closed curve which contains all the singular points of each integrand, and the symbol (K) means that the integration should be done along this curve in a positive sense, and

$$\left.\begin{aligned}
&j=\sqrt{-1}\,,\\
&m=0,\ 1,\ 2,\ 3,\ldots\ldots,\\
&l=\textit{the total length of the line},\\
&g=(LC)^{-\frac{1}{2}}\,,\\
&\varepsilon=\textit{the base of natural logarithm},\\
&f=(z-Z_1)(z-Z_2)\Big/\Big\{(z+Z_1)(z+Z_2)\Big\},\\
&f_1=z/(z+Z_1),\\
&f_2=(z-Z_2)/(z+Z_2),\\
&Z_1,\,Z_2=\begin{array}{l}\textit{concentrated genetalized}\\ \textit{impedances}^1\textit{ at both ends},\end{array}\\
&z=\sqrt{(Lp+R)/(Cp+G)},\\
&q=\sqrt{(Lp+R)(Cp+G)},
\end{aligned}\right\}\qquad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where R, L, G and C are distributed constants of the line, resistance, inductance, leakage and capacity per unit length respectively, and the determination of the radical sign is taken so as to be positive when the argument of p is zero.

2. PROOF OF THE ABOVE FORMULAS.

We suppose that the current flows in the positive direction of x, and that the line is energized by the e.m.f. $E\varepsilon^{-p_0t}$ applied at $x=0$ through the concentrated impedance Z_1 since $t=0$, the other terminal being closed by another impedance Z_2, as is indicated in fig. 1. The differential equations of the current i and the potential v are

$$\left.\begin{aligned}
-\frac{\partial v}{\partial x}&=L\frac{\partial i}{\partial t}+Ri,\\
-\frac{\partial i}{\partial x}&=C\frac{\partial v}{\partial t}+Gv.
\end{aligned}\right\}\qquad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

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1. Let v_2 and i_2 be the potential and current at a terminal of a transmission line, and assume that the current flows from the line into the impedance which connects the line and the earth, then we shall have the following differential equation.

$$Z\left(\frac{d}{dt}\right)i_2=v_2.$$

Replace the differential operator $\frac{d}{dt}$ by the letter p, and we call $Z(p)$ the generalized terminal impedance of the line.

We shall solve these fundamental equations under the initial conditions

$$\left.\begin{aligned} &v=0 \\ &i=0 \qquad \text{at } t=0, \end{aligned}\right\}\qquad\cdots\cdots(5)$$

everywhere in the circuit.

The solutions are found by substituting the contour integrals

$$\left.\begin{aligned} v&=\frac{1}{2\pi j}\int_{(K)} V\varepsilon^{pt}\,dp, \\ i&=\frac{1}{2\pi j}\int_{(K)} I\varepsilon^{pt}\,dp, \end{aligned}\right\}\qquad\cdots\cdots(6)$$

in equations (4) where V and I are certain functions of p to be determined by the initial and the terminal conditions of the line, and the path of integration (K) is a closed curve in the p-plane which encloses all the poles of these functions V and I.

Then by condition (5), the equations for V and I will be

$$\left.\begin{aligned} -\frac{dV}{dx}&=(Lp+R)\,I, \\ -\frac{dI}{dx}&=(Cp+G)\,V. \end{aligned}\right\}\qquad\cdots\cdots(7)$$

Hence we obtain

$$\left.\begin{aligned} V&=A\cosh qx+B\sinh qx, \\ I&=-\frac{1}{z}(A\sin qx+B\cosh qx), \end{aligned}\right\}\qquad\cdots\cdots(8)$$

where A and B are integration constants to be determined by the terminal conditions of the line, and

$$\left.\begin{aligned} q&=\sqrt{(Lp+R)(Cp+G)}, \\ z&=\sqrt{(Lp+R)/(Cp+G)}, \end{aligned}\right\}\qquad\cdots\cdots(9)$$

and *the determination of the radical sign is taken so that it is positive when the argument of p is zero.* ⎫ ⎭ $\cdots\cdots(10)$

At the terminals the following relations hold,

$$\left.\begin{aligned} v_{x=0}+Z_1\left(\frac{d}{dt}\right) i_{x=0} &= E\varepsilon^{-p_0 t}, \\ v_{x=l} &= Z_2\left(\frac{d}{dt}\right) i_{x=l}\ , \end{aligned}\right\} \qquad \text{.........................(11)}$$

or by equation (5), the substitution of (6) in (11) gives the following[1]:

$$\left.\begin{aligned} V_{x=0}+Z_1(p)\, I_{x=0} &= \frac{E}{p+p_0}, \\ V_{x=l} &= Z_2(p)\, I_{x=l}. \end{aligned}\right\} \qquad \text{.........................(12)}$$

From (8) and (12), we obtain

$$\left.\begin{aligned} A &= \frac{E}{p+p_0}\,\frac{\sinh ql+(Z_2/z)\cosh ql}{\Delta}, \\ B &= -\frac{E}{p+p_0}\,\frac{\cosh ql+(Z_2/z)\sinh ql}{\Delta}, \end{aligned}\right\} \qquad \text{............(13)}$$

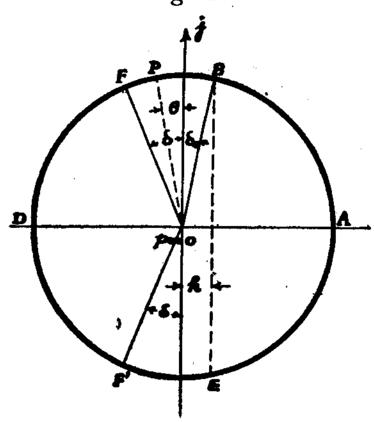
where

$$\left.\begin{aligned} & Z_1=Z_1(p),\ Z_2=Z_2(p), \\ & \Delta=(1+Z_1Z_2/z^2)\sinh ql+(Z_1/z+Z_2/z)\cosh ql \\ & \quad =(1+Z_1/z)(1+Z_2/z)\varepsilon^{ql}-(1-Z_1/z)(1-Z_2/z)\varepsilon^{-ql}. \end{aligned}\right\} \qquad \text{.........(14)}$$

Substituting (13) in (8), we get

$$\left.\begin{aligned} V &= \frac{E}{p+p_0}\,\frac{\sinh q\,(l-x+(Z_2/z)\cosh q\,(l-x)}{\Delta}, \\ I &= \frac{E}{(p+p_0)\,z}\,\frac{\cosh q\,(l-x)+(Z_2/z)\sinh q\,(l-x)}{\Delta}. \end{aligned}\right\} \qquad \text{............(15)}$$

Fig. 2.



As the path of integration of equations (6), we take a circle $ABDEA$ shown in fig. 2 of sufficiently large radius with its origin at $p=0$. The larger the radius of the circle, the more poles of V and I are contained in the domain bounded by this circle. Accordingly we shall consider the case where

1. T.J.I'A. Bromwich, Proc. London Math. Soc. Series 2 Vol. 15 (1916) pp. 406–448.

its radius tends to infinity. Hereafter we divide the path $ABDEA$ into two parts, namely EAB and BDE, and discuss the values of $\int V\varepsilon^{pt}\,dp$ and $\int I\varepsilon^{pt}\,dp$ taken along each of these paths, where BE is a straight line distant h from the imaginary axis, and h is any positive finite quantity independent of p.

Now the functions V and I, given by (15), reduce to the following forms:

$$\left.\begin{aligned} V&=\frac{E}{p+p_0}\,\frac{\varepsilon^{q(2l-x)}-f_2\varepsilon^{qx}}{\varepsilon^{2ql}-f}f_1,\\ I&=\frac{E}{(p+p_0)z}\,\frac{\varepsilon^{q(2l-x)}+f_2\varepsilon^{qx}}{\varepsilon^{2ql}-f}f_1,\end{aligned}\right\}\quad\cdots\cdots(16)$$

where

$$\left.\begin{aligned} f&=(1-Z_1/z)(1-Z_2/z)\Big/\Big\{(1+Z_1/z)(1+Z_2/z)\Big\},\\ f_1&=1/(+Z_1/z),\\ f_2&=(1-Z_2/z)/(1-Z_2/z).\end{aligned}\right\}\quad\cdots\cdots(17)$$

We suppose that the generalized terminal impedances Z_1 and Z_2 consist of concentrated electrical constants. In such a case, they are rational functions of p, and for a sufficiently large value of $|p|$, they can be expressed by the form:

$$Z(Z_1 \text{ or } Z_2)=b_1p+b_0+b_{-1}p^{-1}+b_{-2}p^{-2}+\cdots\cdots, \qquad \cdots\cdots(18)$$

and since $z=\sqrt{(Lp+R)/(Cp+G)}$, we have, when $|p|$ tends to infinity,

(i) if $b_1 \neq 0$,

$$(1-Z/z)/(1+Z/z)=-1+o\left(\frac{1}{p}\right),$$

(ii) if $b_1=0$ and $b_0 \neq 0$,

$$(1-Z/z)/(1+Z/z)=\left(\sqrt{\frac{L}{C}}-b_0\right)\Big/\left(\sqrt{\frac{L}{C}}+b_0\right)+o\left(\frac{1}{p}\right),$$

where it is easily understood, from the properties of the generalized concentrated impedance, that b_0, in this case, is a certain positive quantity, and accordingly we have

$$(1-Z/z)/(1+Z/z)<1,$$

as $|p|$ approaches to infinity.

(iii) If $b_1=0$ and $b_0=0$, then we have

$$(1-Z/z)/(1+Z/z)=1+o\left(\frac{1}{p}\right).$$

Hence in general, we have

$$\left.\begin{aligned}(1-Z_1/z)/(1+Z_1/z)&=a_1+o\left(\frac{1}{p}\right),\\(1-Z_2/z)/(1+Z_2/z)&=a_2+o\left(\frac{1}{p}\right),\end{aligned}\right\} \qquad \text{(19)}$$

where

$$\left.\begin{aligned}0<|a_1|\leqq 1,\\0<|a_2|\leqq 1,\end{aligned}\right\} \qquad \text{(20)}$$

unless $b_1=0$ and $b_0=\sqrt{\frac{L}{C}}$ simultaneously.

Next, we shall consider the common denominator of equations (16) i.e. $(\varepsilon^{2ql}-f)$, along the arc BDE when the radius of the circle $|p|$ tends to infinity.

We assume that $|p|$ increases discontinuously owing to the rule

$$|p|=\frac{1}{l\sqrt{LC}}\,\frac{n\pi}{2}, \qquad \text{(21)}$$

where n is a certain integer suitably chosen, and approaches to infinity to give $|p|$ an infinite value. Accordingly, we can write the value of p corresponding to the point P (see fig. 2), as follows,

$$p=\frac{1}{l\sqrt{LC}}\,\frac{n\pi}{2}\,\varepsilon^{\left(\frac{\pi}{2}+\theta\right)j}. \qquad \text{(22)}$$

On the other hand, for sufficiently large $|p|$, we have from (9),

$$lq=l\sqrt{LC}\left\{p+\rho+o\left(\frac{1}{p}\right)\right\}, \qquad \text{(23)}$$

where

$$\rho=\frac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right). \qquad \text{(24)}$$

Hence from (22) and (23), we obtain

$$lq = X + o_1\left(\frac{1}{p}\right) + j\left\{\frac{n\pi}{2}\cos\theta + o_2\left(\frac{1}{p}\right)\right\}, \quad \text{..............(25)}$$

where

$$X = -\frac{n\pi}{2}\sin\theta + \rho l\sqrt{LC}. \quad \text{..............(26)}$$

And for sufficiently large value of $|p|$, we have on the arc BDE,

$$X \leqq l\sqrt{LC}\,(h+\rho). \quad \text{..............(27)}$$

And from (17) and (19), we obtain

$$f = a_1 a_2 + o_3\left(\frac{1}{p}\right), \quad \text{..............(28)}$$

where

$$0 < |a_1 a_2| \leqq 1, \quad \text{..............(29)}$$

unless $b_1 = 0$ and $b_0 = \sqrt{\frac{L}{C}}$ simultaneously.

Consequently, the relations (25) and (28) give the following:

$$\left|\varepsilon^{2ql} - f\right| = \left|\exp\left[2X + 2o_1\left(\frac{1}{p}\right) + j\left\{n\pi\cos\theta + 2o_2\left(\frac{1}{p}\right)\right\}\right] - a_1 a_2 - o_3\left(\frac{1}{p}\right)\right|.$$

Since ε^X is finite along the arc BDE, when $|p|$ tends to infinity, as is evident from (27), we may neglect, for sufficiently large $|p|$ or n, the terms involving $o_1\left(\frac{1}{p}\right)$, $o_2\left(\frac{1}{p}\right)$ and $o_3\left(\frac{1}{p}\right)$ in the above expression. Therefore we have

$$\lim_{n\to\infty}\left|\varepsilon^{2ql} - f\right| = \sqrt{\left\{\varepsilon^{2X}\cos(n\pi\cos\theta) - a_1 a_2\right\}^2 + \varepsilon^{4X}\sin^2(n\pi\cos\theta)}$$

$$= \sqrt{a_1^2 a_2^2 - 2a_1 a_2\varepsilon^{2X}\cos(n\pi\cos\theta) + \varepsilon^{4X}}. \quad \text{..............(30)}$$

We choose δ indicated in fig. 2, so that

$$\delta = n^{-\frac{2}{3}}. \quad \text{..............(31)}$$

And for a sufficiently large value of n,

$$\delta_0 = 2l\sqrt{LC}\,h/(n\pi). \quad \text{..............(32)}$$

Hence, for sufficiently large n,

$$\delta_0 < \delta. \quad \text{......(33)}$$

Therefore, if P be any point on the arc BF, then

$$|\theta| \leqq \delta = n^{-\frac{2}{3}}.$$

And consequently it follows that

$$\theta^2 \leqq n^{-\frac{4}{3}}, \quad \theta^4 \leqq n^{-\frac{8}{3}}, \ldots\ldots$$

or

$$\frac{n\pi\theta^2}{2} \leqq \frac{\pi}{2} n^{-\frac{1}{3}}, \quad \frac{n\pi}{4!}\theta^4 \leqq \frac{\pi}{4!} n^{-\frac{5}{3}}, \quad \ldots\ldots .$$

Accordingly on th arc BF,

$$\lim_{n\to\infty} \cos(n\pi\cos\theta) = \cos(n\pi). \quad \text{......(34)}$$

Therefore, on the arc BF, when n tends to infinity, we can keep the sign of $-2a_1 a_2 \varepsilon^{2X} \cos(n\pi\cos\theta)$ *always positive by taking n odd or even according as the product* $a_1 a_2$ *is positive or negative.* (35)

Hence on the arc BF, the relation

$$\lim_{n\to\infty} |\varepsilon^{2ql} - f| \geqq |a_1 a_2| \quad \text{......(36)}$$

is always possible, provided n or $|p|$ tends to infinity in the above mentioned manner. Then it comes about that, unless $b_1 = 0$ and $b_0 = \sqrt{\frac{L}{C}}$ simultaneously, $|\varepsilon^{2ql} - f|$ is always larger than a finite quantity not equal to zero. (See (29).) On the arc BF, the following relations hold, as $|p|$ or n tends to infinity:

$$\left.\begin{aligned} &|\varepsilon^{pt}| \leqq \varepsilon^{ht}, \\ &|\varepsilon^{q(2l-x)}| \leqq \varepsilon^{\sqrt{LC}(h+\rho)(2l-x)}, \\ &\left|\frac{1-Z_2/z}{1+Z_2/z}\varepsilon^{qx}\right| \leqq |a_2|\, \varepsilon^{\sqrt{LC}(h+\rho)x}. \end{aligned}\right\} \quad \text{......(37)}$$

And from (18), for sufficiently large $|p|$ or n,

$$\begin{aligned} f_1 &= 1/(1+Z_1/z) \\ &\cong 1/(C_1 p + C_2), \quad \text{......(38)} \end{aligned}$$

where C_1 and C_2 are certain finite quantities that are independent of p, and never equal to zero simultaneously. ...(39)

Hence if n be chosen so that it may satisfy the condition given by (35), we have the following relations from equations (16), (19), (36), (37) and (38).

$$\lim_{n\to\infty}\left|\int_{BF} V\varepsilon^{pt}dp\right| \leqq E\lim_{n\to\infty}\int_{BF}\left|\frac{p}{p+p_0}\right|\frac{|\varepsilon^{q(2l-x)}|+|f_2|\,|\varepsilon^{qx}|}{|\varepsilon^{2ql}-f|}\,|f_1|\,|\varepsilon^{pt}|\left|\frac{dp}{p}\right|$$

$$\leqq E\lim_{n\to\infty}\int_{-\delta_0}^{\delta}\frac{\varepsilon^{\sqrt{LC}(h+\rho)(2l-x)}+|a_2|\,\varepsilon^{\sqrt{LC}(h+\rho)x}}{|a_1 a_2|\,|C_1p+C_2|}\,\varepsilon^{ht}\,d\theta.$$

Therefore no matter whether C_1 is zero or not, it follows that

$$\lim_{n\to\infty}\left|\int_{BF} V\varepsilon^{pt}dp\right| = 0. \quad\text{.........(40)}$$

Just in the same way as above developed, we can prove that

$$\lim_{|p|\to\infty}\left|\int_{F'E} V\varepsilon^{pt}dp\right| = 0, \quad\text{.........(41)}$$

where OF' subtends the same angle δ with the imaginary axis downwards, and n is also so chosen that it may satisfy the condition of (35).

Next at the points F and F', (see fig. 2),

$$\left.\begin{aligned}\lim_{n\to\infty}|\varepsilon^{pt}| &= \lim_{n\to\infty}\exp\left[-\frac{1}{l\sqrt{LC}}\,\frac{n\pi}{2}\sin(n^{-\frac{2}{3}})t\right]\\ &= \exp\left[-\frac{\pi}{l\sqrt{LC}}\,\frac{n^{\frac{1}{3}}}{2}t\right],\\ \lim_{n\to\infty}|\varepsilon^{qx}| &= \exp\left[\left(-\frac{\pi n^{\frac{1}{3}}}{2l}+\rho\sqrt{LC}\right)x\right]\text{etc},\end{aligned}\right\}\quad\text{.........(42)}$$

and since the absolute values of ε^{pt}, ε^{qx} etc. on the arc FDF' are smaller than or equal to those at the points F and F' when n tends to infinity, we can deduce the following relation from equations (42),

$$\lim_{n\to\infty}\left|\int_{FDF'} V\varepsilon^{pt}dp\right| \leqq \lim_{n\to\infty}\int_{FDF'} E\left|\frac{p}{p+p_0}\right|\frac{|\varepsilon^{q(2l-x)}|+|f_2|\,|\varepsilon^{qx}|}{|\varepsilon^{2ql}-f|}\,|f_1|\,|\varepsilon^{pt}|\,dp$$

$$\leqq \lim_{n\to\infty}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} E\,\frac{1+\left|\frac{1}{a_2}\right|}{|a_1|(|C_1 p|\pm|C_2|)}\exp\left[-\frac{\pi(n)^{\frac{1}{3}}}{2l}\left(x+\frac{t}{\sqrt{LC}}\right)+\rho\sqrt{LCx}\right]\times d\theta,$$

since $2l-x\geqq x$. The double sign of the demominator is taken so that it is plus or minus according as C_1 is zero or not. Therefore we have

$$\lim_{n\to\infty}\left|\int_{FDF'} V\varepsilon^{pt}dp=0.\right. \qquad\text{......(43)}$$

Equation (43) *holds for all values of x and t including the case where x and t are simultaneously equal to zero if* $C_1\neq 0$, *and excluding the same case if* $C_1=0$. ......(44)

Hence under the restriction of (44), we have

$$\lim_{n\to\infty}\left|\int_{FDF'} V\varepsilon^{pt}dp\right|=0. \qquad\text{......(45)}$$

Consequently, by (40), (41) and (45), we have

$$\lim_{n\to\infty}\int_{BFDF'E} V\varepsilon^{pt}dp=0. \qquad\text{......(46)}$$

By a process similar to that above developed, we can also prove that

$$\lim_{n\to\infty}\int_{BFDF'E} I\varepsilon^{pt}dp=0. \qquad\text{......(47)}$$

Therefore the equations (6) are reduced to the following forms:

$$\left.\begin{aligned} v&=\frac{1}{2\pi j}\lim_{n\to\infty}\int_{EAB} V\varepsilon^{pt}dp,\\ i&=\frac{1}{2\pi j}\lim_{n\to\infty}\int_{EAB} I\varepsilon^{pt}dp,\end{aligned}\right\} \qquad\text{......(48)}$$

where n is chosen in the manner stated in (35).

Substituting (15) in (48), we get finally,

$$\left.\begin{aligned} v&=\frac{E}{2\pi j}\lim_{n\to\infty}\int_{EAB}\frac{1}{p+p_0}\,\frac{\varepsilon^{-qx}-f_2\,\varepsilon^{q(-2l+x)}}{1-f\varepsilon^{-2ql}}\,f_1\varepsilon^{pt}dp,\\ i&=\frac{E}{2\pi j}\lim_{n\to\infty}\int_{EAB}\frac{1}{(p+p_0)\,z}\,\frac{\varepsilon^{-qx}+f_2\,\varepsilon^{q-(2l+x)}}{1-f\varepsilon^{-2ql}}\,f_1\varepsilon^{pt}dp.\end{aligned}\right\} \qquad\text{......(49)}$$

On the other hand, we have

$$\lim_{n\to\infty} |f| = |a_1 a_2| \leqq 1 \quad \text{...(50)}$$

from (28), and

$$\lim_{n\to\infty} |\varepsilon^{-2ql}| \leqq \varepsilon^{-2(h+\rho)\sqrt{LC}\,l} < 1 \quad \text{.................................(51)}$$

everywhere on the arc EAB. Hence the expansion of $(1-f\varepsilon^{-2ql})^{-1}$ in ascending powers of $f\varepsilon^{-2ql}$, i.e.

$$1+f\varepsilon^{-2ql}+f^2\varepsilon^{-4ql}+\cdots\cdots \quad \text{...............................(52)}$$

converges uniformly on the arc EAB when n approaches to infinity.

Replace the common factor $(1-f\varepsilon^{-2ql})^{-1}$ of v and i in equations (49) by the above power series, then it is possible to integrate the results term by term. Thus we obtain

$$\left.\begin{aligned} v &= \frac{1}{2\pi j}\lim_{n\to\infty}\int_{EAB}\frac{\varepsilon^{pt}}{p+p_0}f_1\left(\varepsilon^{-qx}-f_2\varepsilon^{q(2l+x)}\right)\left(1+f\varepsilon^{-2ql}+f^2\varepsilon^{-4ql}+\cdots\right)dp, \\ i &= \frac{1}{2\pi j}\lim_{n\to\infty}\int_{EAB}\frac{\varepsilon^{pt}}{(p+p_0)z}f_1\left(\varepsilon^{-qx}+f_2\varepsilon^{q(-2l+x)}\right)\left(1+f\varepsilon^{-2ql}+f^2\varepsilon^{-4ql}+\cdots\right)dp. \end{aligned}\right\}(53)$$

The general terms that appear in equations (53) are

$$\left.\begin{aligned} &\lim_{n\to\infty}\int_{EAB} V_{m1}\,dp \quad \text{or} \quad \lim_{n\to\infty}\int_{EAB} V_{m2}\,dp \quad \text{for } v, \\ &\text{and} \\ &\lim_{n\to\infty}\int_{EAB}\frac{V_{m1}}{z}\,dp \quad \text{or} \quad \lim_{n\to\infty}\int_{EAB}\frac{V_{m2}}{z}\,dp \quad \text{for } i, \end{aligned}\right\}\text{...............(53)a}$$

where

$$\left.\begin{aligned} V_{m1} &= \frac{f_1}{p+p_0}f^m\varepsilon^{-(2ml+x)q+pt}, \\ V_{m2} &= \frac{f_1 f_2}{p+p_0}f^m\varepsilon^{-(\overline{2m+1}l-x)q+pt}, \\ m &= 0,\ 1,\ 2,\ 3,\ldots\ldots\ldots\ldots . \end{aligned}\right\}\text{...............................(54)}$$

From (23)

$$\lim_{n\to\infty} q=\sqrt{LC}(p+\rho).$$

Hence along the arc EAB, we have, in the limiting case where n tends to infinity,

$$\left.\begin{aligned}&-(2ml+x)q+pt=p(t-\overline{2ml+x}\sqrt{LC})-\rho(2ml+x)\sqrt{LC},\\&-(2ml-x)q+pt=p\left\{t-(\overline{2m+1}l-x)\sqrt{LC}\right\}-\rho(2ml-x)\sqrt{LC},\end{aligned}\right\}\quad\ldots(55)$$

and f, f_1, f_2 and z approach to $a_1 a_2$, $(C_1 p+C_2)^{-1}$, a_2 and $\sqrt{\frac{L}{C}}$ respectively as n increases without limit. (See (29), (38), (19) and (9).)

Therefore we can easily prove that

$$\left.\begin{aligned}&\lim_{n\to\infty}\int_{EAB} V_{m1}dp=0,\\&\lim_{n\to\infty}\int_{EAB}\frac{V_{m1}}{z}dp=0,\end{aligned}\right\}\text{ for } t-\overline{2ml+x}\sqrt{LC}<0\ \ldots\ldots\ldots\ldots\ldots(56)$$

and

$$\left.\begin{aligned}&\lim_{n\to\infty}\int_{EAB} V_{m2}dp=0,\\&\lim_{n\to\infty}\int_{EAB}\frac{V_{m2}}{z}dp=0,\end{aligned}\right\}\text{ for } t-(\overline{2m+1}\,l-x)\sqrt{LC}<0,\ \ldots\ldots(57)$$

where

$$m=0,\ 1,\ 2,\ 3,\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots .$$

The line integrals $\int V_{m1}dp$ and $\int (V_{m1}/z)\,dp$ when $t-(2ml+x)\sqrt{LC}>0$, and those $\int V_{m2}\,dp$ and $\int (V_{m2}/z)\,dp$ when $t-\left\{2(m+1)l-x\right\}\sqrt{LC}>0$, as n tends to infinity, approach to zero when the path is along EAB. Hence if we denote these integrals by the symbol $\int$ we have, in this case,

$$\lim_{n\to\infty}\int_{EAB}=\lim_{n\to\infty}\int_{EAB}+\lim_{n\to\infty}\int_{BDE}=\lim_{n\to\infty}\int_{ABDEA}.\ \ldots\ldots\ldots\ldots\ldots\ldots(58)$$

On the other hand, the singular points of V_{m1}, V_{m2}, V_{m1}/z and V_{m2}/z are of finite number, and exist in the finite domain, and moreover, these functions V_{m1} etc. are holomorphic functions of p in the exterior domain of any closed curve including these singular points, since they consist of $q=\sqrt{(Lp+R)(Cp+G)}$, $z=\sqrt{(Lp+R)/(Cp+G)}$ and rational functions of p.

Hence we can replace the path $ABDEA$ $(n\to\infty)$ of (58) by any closed curve K involving all the singular points of V_{m1} etc..

Therefore we are led finally to the following results:

$$\left.\begin{aligned}
\lim_{n\to\infty}\int_{EAB}(V_{m1})\,dp&=0 && \text{for} \quad t-(2ml+x)/g<0,\\
&=\int_{(K)}(V_{m1})\,dp && \text{for} \quad t-(2ml+x)/g>0,\\
\lim_{n\to\infty}\int_{EAB}(V_{m2})\,dp&=0 && \text{for} \quad t-\left\{2(m+1)l-x\right\}/g<0,\\
&=\int_{(K)}(V_{m2})\,dp && \text{for} \quad t-\left\{2(m+1)l-x\right\}/g>0,
\end{aligned}\right\} \quad \cdots\cdots(59)$$

where

$$\left.\begin{aligned}
&m=0,\ 1,\ 2,\ 3,\ \cdots\cdots\cdots\cdots,\\
&g=(LC)^{-\frac{1}{2}},\\
&(V_{m1})=V_{m1} \quad \text{or} \quad V_{m1}/g,\\
&(V_{m2})=V_{m2} \quad \text{or} \quad V_{m2}/g.
\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(60)$$

Combining (53) with (59), we see that equations (1) and (2) hold.

From (1)

$$v=0$$

$$i=0 \quad \text{for} \quad t<x/g.$$

Hence we know that the potential and current become zero at $t=0$ along the whole line except at the point $x=0$, when $C_1=0$. This coincides with the actual initial conditions of the line. On the other hand it is evident from the process of deduction that results (1) and (2) satisfy our fundamental equations (4) and (11). Therefore the formulas (1) and (2) are what we require.

3. SOME FEATURES OF PHENOMENA DIRECTLY ESTIMATED FROM THE FUNDAMENTAL FORMULAS.

Before proceeding further, it will be interesting to discuss the brief features of the propagation phenomena in the light of the fundamental equations (1) and (2) for the potential and current.

Substituting 0, 1, 2 etc. in m of the equations (1), we get

$$v=0 \quad \text{for} \quad t<x/g, \qquad\qquad (61)$$

$$v=v_0 \quad \text{tor} \quad (2l-x)/g>t>x/g, \qquad\qquad (62)$$

$$v=v_0+v_1 \quad \text{for} \quad (2l+x)/g>t>(2l-x)/g, \qquad\qquad (63)$$

$$v=v_0+v_1+v_2 \quad \text{for} \quad (4l-x)/g>t>(2l+x)/g \qquad\qquad (64)$$

etc. .

We observe, first, from (61) and (62), that we have a true finite velocity of propagations $g=(LC)^{-\frac{1}{2}}$. No matter what the form of the impressed e.m.f. is at the beginning of the line ($x=0$), its effect does not reach the point x in the line until the time $t=x/g$ has elapsed. Consequently $g=x/t$ is the velocity with which the wave is propagaled. This is the strict consequence of the distributed inductance and capacity of the line, and the volocity depends upon these constants only, since $g=(LC)^{-\frac{1}{2}}$.

The second term v_1 in equation (63) is the reflected wave of v_0 from the other terminal ($x=l$) due to the terminal irregularity which exists there, since $t=(2l-x)/g$ is the time reqnired for an effect applied at the beginning of the line ($x=0$) to reach the point x after having been reflected at $x=l$. And similarly it follows that the third term v_2 in equation (64) is the reflected wave from the sending terminal, etc. .

Hereafter we shall call v_0 the original potential wave and v_1 and v_2 the first and second reflected potential waves, and so on; and similarly with with the current waves.

If the line is infinitely long, we can always keep the relation $(2l-x/g)>t$, however great t may be. Hence the potential at the point x is given by equations (61) and (62) solely, which are potential equations for a semi-infinitely long transmission line.

Similar conclusions may be derived for the current wave from equations (1).

4. THE FIRST METHOD OF EVALUATING v AND i GIVEN BY (1) AND (2).

The forms of potential and current given by (1) and (2) are not convenient for practical use as they stand, since they consist of contour integrals taken in a complex plane. It is necessary to express them by some known functions.

The general terms in the expressions of v and i are, by (2),

$$\frac{E}{2\pi j}\int_{(K)} \frac{\varepsilon^{pt}\varepsilon^{-qx}}{p+p_0} f^m f_1 (f_2)\, dp, \quad\ldots\ldots\ldots\ldots(65)$$

and

$$\frac{E}{2\pi j}\int_{(K)} \frac{\varepsilon^{pt}\varepsilon^{-qx}}{(p+p_0)\, z} f^m f_1 (f_2)\, dp, \quad\ldots\ldots\ldots\ldots(66)$$

where x stands for $(2ml+x)$ and $2(m+1)l-x$ of (2), and

$$(f_2)=f_2 \text{ or } 1. \quad\ldots\ldots\ldots\ldots(67)$$

Since f, f_1 and f_2, being given by equations (3), may be converted into $(z-Z_1)^2(z-Z_2)^2/[(z^2-Z_1^2)(z^2-Z_2^2)]$, $(z-Z_1)/(z^2-Z_1^2)$ and $(z-Z_2)^2/(z^2-Z_2^2)$ respectively, and Z_1, Z_2 and z^2 are rational functions of p, all the functions f, f_1 and f_2 may be written in the form of $T(p)+zU(p)$, where $T(p)$ and $U(p)$ are certain rational functions of p. Moreover if we assume that $\lim_{|p|\to\infty} (Z_1 \text{ or } Z_2) \neq \sqrt{\frac{L}{C}}$, then the functions $T(p)$ and $U(p)$ must be of the form $a_0+\sum_{\nu=1}^{s}\sum_{\lambda=1}^{n}\frac{a_{\nu\lambda}}{(p+p_\nu)^\lambda}$, where p_ν's are the poles of T and U. And consequently the function $\frac{1}{p+p_0} f_m f_1 (f_2)$ may be expressed by

$$\sum_{k=0}^{m}\sum_{\tau=1}^{h}\frac{a_{k\tau}}{(p+p_k)^\tau}+\frac{1}{z}\sum_{\nu=0}^{m'}\sum_{\lambda=1}^{h'}\frac{b_{\nu\lambda}}{(p+p_\nu)^\lambda}.$$

Hence to express v and i of (1) and (2) by certain special functions, it is enough to evaluate two contour integrals $\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}\varepsilon^{-qx}}{(p+p_0)^n}\,dp$ and $\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}\varepsilon^{-qx}}{z(p+p_0)^n}\,dp$ for $n=1, 2, 3,\ldots\ldots\ldots$.

We shall evaluate these contour integrals in the present section.

In the computation of the contour integrals, we assume that

1) *the determination of the radical sign is plus when the argument of p is zero, and* 2) *the path of integration contains all the singular points of the integrand.*

We shall, first, consider the following integrals :

$$T_1=\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{p+G/C}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp, \quad\text{.........(68)}$$

where R, C, G and L are all positive quantities and

$$q=\sqrt{(R+Lp)(G+Cp)}. \quad\text{.........(69)}$$

Putting

$$p'=p+\rho, \quad\text{.........(70)}$$

and

$$\rho=\frac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right), \quad\text{.........(71)}$$

we have

$$\left.\begin{aligned}&Lp+R=L(p'\pm\sigma),\\&Cp+G=C(p'\mp\sigma),\\&q=\sqrt{LC}\sqrt{(p'+\sigma)(p'-\sigma)},\\&\sigma=\frac{1}{2}\left(\frac{R}{L}\sim\frac{G}{C}\right)>0.\end{aligned}\right\} \quad\text{.........(72)}$$

Then the integral T_1 becomes

$$T_1=\frac{1}{2\pi j}\sqrt{\frac{C}{L}}\int_{(K')}\frac{\varepsilon^{-\sqrt{LC}\sqrt{(p'+\sigma)(p'-\sigma)}\,x+p't-\rho t}}{\sqrt{(p'+\sigma)(p'-\sigma)}}\,dp'. \quad\text{.........(73)}$$

Again introducing a new variable ζ such that

$$q=\sqrt{LC}(p'-\zeta), \quad\text{.........(74)}$$

we obtain by the aid of (72)

$$p'=\frac{\zeta^2+\sigma^2}{2\zeta}, \quad\text{.........(75)}$$

$$q=\frac{\sigma^2-\zeta^2}{2g\zeta}, \quad\text{where } g=(LC)^{-\frac{1}{2}}, \quad\text{.........(76)}$$

$$dp'=\frac{\zeta^2-\sigma^2}{2\zeta^2}\,d\zeta. \quad\text{.........(77)}$$

Putting

$$\zeta=r\varepsilon^{j\theta}, \quad\text{.........(78)}$$

$$p'=X+jY, \quad\text{.........(79)}$$

we have

$$p'=\frac{1}{2}\left\{\left(r+\frac{\sigma^2}{r}\right)\cos\theta+j\left(r-\frac{\sigma^2}{r}\right)\sin\theta\right\}, \quad \ldots\ldots\ldots\ldots (80)$$

$$q=\frac{\sqrt{LC}}{2}\left\{-\left(r-\frac{\sigma^2}{r}\right)\cos\theta-j\left(r+\frac{\sigma^2}{r}\right)\sin\theta\right\}, \ldots\ldots\ldots (81)$$

and

$$\left.\begin{aligned} X&=\frac{1}{2}\left(r+\frac{\sigma^2}{r}\right)\cos\theta, \\ Y&=\frac{1}{2}\left(r-\frac{\sigma^2}{r}\right)\sin\theta. \end{aligned}\right\} \ldots\ldots\ldots\ldots (82)$$

From (78) we see that, for a constant value of r, ζ deseribes a system of concentric cireles with centre at the origin. The corresponding images in the p'-plane may be obtained by eliminating θ from the equations (82). The result is

$$\frac{4X^2}{\left(r+\frac{\sigma^2}{r}\right)^2}+\frac{4Y^2}{\left(r-\frac{\sigma^2}{r}\right)^2}=1. \ldots\ldots\ldots\ldots (83)$$

This equation represent a system of confocal ellipses with $\pm\sigma$ as the common foci. In the special case where ζ describes a circle of radius σ, viz., when

$$r\sim\frac{\sigma^2}{r}=0,$$

the image in the p'-plane becomes a segment between the common foci. Therefore as long as

Fig. 3.

ζ—plane

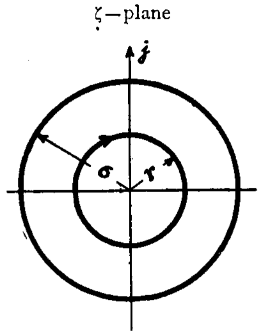
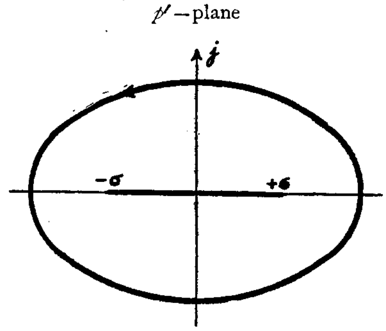


Fig. 4.

p'—plane



$$r \neq \sigma, \quad \text{...} (84)$$

the points $p' = \pm\sigma$ are fully in the interior of the elliptic path, the image in the p'-plane of the circle of radius r described by ζ. Here we remark that the points $p' = \pm\sigma$ are the singular points of the integrand of equation (73).

When the argument of p is zero, p' must be positive, for by (70) we have $p' = p + \rho$, and $\rho \geq 0$. In order that p' may be positive, θ must be equal to $2n\pi$, by (80), where n is any integer. Hence we obtain $\theta = 2n\pi$ when the argument of p is zero. On the other hand, by assumption 1), q must be positive when the argument of p is zero. Therefore q must be also positive when $\theta = 2n\pi$. Accordingly substitute $\theta = 2n\pi$ in (81), then the value of q obtained thereby should be positive. Thus we have the following relation :—

$$-\left(r - \frac{\sigma^2}{r}\right) > 0,$$

or

$$r < \sigma. \quad \text{...} (85)$$

Since condition (84) is satisfied by (85), we may say that relation (85) is the necessary and sufficient condition for r in order to satisfy assumstion 1) as well as our requirement that the points $\pm\sigma$ should be contained in the domain bounded by the ellipse given by (83). Hence if the ellipse shown by (83) be taken as the path of integration of T_1 in equation (73), r must be taken smaller than σ, or in other words, if p' be transformed into ζ, the path of integration of T_1, taken in the ζ-plane, should be a circle having a radius smaller than σ.

It follows from equation (80) that, if ζ describes a circle with radius r in the negative sense, then p' describes an elliptic path in the positive sense so long as relation (85) is satisfied. Hence the integral obtained by substituting (75), (76) and (78) in equation (73) should be integrated in the negative sense along any circular path K_ζ with radius smaller than σ. Hence we have

$$T_1=\sqrt{\frac{C}{L}}\,\varepsilon^{-pt}\frac{1}{2\pi j}\int_{(K_\zeta-)} d\zeta\,\frac{2\zeta}{\sigma^2-\zeta^2}\,\frac{\zeta^2-\sigma^2}{2\zeta^2}\,exp\left\{-\frac{\sigma^2-\zeta^2}{2\zeta}\,\frac{x}{g}+\frac{\zeta^2+\sigma^2}{2\zeta}\,t\right\}$$

$$=\sqrt{\frac{C}{L}}\,\varepsilon^{-pt}\frac{1}{2\pi j}\int^{(0+)}\frac{d\zeta}{\zeta}\,exp\left[\frac{1}{2}\left\{\left(t+\frac{x}{g}\right)\zeta+\left(t-\frac{x}{g}\right)\frac{\sigma^2}{\zeta}\right\}\right], \quad\ldots\ldots\ldots(86)$$

where $(K_\zeta-)$ denotes that the integration should be done in the negative sense along the closed path K_ζ.

Putting

$$u=\left(t+\frac{x}{g}\right)\zeta/2,$$

we have finally

$$T_1=\sqrt{\frac{C}{L}}\,\varepsilon^{-pt}\frac{1}{2\pi j}\int^{(0+)}\frac{du}{u}\,exp\left(u+\frac{t^2-x^2/g^2}{4u}\,\sigma^2\right), \quad\ldots\ldots\ldots\ldots\ldots\ldots(87)$$

because t, x and g are all positive quantities. And since

$$\frac{1}{2\pi j}\int^{(0+)}\frac{du}{u}\,exp\left(u+\frac{t^2-x^2/g^2}{4u}\,\sigma^2\right)=I_0\left(\sigma\sqrt{t^2-x^2/g^2}\right),[1] \quad\ldots\ldots\ldots\ldots(88)$$

where I_0 is Bessel function of the first kind and of order zero with imaginary argument, we have

$$\left.\begin{aligned}T_1&=\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{p+G/C}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp\\&=\sqrt{\frac{C}{L}}\,\varepsilon^{-pt}\,I_0\left(\sigma\sqrt{t^2-x^2/g^2}\right).\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(89)$$

Next we shall consider the following integral:

$$T_2=\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{p+p_0}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(90)$$

---

1. In general we have (Whittaker and Watson: Modern Analysis p. 355,)

$$J_n(z)=\frac{1}{2\pi j}\left(\frac{1}{2}z\right)^n\int^{(0+)}t^{-n-1}exp\left(t-\frac{z^2}{4t}\right)dt.$$

Putting $n=0$ and $z=ja$, we obtain

$$I_0(a)=\frac{1}{2\pi j}\int^{(0+)}t^{-1}e\cdot p\left(t+\frac{a^2}{4t}\right)dt.$$

By a slight change we have

$$T_2 = T_1 + (G/C - p_0)\int_{(K)} \frac{\varepsilon^{-qx+pt}}{(p+p_0)(p+G/C)}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}} dp. \quad \text{...........(91)}$$

Since the integrand of equation (89) is a continuous function of p and t along the path of integration (K), we may integrate both its sides with respect to t after multiplying them by $\varepsilon^{p_0 t}$. Thus we have

$$\frac{1}{2\pi j}\int_{(K)} \frac{\varepsilon^{-qx+(p+p_0)t}}{(p+p_0)(p+G/C)}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}} dp$$

$$-\frac{1}{2\pi j}\int_{(K)} \frac{\varepsilon^{-qx+(p+p_0)x/g}}{(p+p_0)(p+G/C)}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}} dp$$

$$= \sqrt{\frac{C}{L}}\int_{x/g}^{t} \varepsilon^{(p_0-\rho)t} I_0(\sigma\sqrt{t^2 - x^2/g^2})\, dt, \quad \text{...........................(92)}$$

or

$$\frac{1}{2\pi j}\int_{(K)} \frac{\varepsilon^{-qx+pt}}{(p+p_0)(p+G/C)}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}} dp$$

$$= \sqrt{\frac{C}{L}}\,\varepsilon^{-p_0 t}\int_{x/g}^{t} \varepsilon^{(p_0-\rho)t} I_0(\sigma\sqrt{t^2 - x^2/g^2})\, dt + \varepsilon^{-p_0(t-x/g)}\, T_2', \quad \text{.........(93)}$$

where

$$T_2' = \frac{1}{2\pi j}\int_{(C)} \frac{\varepsilon^{-qx+px/g}}{(p+p_0)(p+G/C)}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}} dp. \quad \text{.........................(94)}$$

Evidently in the finite exterior domain of the path (K), the integrand of T_2' is holomorphic; hence the value of T_2' remains unchanged when the path is replaced by a circle of infinitely increasing radius with its centre at the origin. Denoting such a circular path by K_k, we have

$$2\pi\,|\,T_2'\,| = \lim_{|p|\to\infty}\left|\int_{(K_k)} \frac{\varepsilon^{-qx+px/g}}{(p+p_0)(p+G/C)}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}} dp\right|$$

$$\leqq \lim_{|p|\to\infty}\int_{(K_k)} \frac{|\,\varepsilon^{-qx+px/g}\,|}{|\,p+p_0\,|\,|\,p+G/C\,|}\left|\frac{G+Cp}{R+Lp}\right|^{\frac{1}{2}} |\,dp\,|$$

$$= \lim_{|p|\to\infty}\frac{1}{|\,p\,|}\int_{(K_k)} \frac{|\,\varepsilon^{-qx+px/g}\,|}{|\,1+p_0/p\,|\,|\,1+(G/C)/p\,|}\left|\frac{C+G/p}{L+R/p}\right|^{\frac{1}{2}} \frac{dp}{p}. \quad \text{...(95)}$$

On the other hand, for a sufficiently great value of $|p|$, we have, by assumption 1),

$$-qx+px/g=\frac{-(RC+GL)-RG/p}{\sqrt{\{1+R/(Lp)\}\{1+G/(Cp)\}}+1}\frac{x}{\sqrt{LC}}.$$

Hence

$$\lim_{|p|\to\infty}(-qx+px/g)=-\frac{RC+GL}{2\sqrt{LC}}x. \qquad (96)$$

Therefore we have

$$2\pi\,|\,T_2'\,|\leqq\lim_{|p|\to\infty}\frac{1}{|p|}\int_0^{2\pi}\varepsilon^{-\frac{RC+GL}{2\sqrt{LC}}x}\cdot\sqrt{\frac{C}{L}}\cdot d\theta=0,$$

or

$$T_2'=0. \qquad (97)$$

Substituting (97) in (93), we get

$$\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{(p+p_0)(p+G/C)}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp$$

$$=\sqrt{\frac{C}{L}}\,\varepsilon^{-p_0t}\int_{x/g}^{t}\varepsilon^{(p_0-\rho)t}I_0(\sigma\sqrt{t^2-x^2/g^2})\,dt. \qquad (98)$$

It follows from (91) and (98) that

$$T_2=\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{p+p_0}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp$$

$$=\sqrt{\frac{C}{L}}\,\varepsilon^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/g^2})$$

$$+\left(\frac{G}{C}-p_0\right)\sqrt{\frac{C}{L}}\,\varepsilon^{-p_0t}\int_{x/g}^{t}\varepsilon^{(p_0-\rho)t}I_0(\sigma\sqrt{t^2-x^2/g^2})\,dt. \qquad (99)$$

Since the integrand on the left hand side of equation (98) and its derivatives are continuous with respect to x and p on the path of integration (K), and the term on the right hand side is also continuous with respect to x, we may differentiate both sides with respect to x. Such differentiation gives the following :—

$$T_3=\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{p+p_0}\,dp$$

$$=\varepsilon^{-p_0t+(p_0-\rho)x/g}+\frac{\sigma x}{g}\,\varepsilon^{-p_0t}\int_{x/g}^{t}\varepsilon^{(p_0-\rho)t}\,\frac{I_1(\sigma\sqrt{t^2-x^2/g^2})}{\sqrt{t^2-x^2/g^2}}\,dt,\quad\ldots\ldots\ldots\ldots(100)$$

where I_1 is Bessel function of the first kind and of unity order with imaginary argument.

Equation (99) may be written as follows:

$$T_2=\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{p+p_0}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp$$

$$=\sqrt{\frac{C}{L}}\,\varepsilon^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/g^2})+\left(\frac{G}{C}-p_0\right)\sqrt{\frac{C}{L}}\int_{x/g}^{t}\varepsilon^{p_0(\tau-t)-\rho\tau}I_0(\sigma\sqrt{\tau^2-x^2/g^2})\,d\tau.$$

Since the integrand on the left hand side of the above equation and its derivatives with respect to p_0 are continuous functions of p_0 and p along its path of integration, and the terms on the right hand side are also continuous in regard to p_0, we differentiate both sides $(n-1)$ times with respect to p_0, and we obtain the following result:

$$T_4=\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{(p+p_0)^n}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp$$

$$=\frac{1}{(n-1)!}\left\{\left(\frac{G}{C}-p_0\right)\sqrt{\frac{C}{L}}\,\varepsilon^{-p_0t}\int_{x/g}^{t}(t-\tau)^{n-1}\,\varepsilon^{(p_0-\rho)\tau}\,I_0(\sigma\sqrt{\tau^2-x^2/g^2})\,d\tau\right.$$

$$\left.+\sqrt{\frac{C}{L}}\,(n-1)\,\varepsilon^{-p_0t}\int_{x/g}^{t}(t-\tau)^{n-2}\,\varepsilon^{(p_0-\rho)\tau}\,I_0(\sigma\sqrt{\tau^2-x^2/g^2})\,d\tau\right\}\ldots(101)$$

for $n\geq 2$.

The integrand on the left hand side and its derivatives on the path of integration, and the terms on the right hand side are continuous functions of p and p_0. Hence, differentiating both sides $(n-1)$ times with respect to p_0 we get,

$$T_5=\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{(p+p_0)^n}\,dp$$

$$=\frac{1}{(n-1)!}\left\{(t-x/g)^{n-1}\,\varepsilon^{-p_0t+(p_0-\rho)x/g}\right.$$

$$\left.+\frac{\sigma x}{g}\,\varepsilon^{-p_0t}\int_{x/g}^{t}(t-\tau)^{n-1}\,\varepsilon^{(p_0-\rho)\tau}\,\frac{I_1(\sigma\sqrt{\tau^2-x^2/g^2})}{\sqrt{\tau^2-x^2/g^2}}\,d\tau\right\}.\quad\ldots\ldots\ldots\ldots\ldots(102)$$

The results obtained above may be summarized as follows:—

$$\left.\begin{aligned}
&\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qn+pt}}{(p+p_0)^n}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp \\
&=\sqrt{\frac{C}{L}}\varepsilon^{-pt}I_0(\sigma\sqrt{t^2-x^2/g^2})+\left(\frac{G}{C}-p_0\right)\sqrt{\frac{C}{L}}\varepsilon^{-p_0t}\int_{x/g}^{t}\varepsilon^{(p_0-p)t}I_0(\sigma\sqrt{t^2-x^2/g^2})dt \\
&\text{for } n=1, \\
&\text{,,}=\frac{1}{(n-1)!}\sqrt{\frac{C}{L}}\varepsilon^{-p_0t}\left\{\left(\frac{G}{C}-p_0\right)\int_{x/g}^{t}(t-\tau)^{n-1}\varepsilon^{(p_0-p)\tau}I_0(\sigma\sqrt{\tau^2-x^2/g^2})d\tau\right. \\
&\qquad\left.+(n-1)\int_{x/g}^{t}(t-\tau)^{n-2}\varepsilon^{(p_0-p)\tau}I_0(\sigma\sqrt{\tau^2-x^2/g^2})d\tau\right\} \\
&\text{for } n\geqq 2,
\end{aligned}\right\}\quad(103)$$

and

$$\left.\begin{aligned}
&\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{(p+p_0)^n}dp \\
&=\frac{1}{(n-1)!}\left\{(t-x/g)^{n-1}\varepsilon^{-p_0t+(p_0-p)x/g}\right. \\
&\qquad\left.+\frac{\sigma x}{g}\varepsilon^{-p_0t}\int_{x/g}^{t}(t-\tau)^{n-1}\varepsilon^{(p_0-p)\tau}\frac{I_1(\sigma\sqrt{\tau^2-x^2/g^2})}{\sqrt{\tau^2-x^2/g^2}}d\tau\right\}
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(104)$$

for $n\geqq 1$, where we assume that $0!=1$.

The formulas (103) and (104) are what are required and will convert v and i given by (1) and (2) into elementary functions and integrals involving them. The forms of T_2, T_3, T_4 and T_5 are easily computable with the aid of a planimeter or integraph, or by numerical integration, since $I_0(x)$, $I_1(x)$ and ε^x are all tabulated with respect to x, although it is impossible to express these integrals by finite terms of elementary functions.

5. THE SECOND METHOD OF EVALUATING v AND i GIVEN BY (1) AND (2).

In the preceding section we referred to the contour integrals and to their evaluation, be means of which v and i in (1) and (2) can be brought to elementary known functions. But sometimes the above results of integration become practically inconvenient. For instance if p_0 be a real negative quantity, then the factor ε^{-p_0t} takes a great value, while $\varepsilon^{(p_0-\rho)t}I_0(\sigma\sqrt{t^2-x^2/g^2})$ retains a small magnitude, for suitably chosen t and x. Hence some errors cannot be avoided in the numerical calculation of $\int_{x/g}^{t}\varepsilon^{(p_0-\rho)t}I_0(\sigma\sqrt{t^2-x^2/g^2})$, and the amount of the errors will be magnified when multiplied by ε^{-p_0t}. In such a case another evaluation of v and i is necessary. In the present section we shall show another meth-d of evaluating v and i by an infinite series of I_n functions.

As discussed in the preceding section, the terms v_m and i_m, which compose v and i, are generally formed of

$$T_6=\frac{\varepsilon^{-pt}}{2\pi j}\int_{(K)}H(p,z)\,\varepsilon^{-qx+pt}\,dp, \quad\text{.................................(105)}$$

where H is a rational funtion of p and z.

Transform p to p' by relation (70), then we have

$$T_6=\frac{\varepsilon^{-pt}}{2\pi j}\int_{(K)}H\left\{(p'-\rho),\sqrt{\frac{L}{C}}\left(\frac{p'+\sigma}{p'-\sigma}\right)^{\frac{1}{2}}\right\}dp', \quad\text{....................(106)}$$

where we assume that

$$\sigma=\frac{1}{2}\left(\frac{R}{L}-\frac{G}{C}\right)>0.$$

The further transformation of p' to another new variable ζ, by relation (74), gives (75), (76) and (77). Substituting (75), (76) and (77) in (106), we get the following:

$$T_6=\frac{\varepsilon^{-pt}}{2\pi j}\int_{K_\zeta}H\left\{\frac{1}{2}\left(\zeta+\frac{\sigma^2}{\zeta}\right)-\rho,\ \sqrt{\frac{L}{C}}\,\frac{\sigma+\zeta}{\sigma-\zeta}\right\}\frac{\zeta^2-\sigma^2}{2\zeta^2}$$
$$exp\left[\frac{1}{2}\left\{\left(t+\frac{x}{g}\right)\zeta+\left(t-\frac{x}{g}\right)\frac{\sigma^2}{\zeta}\right\}\right]d\zeta, \quad\text{.................(107)}$$

where the path of integration K_ζ is, as discussed in the preceding section, a circle with radius smaller than σ and the center at the origin, and the integration should be done in the negative sense along this circle. The smaller the radius of the circle r compared with σ, the greater are the axes of the ellipse shown by (83), which is the path of integration on the

p'-plane of equation (105). Hence taking r sufficiently small, the singular points of the integrand in equation (106) are fully included in the elliptic path. Therefore we may take, as the path of integration K_ζ in (107), a circle of infinitesimal radius with centre at the origin. Thus we get from (107) the following:

$$T_6=\frac{\varepsilon^{-pt}}{2\pi j}\int^{(0+)} H\left\{\frac{1}{2}\left(\zeta+\frac{\sigma^2}{\zeta}\right)-\rho_1\sqrt{\frac{L}{C}}\cdot\frac{\sigma-\zeta}{\sigma-\zeta}\right\}\frac{\sigma^2-\zeta^2}{2\zeta^2}$$
$$exp\left[\frac{1}{2}\left\{\left(t+\frac{x}{g}\right)\zeta+\left(t-\frac{x}{g}\right)\frac{\sigma^2}{\zeta}\right\}\right]d\zeta. \quad\ldots\ldots\ldots\ldots (108)$$

Here H, a rational function of ζ, should have the form of

$$H=P_1(\zeta)+\frac{P_2(\zeta)}{P_3(\zeta)},$$

where P_1, P_2 and P_3 are polynomials of ζ, and the degree of P_2 with respect to ζ is lower than that of P_3.

Assume that $P_3(\zeta)=1+a_{n-1}\zeta+\cdots\cdots\cdots+a_0\zeta^n$, then we can expand $\frac{1}{P_3(\zeta)}$ in an ascending power series of $(a_{n-1}\zeta+\cdots\cdots\cdots+a_0\zeta^n)$, which converges uniformly on the circular path of integration of (108), since it is always possible, as previously disscussed, to choose the radius of the circle r so as to keep the relation of $|a_{n-1}\zeta+\cdots\cdots\cdots+a_0\zeta^n|<1$. Therefore $\frac{1}{P_3(\zeta)}$ is expansible in the ascending power of ζ on the above path.

Or resolving into partial fractions, we get

$$H=P_1+\sum_{n=1}^{s}\sum_{m=1}^{n}\frac{A_{mn}}{(\zeta_n-\zeta)^n},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (110)$$

where ζ_n is the n-ple root of $P_3(\zeta)=0$, and A_{mn} is a constant independent of ζ. Also in this case, the expansion of $\frac{1}{(\zeta_n-\zeta)^m}$ in the ascending power of $\frac{\zeta}{\zeta_n}$ on the path of integration of (108), is possible. Hence in the above two cases, H may be expanded finally on the path of integration of (108) in the following power series:—

$$H=\sum_{n=m}^{\infty}g_n\zeta^n.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (111)$$

Substituting this H in (108), we know that our question is reduced to a single problem, namely, how to calculate the contour integral

$$T_7=\frac{1}{2\pi j}\int^{(0+)}\zeta^{n-1}\,exp\left[\frac{1}{2}\left\{\left(t+\frac{x}{g}\right)\zeta+\left(t-\frac{x}{g}\right)\frac{\sigma^2}{\zeta}\right\}\right]d\zeta, \quad\ldots\ldots(112)$$

which, by transforming ζ to u by the relation $u=(t+x/g)\zeta/2$, can easily be evaluated as follows :

$$\left.\begin{aligned}T_7&=\frac{2^n}{(t+x/g)^n}\,\frac{1}{2\pi j}\int^{(0+)}u^{n-1}\,exp\left\{u+\frac{(t^2-x^2/g^2)\,\sigma^2}{4u}\right\}du\\&=\sigma^n\left(\frac{t-x/g}{t+x/g}\right)^{\frac{n}{2}}I_{\pm n}(\sigma\sqrt{t^2-x^2/g^2}),\end{aligned}\right\}\quad\ldots\ldots(113)$$ [1]

as t, g and x are positive quantities and $I_n=I_{-n}$.

Hence we know the following rule :—

(A) *To evaluate the contour integral of the form* (105), *transform* p *to* ζ *by the relations* (70) *and* (74), *then we get* (108). *Expand* H *of* (108) *in the ascending power of* ζ, *and substitute the result in* (108), *then the integrand of* (108) *becomes a power series of* ζ *which is integrated immediately term by term by the relation* (113). *Thus the required integral shown by* (105) *is evaluated in a series of* I_n *functions.*

The above is the same as the following, but the latter is sometimes more convenient than the former.

Transform ζ in (108) to $\sigma\zeta$, then we get

$$T_6=\frac{\varepsilon^{-\rho t}}{1\pi j}\int^{(0+)}H\left\{\frac{\sigma}{2}\left(\zeta+\frac{1}{\zeta}\right)-\rho,\ \sqrt{\frac{L}{C}}\,\frac{1+\zeta}{1-\zeta}\right\}\frac{\sigma(1-\zeta^2)}{2\zeta^2}$$
$$exp\left[\frac{\sigma}{2}\left\{\left(t+\frac{x}{g}\right)\zeta+\left(t-\frac{x}{g}\right)\frac{1}{\zeta}\right\}\right]d\zeta. \quad\ldots\ldots(114)$$

Also in this case, H is expansible in the ascending power of ζ on the path of integration of the above integral. Hence the question is reduced to the calculation of

$$T_8=\frac{1}{2\pi j}\int^{(0+)}\zeta^{n-1}\,exp\left[\frac{\sigma}{2}\left\{\left(t+\frac{x}{g}\right)\zeta+\left(p-\frac{x}{g}\right)\frac{1}{\zeta}\right\}\right]d\zeta, \quad\ldots\ldots\ldots\ldots(115)$$

---

1. See foot note (1) in the preceding section.

which is calculated as follows:

$$T_8=\left(\frac{t-x/g}{t+x/g}\right)^{\frac{n}{2}} I_{\pm n}(\sigma\sqrt{t^2-x^2/g^2}). \qquad\qquad (116)$$

Hence we get another rule for the integral of (105).

(B) *Transform* p *to* ζ *by the relation* $p=\frac{\sigma}{2}\left(\zeta+\frac{1}{\zeta}\right)-\rho$, *then the integral of* (105) *is transformed to* (114). *Expand* H *of* (114) *in the ascending power of* ζ, *then the integrand of* (114) *except the exponential function becomes a power series of* ζ, *which is integrated immediately term by term by* (116), *and thus we get the evalution of the required integral* (105).

Rules (A) and (B) and the contour integrals calculated in the preceding section may play important rôles in the investigation of electrical transient phenomena in a transmission line circuit, which shall be minutely discussed in the following sections, and these two rules must be interesting from the mathematical point of view, because they propose a method of expanding certain functions in series of I_n functions.

6. POTENTIAL AND CURRENT AT THE POINT x FOR THE INTERVAL $(2l-x)/g>t>0$ DUE TO THE E.M.F. $E\varepsilon^{-p_0t}$ AT $x=0$, WHEN THE IMPEDANCE Z_1 IS ZERO.

An e.m.f. is directly applied to the line at the terminal $x=0$, then we have $Z_1=0$ and $f_1=1$. Hence by (1) and (2), the current and potential at the point x, due to the e.m.f. $E\varepsilon^{-p_0t}$, are given by the following equations:

$$\left.\begin{aligned}
v&=0 \qquad \text{for} \quad t<x/g,\\
&=\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}\,\varepsilon^{-qx}}{p+p_0}\,dp \quad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g},\\
i&=0 \qquad \text{for} \quad t<x/g,\\
&=\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}\,\varepsilon^{-qx}}{p+p_0}\left(\frac{G+Cp}{R+Lp}\right)^{\frac{1}{2}}dp \quad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g}.
\end{aligned}\right\} \qquad (117)$$

Substituting (103) and (104) in the above equations, we obtain the required solutions in known functions. Thus we have

$$\left.\begin{aligned}
v&=0 \qquad \text{for} \quad t<x/g,\\
&=E\varepsilon^{-p_0t+(p_0-\rho)x/g}+E\frac{\sigma x}{g}\varepsilon^{-p_0t}\int_{x/g}^{t}\varepsilon^{(p_0-\rho)\tau}\frac{I_1(\sigma\sqrt{\tau^2-x^2/g^2})}{\sqrt{\tau^2-x^2/g^2}}d\tau\\
&\qquad\qquad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g},\\
i&=0 \qquad \text{for} \quad t<x/g,\\
&=E\sqrt{\frac{C}{L}}\varepsilon^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/g^2})+E\left(\frac{G}{C}-p_0\right)\sqrt{\frac{C}{L}}\varepsilon^{-p_0t}\\
&\qquad\int_{x/g}^{t}\varepsilon^{(p_0-\rho)\tau}I_0(\sigma\sqrt{\tau^2-x^2/g^2})d\tau \quad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g}.
\end{aligned}\right\}\cdots(118)$$

Putting $p_0=0$ in (118), we get the potential and the current due to the direct e.m.f. E, and their values are

$$\left.\begin{aligned}
v&=0 \qquad \text{for} \quad t<x/g,\\
&=E\varepsilon^{-\rho x/g}+E\frac{\sigma x}{g}\int_{x/g}^{t}\varepsilon^{-\rho\tau}\frac{I_1(\sigma\sqrt{\tau^2-x^2/g^2})}{\sqrt{\tau^2-x^2/g^2}}d\tau \quad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g},\\
i&=0 \qquad \text{for} \quad t<x/g,\\
&=E\sqrt{\frac{C}{L}}\varepsilon^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/g^2})+E\frac{G}{C}\sqrt{\frac{C}{L}}\int_{x/g}^{t}\varepsilon^{-\rho\tau}I_0(\sigma\sqrt{\tau^2-x^2/g^2})d\tau\\
&\qquad\qquad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g}.
\end{aligned}\right\}\cdots(119)$$

These results coincide with those already obtained by J.R. Carson and the present author. We know from the above equations that the potential and the current at the point x increase discontinuously from null to $E\varepsilon^{-\rho x/g}$ and $E\sqrt{\frac{C}{L}}\,\varepsilon^{-\rho x/g}$ respectively just after the propagating waves have reached the point x.

Next we shall consider the case where the applied e.m.f. has the damped oscillatory form of

$$e=E_0\varepsilon^{-\alpha t}\sin(\omega t+\varphi).\qquad\cdots(120)$$

In this case we may write

$$e=\text{imaginary part of } E_0\,\varepsilon^{j\varphi}\,\varepsilon^{(-\alpha+j\omega)t}\,.$$

Hence the potential and the current at the point x, due to this e.m.f. are obtained as follows. Substitute $E=E_0\varepsilon^{j\varphi}$ ond $p_0=\alpha-j\omega$ in v and i of (118), then the imaginary parts of the results give the required solutions. Thus we obtain

$$\left.\begin{aligned}
v&=0 \qquad \text{for} \quad t<x/g,\\
&=\text{imaginary part of}\Big[E_0\,\varepsilon^{j\varphi}\,\varepsilon^{-(\alpha-j\omega)(t-x/g)-px/g}\\
&\quad+E_0\varepsilon^{j\varphi}\frac{\sigma x}{g}\varepsilon^{-(\alpha-j\omega)t}\int_{x/g}^{t}\varepsilon^{(\alpha-j\omega)\tau-p\tau}\frac{I_1(\sigma\sqrt{\tau^2-x^2/g^2})}{\sqrt{\tau^2-x^2/g^2}}d\tau\Big]\\
&=E_0\,\varepsilon^{-\alpha(t-x/g)-px/g}\sin\{\omega(t-x/g)+\varphi\}\\
&\quad+E_0\frac{\sigma x}{g}\int_{x/g}^{t}\varepsilon^{-\alpha(t-\tau)-p\tau}\sin\{\omega(t-\tau)+\varphi\}\frac{I_1(\sigma\sqrt{\tau^2-x^2/g^2})}{\sqrt{\tau^2-x^2/g^2}}d\tau\\
&\qquad\qquad\text{for}\quad \frac{2l-x}{g}>t>\frac{x}{g},\\
i&=0 \qquad \text{for}\quad t<x/g,\\
&=\text{imaginary part of}\Big[E_0\varepsilon^{j\varphi}\sqrt{\frac{C}{L}}\varepsilon^{-pt}I_0(\sigma\sqrt{t^2-x^2/g^2})\\
&\quad+E_0\,\varepsilon^{j\varphi}\Big(\frac{G}{C}-\alpha+j\omega\Big)\sqrt{\frac{C}{L}}\varepsilon^{-(\alpha-\omega)t}\int_{x/g}^{t}\varepsilon^{(\alpha-j\omega)\tau-p\tau}I_0(\sigma\sqrt{\tau^2-x^2/g_2})d\tau\Big]\\
&=E_0\sqrt{\frac{C}{L}}\sin\varphi\varepsilon^{-pt}I_0(\sigma\sqrt{t^2-x^2/g^2})\\
&\quad+E_0\sqrt{\frac{C}{L}}\int_{x/g}^{t}\varepsilon^{-\alpha(t-\tau)-p\tau}\Big\{\Big(\frac{G}{C}-\alpha\Big)\sin(\omega.\overline{t-\tau}+\varphi)\\
&\quad+\omega\cos(\omega.\overline{t-\tau}+\varphi)\Big\}I_0(\sigma\sqrt{\tau^2-x^2/g^2})d\tau \quad \text{for}\quad \frac{2l-x}{g}>t>\frac{x}{g}.
\end{aligned}\right\}(121)$$

The potential and the current at the point x just after the propagating waves have reached that point, are got by putting $t=x/g$ in the above formulas corresponding to $(2l-x)/g>t>x/g$. Hence we have

$$\left.\begin{aligned}
V_{t=x/g+0}&=E_0\sin\varphi\varepsilon^{-px/g},\\
i_{t=x/g+0}&=E_0\sin\varphi\,\varepsilon^{-px/g}\sqrt{\frac{C}{L}}.
\end{aligned}\right\}$$

The transient value of the e.m.f. at the beginning instant of application is by (120),

$$e_{t=0}=E_0 \sin \varphi.$$

From (122) and (123) we see that the fronts of the potential and the current waves due to any damped oscillatory e.m.f., are uniquely determined by the transient magnitude of the e.m.f. at the beginning instant of application, and are quite independent of the damping constant as well as of the frequency of the applied e.m.f., and they propagate along the transmission line, being damped by the factor $\varepsilon^{-\rho x/g}$.

Next, the potential and the current at the point x due to the sinusoidal e.m.f.

$$e=E_0 \sin (\omega t+\varphi)$$

are given by substituting $\alpha=0$ in (121), namely,

$$\left.\begin{aligned}
v&=0 \qquad \text{for} \quad t<x/g\\
&=E_0\varepsilon^{-\rho x/g}\sin\{\omega(t-x/g)+\varphi\}+E_0\frac{\sigma x}{g}\int_{x/g}^{t}\varepsilon^{-\rho\tau}\sin\{\omega(t-\tau)-\varphi\}\frac{I_1(\sigma\sqrt{\tau^2-x^2/g^2})}{\sqrt{\tau^2-x^2/g^2}}d\tau\\
&\qquad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g},\\
i&=0 \qquad \text{for} \quad t<x/g\\
&=E_0\sqrt{\frac{C}{L}}\sin\varphi\varepsilon^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/g^2})+E_0\sqrt{\frac{C}{L}}\int_{x/g}^{t}\varepsilon^{-\rho t}\Big\{\frac{G}{C}\sin(\omega.\overline{t-\tau}+\varphi)\\
&\quad+\omega\cos(\omega.\overline{t-\tau}+\varphi)\Big\}I_0(\sigma\sqrt{\tau^2-x^2/g^2})d\tau \qquad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g}.
\end{aligned}\right\}\text{(125)}$$

We shall discuss the mode of the current wave more precisely. Neglectiug, for the sake of simplicity, the line leakage, we get from the above

$$\left.\begin{aligned}
i&=0 \qquad \text{for} \quad t<x/g,\\
&=E_0\sqrt{\frac{C}{L}}\sin\varphi\,\varepsilon^{-\lambda t}I_0(\lambda\sqrt{t^2-x^2/g^2})\\
&\quad+E_0\omega\sqrt{\frac{C}{L}}\int_{x/g}^{t}\varepsilon^{-\lambda\tau}\cos(\omega.\overline{t-\tau}+\varphi)I_0(\lambda\sqrt{\tau^2-x^2/g^2})d\tau\\
&\qquad \text{for} \quad \frac{2l-x}{g}>t>\frac{x}{g},
\end{aligned}\right\}\cdots\cdots\cdots\cdots(126)$$

where

$$\lambda = \frac{R}{2L}. \qquad \text{.........(127)}$$

When the applied e.m.f. is

$$e = E_0 \sin \omega t, \qquad \text{.........(128)}$$

then, putting $\varphi = 0$ in (126), we obtain

$$\left.\begin{aligned} i &= 0 \quad \text{for} \quad t < x/g, \\ &= E_0 \omega \sqrt{\frac{C}{L}} \int_{x/g}^{t} \varepsilon^{-\lambda\tau} \cos\{\omega(t-\tau)\} I_0(\lambda\sqrt{\tau^2 - x^2/g^2})\,d\tau \\ &\qquad \text{for} \quad \frac{2l-x}{g} > t > \frac{x}{g}, \end{aligned}\right\} \qquad \text{.........(129)}$$

and when the applied e.m.f. is

$$e = E_0 \cos \omega t,$$

the substitution of $\varphi = \frac{\pi}{2}$ in (126) gives

$$\left.\begin{aligned} i &= 0 \quad \text{for} \quad t < x/g, \\ &= E_0 \sqrt{\frac{C}{L}} \varepsilon^{-\lambda t} I_0(\lambda\sqrt{t^2 - x^2/g^2}) - E_0 \omega \sqrt{\frac{C}{L}} \int_{x/g}^{t} \varepsilon^{-\lambda\tau} \sin \omega(t-\tau) I_0(\lambda\sqrt{\tau^2 - x^2/g^2})\,d\tau \\ &\qquad \text{for} \quad \frac{2l-x}{g} > t > \frac{x}{g}. \end{aligned}\right\} (131)$$

The full lines of fig. 5 show the current at $x = 200$ kilometers for several values of φ, due to the sinusoidal e.m.f. $e = \sin(\omega t + \varphi)$ k.v. and the chain line shows the same current due to the direct e.m.f. of 1 k.v.. These are computed, by equations (126), (129) and (131), for the line with the constants

$$R = 10\Omega/km., \quad L = 2{,}5\ mh./km, \quad C = 0{,}005\ \mu f./km., \quad G = 0, \qquad \text{......(132)}$$

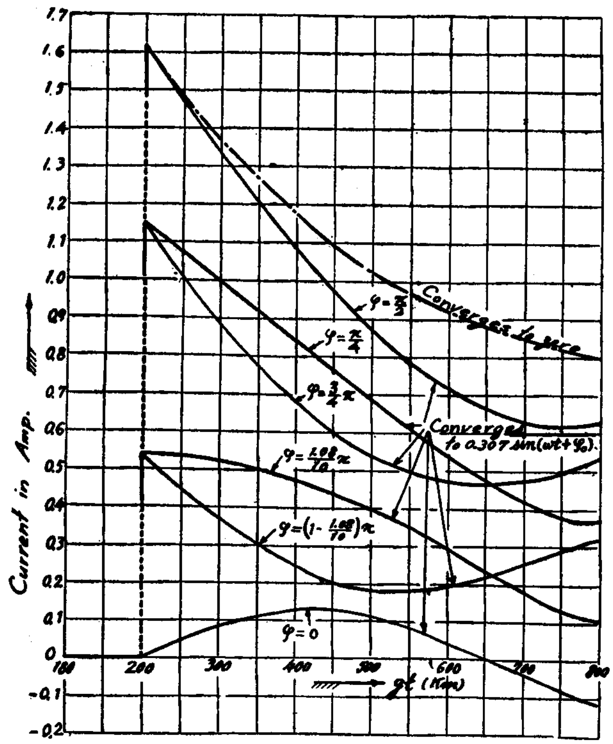
the frequency of the applied e.m.f. being 285 cycles per second.

Next put $p_0 = -j\omega$, $E = E_0 \varepsilon^{j\varphi}$ and $t = \infty$ in the i formula of (118) corresponding to $(2l-x)/g > t > x/g$, then the imaginary part of the result thus obtained represents the steady value of the current at the point x of

Fig. 5.

——— : Current induced at $x=200$ k.m. due to $e=\sin(\omega t+\varphi)$ k.v.

— - — : Current induced at $x=200$ k.m. due to $e=1$ k.v.



a semi-infinitely long transmission line due to the sinusoidal e.m.f. $e=E_0 \sin(\omega t+\varphi)$.

$$(i_0)_{t=\infty}=\text{imaginary part of}\left[E_0 \varepsilon^{j\varphi}\left(\frac{G}{C}+j\omega\right)\sqrt{\frac{C}{L}}\,\varepsilon^{j\omega t}\int_{x/g}^{\infty}\varepsilon^{-(j\omega+\rho)\tau}I_0(\sigma\sqrt{\tau^2-x^2/g^2})d\tau\right]$$

$$=\text{imaginary part of}\left[E_0 \varepsilon^{j(\omega t+\varphi)}\sqrt{\frac{Cj\omega+G}{Lj\omega+R}}\,\varepsilon^{-x\sqrt{(Lj\omega+R)(Cj\omega+G)}}\right].$$

The last part of the above equation may be written as follows:

$$I\varepsilon^{j(\omega t+\varphi_0)}=I\cos(\omega t+\varphi_0)+jI\sin(\omega t+\varphi_0).\text{—}$$

Hence we get

$$(i_0)_{t=\infty}=I\sin(\omega t+\varphi_0), \quad \text{......} \quad (133)$$

where φ_0 depends upon ω, φ, L, C, R, G and x, and I is given by

$$\left.\begin{aligned} I&=\left| E_0\varepsilon^{j(\omega t+\varphi)}\sqrt{\frac{Cj\omega+G}{Lj\omega+R}}\varepsilon^{-x\sqrt{(Lj\omega+R)(Cj\omega+G)}}\right| \\ &=E_0\left(\frac{C^2\omega^2+G^2}{L^2\omega^2+R^2}\right)^{\frac{1}{4}}\exp\left[-x\sqrt{\frac{1}{2}\{\sqrt{(L^2\omega^2+R^2)(C^2\omega^2+G^2)}+RG-\omega^2LC\}}\right]\end{aligned}\right\}(134)$$

When the direct e.m.f. E_0 is applied, we obtain the current at x by putting $\omega=0$ in the above formula. Thus we have

$$I_{d.c.}=E_0\sqrt{\frac{G}{R}}\varepsilon^{-x\sqrt{RG}}\ldots\ldots(135)$$

If the line leakage is neglected,

$$\left.\begin{aligned} I_{G=0}&=E_0\frac{\sqrt{C\omega}}{(L^2\omega^2+R^2)^{\frac{1}{4}}}\exp\left[-x\sqrt{\frac{1}{2}\{C\omega\sqrt{L^2\omega^2+R^2}-LC\omega^2\}}\right], \\ [I_{d.c.}]_{G=0}&=0. \end{aligned}\right\}\ldots\ldots(136)$$

Substituting the numerical values given by (132) in (136), we get

$$\left.\begin{aligned} I_{G=0}&=0{,}905\times 10^{-3}\,\varepsilon^{-0{,}0054x}\,E_0 \text{ amp.}, \\ [I_{d.c.}]_{G=0}&=0. \end{aligned}\right\}\ldots\ldots(137)$$

From fig. 5 we know the following :—

The current at the point $x=200$ *k.m.* remains null, until the time $t=x/g$ has elapsed, which is spent by the effect sent from $x=0$ till it reaches the point considered. At $t=x/g$, except the case $\varphi=0$, currents increase suddenly to the values shown in the figure, and vary continuously thereafter. Fig. 5 is plotted, assuming that the direct e.m.f. and the amplitude of the sinusoidal e.m.f. are each 1 k.v.. The figure shows that the direct e.m.f. induces the greatest current for a short duration after the wave reaches the point under consideration. But if we neglect the line leakage, the current due to the direct e.m.f. dies away as time passes on, and it converges to zero in its stationary state.

We shall consider the case where the sinusoidal e.m.f. $e=\sin(\omega t+\varphi)$ k.v. of a frequency of 285 cycles is applied.

The figure shows that the wave front of the current becomes greatest when

$\varphi=\pi/2$, i.e., when the e.m.f. is applied to the line at its maximum value. From the curves corresponding to $\varphi=\pi/4$, $\varphi=3\pi/4$ etc., we know that the wave fronts of the generated currents depend only upon the transient value of the e.m f. at the first instant of application. Also we know that the nearer φ approaches to zero from $\pi/2$, the more diminished is the rate at which the induced currents near the wave fronts die away, and when $\varphi=0$, the current even increases from the value of the front for a short time. This is because the nearer φ approaches to zero from $\pi/2$, the longer the increasing state of the applied e.m.f. continues.

Thus the transient value of the current due to a sinusoidal e.m.f. takes various forms, in proportion to the transient value of the e.m.f. at the beginning of application. But as time passes on, currents in such forms converge ultimately to a single oscillating value with amplitude of 0.307 amperes as seen from (137).

7. REFLECTION OF INCOMING WAVES BY CRITICAL RESISTANCE.

When the dissipation constants of the line, R and G are null, the incoming electric waves are completely absorbed in a resistance which terminates the line to the earth with the magnitude $\sqrt{L/C}$, and thus reflected waves are completely rejected. The analytical verification of this fact has already been accomplished by many authorities, and the terminal resistance with the above mentioned magnitude is called the critical resistance. This theory is often applied to many experiments in order to exclude reflected waves. But we must notice that it is true only when the dissipation constants of the line are omitted, and nothing is known about the case where these constants are taken into account. Nowadays, many experiments on transmission lines are pursued, however, under the assumption that the incoming waves may be completely excluded by the critical resistance, even when the line resistance reaches some amount. But I cannot agree with this assumption. The nature of the critical resistance should be more minutely investigated theoretically.

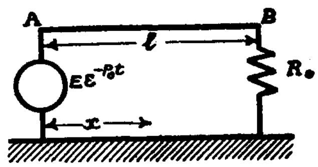
In the electric circuit, whose receiving end is terminated by the resistance R_0 as shown in fig. 6, the first reflected potential wave at the point x produced by the terminal resistance due to the e.m.f. $E\varepsilon^{-p_0t}$ applied at $x=0$ since $t=0$ is given by

$$v_1=-\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{-q(2l-x)+pt}}{p+p_0}f_1f_2dp \quad \text{for} \quad t>(2l-x)/g, \quad \dots\dots\dots\dots(138)$$

where, if we neglect the line leakage for the sake of simplicity,

$$\left.\begin{aligned} z&=\sqrt{(Lp+R)/(Cp)},\\ q&=\sqrt{(Lp+R)Cp},\\ f_1&=1,\\ f_2&=\frac{z-R_0}{z+R_0}, \end{aligned}\right\} \quad \dots\dots\dots\dots(139)$$

Fig. 6



Putting $x=l$ in (138), we get the first reflected potential wave at the point B, which is given by

$$v_1=-\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{-qx+pt}}{p+p_0}\,\frac{z-R_0}{z+R_0}\,dp \qquad \text{for} \quad t>l/g. \quad \dots\dots\dots\dots(140)$$

Putting, further, $l=0$ in (140), we get the first reflected potential wave at B, when the incoming potential wave at the same point takes the form of $E\varepsilon^{-p_0t}$ for $t\geq 0$.

$$v_1=-\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}}{p+p_0}\,\frac{z-R_0}{z+R_0}\,dp \qquad \text{for} \quad t>0. \quad \dots\dots\dots\dots(141)$$

When the incoming potential wave is rectangular, we get the reflected potential wave at B, by putting $p_0=0$ in the above equation.

$$v_1=-\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}}{p+p_0}\,\frac{z-R_0}{z-R_0}\,dp \qquad \text{for} \quad t>0. \quad \dots\dots\dots\dots(142)$$

A) Reflected potential wave due to incoming rectangular potential wave.

For the sake of brevity, we assume that the form of the incoming potential wave is rectangular, then the reflected potential wave at B is given by (142). Next we shall consider the case where R_0 possesses the critical value, i.e.

$$R_0=\sqrt{\frac{L}{C}}. \quad\cdots\cdots(143)$$

Putting (139) and (143) in (142), we get

$$v_1=-\frac{E}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}}{p}\frac{\left(\frac{Lp+R}{Cp}\right)^{\frac{1}{2}}-\sqrt{\frac{L}{C}}}{\left(\frac{Lp+R}{Cp}\right)^{\frac{1}{2}}-\sqrt{\frac{L}{C}}}dp$$

$$=\frac{E}{2\pi j}\int_{(K)}\left\{2\sqrt{\frac{L}{C}}\frac{1}{p}\left(\frac{Cp}{Lp+R}\right)^{\frac{1}{2}}+\frac{2L}{R}\sqrt{\frac{L}{C}}\left(\frac{Cp}{Lp+R}\right)^{\frac{1}{2}}-\frac{2L}{R}-\frac{1}{p}\right\}\varepsilon^{pt}dp. \quad(144)$$

On the other hand, by (103), we have

$$\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}}{p}\left(\frac{Cp}{Lp+R}\right)^{\frac{1}{2}}dp=\sqrt{\frac{C}{L}}\,\varepsilon^{-\lambda t}I_0(\lambda t), \quad\cdots\cdots(145)$$

where

$$\lambda=\frac{R}{2L}. \quad\cdots\cdots(127)$$

And we know that

$$\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}}{p}dp=1, \qquad \frac{1}{2\pi j}\int_{(K)}\varepsilon^{pt}dp=0. \quad\cdots\cdots(146)$$

Differentiating both sides of (145) with respect to t, we obtain finally,

$$\frac{1}{2\pi j}\int_{(K)}\varepsilon^{pt}\left(\frac{Cp}{Lp+R}\right)^{\frac{1}{2}}dp=\sqrt{\frac{C}{L}}\,\lambda\varepsilon^{-\lambda t}\{-I_0(\lambda t)+I_1(\lambda t)\}. \quad\cdots\cdots(147)$$

Hence substituting (145), (146) and (147) in (144), we get

$$v_1=E\left[\varepsilon^{-\lambda t}\{I_0(\lambda t)+I_1(\lambda t)\}-1\right]. \quad\cdots\cdots(148)$$

From (148) we get the following conclusions, when the form of the incoming potential wave is rectangular.

1) The reflected potential wave produced by the critical resistance is determined solely by the ratio $\lambda=R/(2L)$, and is independent of the electrostatic capacity C of the line.

2) The critical resistance flattens the front of the reflected potential wave instantly, and discontinuity in the front is completely rejected, even

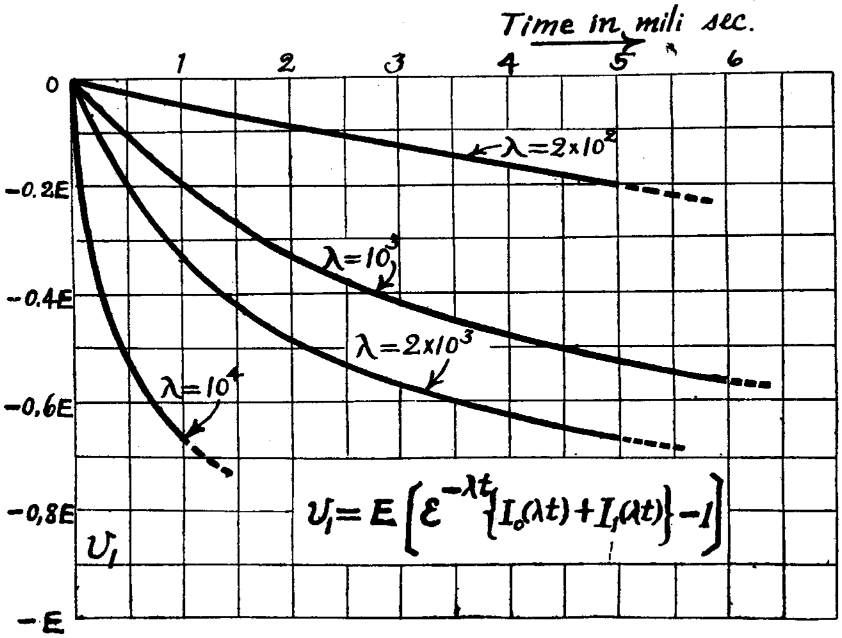
when the incoming potential wave takes the rectangular form ; because, for sufficiently small t, we may write

$$v_1 \cong \frac{\lambda t}{2}. \quad \text{.........} (139)$$

Thus we know that the critical resistance is effective for flattening the front of the reflected wave.

Fig. 7.

Reflected potential wave produced by critical resistance when rectangular potential wave E comes in



3) Fig. 7 is plotted from equation (148), which shows that the reflected potential wave increases with time, when the ratio $\lambda = R/(2L)$ becomes great. Hence when R/L is large, the value of the reflected wave is not so small that it may be possibly neglected, even just after the reflection takes place. Hence we cannot assert generally that an experiment pursued under the assumption that the critical resistance may be effective for the rejection of reflected waves, can really show the actual phenomena without errors.

4) In the stationary condition, the reflected wave takes the form

$$\lim_{t\to\infty} v_1 = E\left[\lim_{t\to\infty} \varepsilon^{-\lambda t}\{I_0(\lambda t)+I_1(\lambda t)\}-1\right]$$
$$= -E. \qquad\qquad (150)$$

Hence we know that the reflected potential wave approaches to the same magnitude as the incoming wave but with different sign, as time goes on.

B) *Reflected potential wave due to the incoming potential wave.*

In this case, the reflected potential wave is given by (141), i.e.,

$$\left.\begin{aligned} v_1 &= -\frac{E}{2\pi j}\int_{(K)} \frac{\varepsilon^{pt}}{p+p_0} \frac{\left(\frac{Lp+R}{Cp}\right)^{\frac{1}{2}}-\sqrt{\frac{L}{C}}}{\left(\frac{Lp+R}{Cp}\right)^{\frac{1}{2}}+\sqrt{\frac{L}{C}}}dp \\ &= \frac{E}{2\pi j}\int_{(K)}\left\{\frac{2(R-Lp_0)}{R}\sqrt{\frac{L}{C}}\frac{1}{p+p_0}\left(\frac{Cp}{Lp+R}\right)^{\frac{1}{2}}\right. \\ &\quad \left.+\frac{2L}{R}\sqrt{\frac{L}{C}}\left(\frac{Cp}{Lp+R}\right)^{\frac{1}{2}}-\frac{2L}{R}-\frac{R-2Lp_0}{R}\frac{1}{p+p_0}\right\}dp \\ &\text{for} \qquad t>0. \end{aligned}\right\} \qquad (151)$$

But from (103) we have

$$\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}}{p+p_0}\left(\frac{Cp}{Lp+R}\right)^{\frac{1}{2}}dp = \sqrt{\frac{C}{L}}\varepsilon^{-\lambda t}I_0(\lambda t) - p_0\sqrt{\frac{C}{L}}\varepsilon^{-p_0 t}\int_0^t \varepsilon^{(p_0-\lambda)t}I_0(\lambda t)\,dt, \quad (152)$$

and evidently

$$\frac{1}{2\pi j}\int_{(K)}\frac{\varepsilon^{pt}}{p+p_0}\,dp = \varepsilon^{-p_0 t}. \qquad\qquad (153)$$

Substituting (146), (152) and (153) in (151), we have finally

$$\left.\begin{aligned} V_1 &= -E\left\{-\varepsilon^{-\lambda t}I_1(\lambda t)+\frac{p_0-\lambda}{\lambda}\varepsilon^{-\lambda t}I_0(\lambda t)\right. \\ &\quad \left.+\frac{(2\lambda-p_0)p_0}{\lambda}\varepsilon^{-p_0 t}\int^t \varepsilon^{(p_0-\lambda)t}I_0(\lambda t)dt+\frac{\lambda-p_0}{\lambda}\varepsilon^{-p_0 t}\right\} \\ &\text{for} \qquad t>0. \end{aligned}\right\} \qquad (154)$$

In the special case where the incoming potential wave has the form of $E_0 \sin(\omega t+\varphi)$ for $t \geqq 0$, we shall get the first reflected potential wave by taking the imaginary part of the result obtained by the substitution of $E=E_0\varepsilon^{j\varphi}$ and $p_0=-j\omega$ in the above equation. Hence we have

$$v_1=\text{imaginary part of } E_0\varepsilon^{j\varphi}\Big\{\varepsilon^{-\lambda t}I_1(\lambda t)+\frac{j\omega+\lambda}{\lambda}\varepsilon^{-\lambda t}I_0(\lambda t)$$
$$+\frac{(2\lambda+j\omega)j\omega}{\lambda}\varepsilon^{-p_0t}\int_0^t\varepsilon^{(p_0-\lambda)t}I_0(\lambda t)\,dt-\frac{\lambda+j\omega}{\lambda}\varepsilon^{j\omega t}\Big\}, \quad\text{..................(155)}$$

or

$$\left.\begin{aligned}v_1=E_0\Big\{&\sin\varphi\,\varepsilon^{-\lambda t}I_1(\lambda t)+\frac{\omega\cos\varphi+\lambda\sin\varphi}{\lambda}\varepsilon^{-\lambda t}I_0(\lambda t)\\&+\frac{\omega}{\lambda}\int_0^t\Big[2\lambda\cos(\omega\overline{t-\tau}+\varphi)-\omega\sin(\omega\overline{t-\tau}+\varphi)\Big]\varepsilon^{-\lambda\tau}I_0(\lambda\tau)\,d\tau\\&-\frac{\omega\cos(\omega t+\varphi)+\lambda\sin(\omega t+\varphi)}{\lambda}\Big\}\\ \text{for}\quad &t>0.\end{aligned}\right\}\text{......(156)}$$

Putting $t=0$ in the above equation, we get

$$(v_1)_{t=0}=0. \quad\text{...(157)}$$

Therefore we know that the front of the reflected wave is immediately flattened by the critical resistance, even when the front of the incoming potential wave is discontinuous.

C) *Reflected potential wave due to incoming potential wave of extra high frequency.*

The reflected wave is also given by (156). But this formula is somewhat inconvenient for numerical calculation when ω/λ becomes large. We shall transform it into an easily calculable form.

By integration by parts, we have

$$\int_0^t\varepsilon^{(p_0-\lambda)t}I_0(\lambda t)\,dt=\frac{1}{p_0-\lambda}\Big\{\varepsilon^{(p_0-\lambda)t}I_0(\lambda t)-1-\lambda\int_0^t\varepsilon^{(p_0-\lambda)t}I_1(\lambda t)\,dt\Big\}. \quad\text{.........(158)}$$

Putting this relation in (154), we have finally

$$\left.\begin{aligned}v_1=E\Big\{&\varepsilon^{-\lambda t}I_1(\lambda t)-\frac{\lambda}{p_0-\lambda}\varepsilon^{-\lambda t}I_0(\lambda t)+\frac{\lambda}{p_0-\lambda}\varepsilon^{-p_0t}\\&+\frac{(2\lambda-p_0)p_0}{(p_0-\lambda)}\varepsilon^{-p_0t}\int_0^t\varepsilon^{(p_0-\lambda)t}I_1(\lambda t)\,dt\Big\}\\ \text{for}\quad &t>0.\end{aligned}\right\}\text{..................(159)}$$

This is the first reflected potential wave corresponding to the original wave $E\varepsilon^{-p_0t}\ (t\geq 0)$. And the wave produced by the first reflection when $E_0 \sin(\omega t+\varphi)\ (t\geq 0)$ come in, is given by the imaginary part of the nesult obtained by the substitution of $E=E_0\varepsilon^{j\varphi}$ and $p_0=-j\omega$ in the above equation. Thus we have

$$\left.\begin{aligned} v_1 = E_0\Big\{ & \sin\varphi\,\varepsilon^{-\lambda t} I_1(\lambda t) + \frac{\lambda(\lambda\sin\varphi-\omega\cos\varphi)}{\lambda^2+\omega^2}\,\varepsilon^{-\lambda t} I_0(\lambda t) \\ & - \frac{\lambda[\lambda\sin(\omega t+\varphi)-\omega\cos(\omega t+\varphi)]}{\lambda^2+\omega^2} \\ & + \frac{\omega}{\lambda^2+\omega^2}\int_0^t \Big[\lambda\omega\sin(\omega\overline{t-\tau}+\varphi)+(\omega^2+2\lambda^2)\cos(\omega\overline{t-\tau}+\varphi)\Big]\varepsilon^{-\lambda\tau} I_1(\lambda\tau)\,d\tau\Big\} \\ \text{for}\quad & t>0. \end{aligned}\right\} \quad (160)$$

When $\lambda/|p_0|$ and λt are negligibly small compared with unity, we have from (159),

$$v_1 \cong -\frac{\lambda}{2p_0}(1-\varepsilon^{-p_0t})\,E \qquad \text{for}\quad t>0, \quad \ldots\ldots\ldots\ldots(161)$$

because, in this case, we have

$$\varepsilon^{-\lambda t} I_1(\lambda t) \cong \frac{\lambda t}{2}, \qquad \frac{\lambda}{p_0-\lambda}\,\varepsilon^{-\lambda t} I_0(\lambda t) \cong \frac{\lambda}{p_0}, \qquad \frac{\lambda}{p_0-\lambda}\,\varepsilon^{-p_0t} \cong \frac{\lambda}{p_0}\,\varepsilon^{-p_0t},$$

$$\frac{(2\lambda-p_0)\,p_0}{p_0-\lambda}\,\varepsilon^{-p_0t}\int_0^t \varepsilon^{(p_0-\lambda)t} I_1(\lambda t)\,dt \cong -p_0\,\varepsilon^{-p_0t}\int_0^t \varepsilon^{p_0t}\frac{\lambda t}{2}\,dt$$

$$= -\frac{\lambda}{2p_0}\Big\{(p_0t-1)+\varepsilon^{-p_0t}\Big\}.$$

Bquation (161) corresponds to the case where the incoming potential wave takes the form of $E_0\varepsilon^{-p_0t}$. Therefore to get the first reflected potential wave when $E_0\sin(\omega t+\varphi)\ (t\geq 0)$ comes in, put $E=E_0\varepsilon^{j\varphi}$ and $p_0=-j\omega$ in (161), then its imaginary part gives the required solution, i.e.,

$$v_1 \cong \frac{\lambda}{2\omega}\Big\{\cos(\omega t+\varphi)-\cos\varphi\Big\}E_0 \qquad \text{for}\quad t>0. \quad \ldots\ldots\ldots(162)$$

When the incoming wave has infinitely high frequency, we get the reflected potential wave by putting $\omega=\infty$ in (160), i.e.,

$$\lim_{\omega\to\infty} v_1 = E_0\Big\{\sin\varphi\,\varepsilon^{-\lambda t} I_1(\lambda t)$$

$$+\lim_{\omega\to\infty}\frac{\omega}{\omega^2+\lambda^2}\int_0^t \Big[\lambda\omega\sin(\omega.\overline{t-\tau}+\varphi)+(\omega^2+2\lambda^2)\cos(\omega.\overline{t-\tau}+\varphi)\Big]\varepsilon^{-\lambda\tau} I_1(\lambda\tau)\,d\tau\Big\}$$

$$= E_0\Big\{\sin\varphi\,\varepsilon^{-\lambda t} I_1(\lambda t)$$

$$-\lim_{\omega\to\infty}\Big[\frac{-\omega\lambda}{\lambda^2+\omega^2}\cos(\omega.\overline{t-\tau}+\varphi)\,\varepsilon^{-\lambda t} I_1(\lambda t)+\frac{\omega\lambda}{\lambda^2+\omega^2}\int_0^t \cos(\omega.\overline{t-\tau}+\varphi)\frac{d}{d\tau}(\varepsilon^{-\lambda\tau} I_1(\lambda\tau))d\tau$$

$$+\frac{2\lambda^2+\omega^2}{\lambda^2+\omega^2}\sin\varphi\,\varepsilon^{-\lambda t} I_0(\lambda t)$$

$$+\frac{2\lambda^2+\omega^2}{\omega(\lambda^2+\omega^2)}\Big(\cos\varphi\,\frac{d}{dt}\,(\varepsilon^{-\lambda t} I_0(\lambda t))-\frac{\lambda}{2}\cos(\omega t+\varphi)\Big)$$

$$-\frac{\omega^2+2\lambda^2}{\omega(\lambda^2+\omega^2)}\int_0^t \cos(\omega\overline{t-\tau}+\varphi)\frac{d^2}{d\tau^2}\Big(\varepsilon^{-\lambda\tau} I_1(\lambda\tau)\Big)d\tau\Big]\Big\}$$

$$= E_0\Big\{+\sin\varphi\,\varepsilon^{-\lambda t} I_1(\lambda t)-\sin\varphi\,\varepsilon^{-\lambda t} I_1(\lambda t)\Big\},$$

or

$$\lim_{\omega\to\infty} v_1 = 0. \quad\text{...}(163)$$

From the above results, we get the following conclusions :—

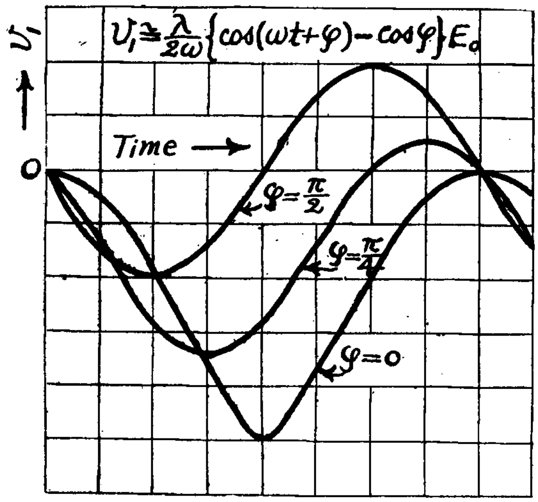
1) The higher the frequency of the incoming wave, the smaller the magnitude of the reflected wave becomes, and the absorption of the incoming wave by the critical resistance is completely realized for a wave of extra high frequency. From (162) we see that the magnitude of the reflected potential wave decreases in proportion as the frequency of the incoming wave increases. Thus we know that the higher the frequency of the incoming wave, the smaller the magnitude of the reflected wave produced by the critical resistance becomes.

2) Fig. 8 is the graph of (162), which shows that the reflected wave near its wave front has the greatest magnitude when $\varphi=0$, and the smallest when $\varphi=\pi/2$, and its magnitude decreases as φ approaches to $\pi/2$ from zero. The front of the incoming wave is $E_0 \sin\varphi$, which increases as φ varies from zero to $\pi/2$. Hence we know that, when a sinusoidal potential wave of certain amplitude comes in, the smaller its wave front is, the greater the magnitude of the reflected wave produced by the critical resistance near the wave front.

3) From (160), it is known that the reflected potential wave is uniquely determined by the ratio $\lambda=R/(2L)$ as well as the frequency and

Fig. 8.

Reflected potential wave produced by critical resistance when sinusoidal potential wave of high frequency $E_0 \sin(\omega t+\varphi)$ comes in



the phase angle of the incoming wave, and it is independent of the capacity of the line.

D) *Relation between the frequency of the incoming wave and the steady reflected potential wave.*

Next we shall consider the steady value of the first reflected potential wave. Putting $t=\infty$ in (154), we obtain its steady value due to the e.m.f. $E\varepsilon^{-p_0t}$, and it is given by

$$(v_1)_{t=\infty}=-E\left\{\frac{(2\lambda-p_0)p_0}{\lambda}\varepsilon^{-p_0t}\int_0^\infty \varepsilon^{(p_0-\lambda)t}I_0(\lambda t)\,dt+\frac{\lambda-p_0}{\lambda}\varepsilon^{-p_0t}\right\},\ldots\ldots \quad (164)$$

if the real part of $p_0 \geqq 0$.

Now we have

$$\int_0^\infty \varepsilon^{-(j\omega+\lambda)t}I_0(\lambda t)\,dt=-\frac{j}{\omega\sqrt{u}}\left(\cos\frac{\theta}{2}+j\sin\frac{\theta}{2}\right),\ldots\ldots\ldots\ldots\ldots\ldots(165)$$

where

$$\left.\begin{aligned}\theta&=\tan^{-1}\left(2\frac{\lambda}{\omega}\right),\\ u&=\sqrt{1+4\frac{\lambda^2}{\omega^2}}.\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(166)$$

The steady reflected potential wave due to the incoming potential wave $E_0 \sin(\omega t+\varphi)$ $(t \geqq 0)$ is given by the imaginary part of the result obtained by the substitution of $E=E_0\varepsilon^{j\varphi}$ and $p_0=-j\omega$ in (164), and it is calculated with the aid of (165) as follows.

$$(v_1)_{t=\infty}=\text{imaginary part of } E_0\varepsilon^{j\varphi}\left\{\frac{(2\lambda+j\omega)j\omega}{\lambda}\varepsilon^{j\omega t}\int_0^\infty \varepsilon^{-(j\omega+\lambda)t} I_0(\lambda t)\,dt-\frac{\lambda+j\omega}{\lambda}\varepsilon^{j\omega t}\right\}$$

$$=-E_0\left\{\left(1-\frac{\omega\sqrt{u}}{\lambda}\sin\frac{\theta}{2}\right)\sin\omega t+\left(\frac{\omega}{\lambda}-\frac{\omega\sqrt{u}}{\lambda}\cos\frac{\theta}{2}\right)\cos\omega t\right\}. \quad \ldots(167)$$

And its amplitude is given by

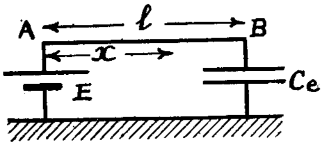
$$V_1=E_0\sqrt{1-\frac{2}{\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{R}{\omega L}\right)^2}+1\right)}+1}} \quad \ldots\ldots\ldots\ldots(168)$$

This formula shows that the amplitude of the steady first reflected potential wave is uniquely determined by the ratio $R/(\omega L)$, and it increases from zero to E_0 as $R/(\omega L)$ varies from zere to infinity.

8. REFLECTION OF INCOMING WAVES BY TERMINAL CONDENSER.

We shall show an example of the second method described in section 5, which will bring v and i, given by (1) and (2), into real known functions.

Fig. 9.



We assume that the receiving end of the transmission line is terminated by a condenser of capacity C_e, and that the direct e.m.f. E is directly applied at the beginning of the line. Then the first reflected potential and current waves at x and t are, by (1) and (2),

$$\left.\begin{aligned} v_1&=-\frac{E}{2\pi j}\int_{(K)}\frac{f_1 f_2}{p}\varepsilon^{-qy+pt}\,dp, &&\text{for } t>y/g\\ i_1&=\frac{E}{2\pi j}\int_{(K)}\frac{f_1 f_2}{p}\varepsilon^{-qy+pt}\,dp, &&\text{for } t>y/g\end{aligned}\right\} \quad \ldots\ldots\ldots(169)$$

where

$$y=2l-x,\quad f_1=1 \quad \text{and} \quad f_2=\frac{z-(C_e p)^{-1}}{z+(C_e p)^{-1}}. \qquad \text{.................}(170)$$

Assuming that no line leakage exists, we get

$$z=\left(\frac{R+Lp}{Cp}\right)^{\frac{1}{2}}. \qquad \text{.................}(171)$$

Apply rule (B) of section 5, and change p to ζ, then by (171) f_2 is tranformed to

$$f_2=1-\frac{4m\zeta}{1+2m\zeta-\zeta^2}, \qquad \text{.................}(172)$$

where

$$\lambda=\frac{R}{2L}, \qquad m=\frac{1}{C_e\lambda}\sqrt{\frac{C}{L}}. \qquad \text{.................}(173)$$

Hence by (114), v_1 and i_1 are reduced to the following forms:

$$\left.\begin{aligned} v_1&=-E\varepsilon^{-\lambda t}\frac{1}{2\pi j}\int^{(0+)}\left(1-\frac{4m\zeta}{1+2m\zeta-\zeta^2}\right)\frac{1+\zeta}{(1-\zeta)\zeta} \\ &\qquad exp\left[\frac{\lambda}{2}\left\{\left(t+\frac{y}{g}\right)\zeta+\left(t-\frac{y}{g}\right)\frac{1}{\zeta}\right\}\right]d\zeta \\ &\qquad\qquad \text{for} \quad t>y/g, \\ i_1&=E\sqrt{\frac{C}{L}}\,\varepsilon^{-\lambda t}\frac{1}{2\pi j}\int^{(0+)}\left(1-\frac{4m\zeta}{2+2m\zeta-\zeta^2}\right)\frac{1+\zeta}{1-\zeta} \\ &\qquad exp\left[\frac{\lambda}{2}\left\{\left(t+\frac{y}{g}\right)\zeta+\left(t-\frac{y}{g}\right)\frac{1}{\zeta}\right\}\right]d\zeta \\ &\qquad\qquad \text{for} \quad t>y/g. \end{aligned}\right\} \qquad \text{...........}(174)$$

Let the roots of $\zeta^2-2m\zeta-1=0$ be ζ_1 and ζ_2, then we have

$$\left.\begin{aligned} \zeta_1&=m+\sqrt{m^2+1}, \\ \zeta_2&=m-\sqrt{m^2+1}. \end{aligned}\right\} \qquad \text{.................}(175)$$

Since

$$\frac{1+\zeta}{(1-\zeta)(1-\zeta/\zeta_1)(1-\zeta/\zeta_2)}=\frac{1}{m}\frac{1}{1-\zeta}+\frac{A}{1-\zeta/\zeta_1}+\frac{B}{1-\zeta/\zeta_2},$$

$$\frac{1-\zeta}{(1-\zeta)\zeta}=\frac{1}{\zeta}+\frac{2}{1-\zeta},$$

where

$$A=\frac{1-\zeta_2}{(1-\zeta_1)(\zeta_1-\zeta_2)},\qquad B=\frac{\zeta_1-1}{(1-\zeta_2)(\zeta_1-\zeta_2)},$$

the integrals of (174) are transformed to the following :—

$$\left.\begin{aligned}
v_1&=-E\varepsilon^{-\lambda t}\frac{1}{2\pi j}\int^{(0+)}\left\{\frac{1}{\zeta}+\left(2-\frac{4}{m}\right)\frac{1}{1-\zeta}-\frac{4A}{1-\zeta/\zeta_1}-\frac{4B}{1-\zeta/\zeta_2}\right\}\\
&\qquad exp\left[\frac{\lambda}{2}\left\{\left(t+\frac{y}{g}\right)\zeta+\left(t-\frac{y}{g}\right)\frac{1}{\zeta}\right\}\right]d\zeta\\
&\qquad\qquad \text{for}\quad t>y/g,\\
i_1&=E\sqrt{\frac{C}{L}}\varepsilon^{-\lambda t}\frac{1}{2\pi j}\int^{(0+)}\left\{\frac{1}{\zeta}-\frac{2m}{\sqrt{m^2+1}}\left(\frac{\zeta_1}{1-\zeta/\zeta_2}-\frac{\zeta_1}{1-\zeta/\zeta_1}\right)\right.\\
&\qquad exp\left[\frac{\lambda}{2}\left\{\left(t+\frac{y}{g}\right)\zeta+\left(t-\frac{y}{g}\right)\frac{1}{\zeta}\right\}\right]d\zeta\\
&\qquad\qquad \text{for}\quad t>y/g.
\end{aligned}\right\}\quad\ldots(176)$$

By rule (B), expand $\frac{1}{1-\zeta}$, $\frac{1}{1-\zeta/\zeta_1}$ and $\frac{1}{1-\zeta/\zeta_2}$ in the ascending power of ζ, and we get

$$\left.\begin{aligned}
v_1&=-E\varepsilon^{-\lambda t}\frac{1}{2\pi j}\int^{(0+)}\left\{\frac{1}{\zeta}+\left(2-\frac{4}{m}\right)\sum_{n=0}^{\infty}\zeta^n-4A\sum_{n=0}^{\infty}\left(\frac{\zeta}{\zeta_1}\right)^n-4B\sum_{n=0}^{\infty}\left(\frac{\zeta}{\zeta_2}\right)^n\right\}\\
&\qquad exp\left[\frac{\lambda}{2}\left\{\left(t+\frac{y}{g}\right)\zeta+\left(t-\frac{y}{g}\right)\frac{1}{\zeta}\right\}\right]d\zeta\qquad\text{for}\quad t>y/g,\\
i_1&=E\sqrt{\frac{C}{L}}\varepsilon^{-\lambda t}\frac{1}{2\pi j}\int^{(0+)}\left[\frac{1}{\zeta}-\frac{2m}{\sqrt{m^2+1}}\left\{\zeta_1\sum_{n=0}^{\infty}\left(\frac{\zeta}{\zeta_2}\right)^n-\zeta_2\sum_{n=0}^{\infty}\left(\frac{\zeta}{\zeta_1}\right)^n\right\}\right]\\
&\qquad exp\left[\frac{\lambda}{2}\left\{\left(t+\frac{y}{g}\right)\zeta+\left(t-\frac{y}{g}\right)\frac{1}{\zeta}\right\}\right]d\zeta\qquad\text{for}\quad t>y/g.
\end{aligned}\right\}\quad\ldots(177)$$

Since $\zeta_1\zeta_2=-1$, the above v_1 and i_1 are, after termwise integration with the aid of (116), reduced to the following forms :—

$$\left.\begin{aligned} v_1=&-E\varepsilon^{-\lambda t}\Big[I_0\{\lambda\xi_y(t)\}+\Big(2-\frac{4}{m}\Big)\sum_{n=1}^{\infty}\gamma_y^n I_n\{\lambda\xi_y(t)\} \\ &+4A\zeta_1\sum_{n=1}^{\infty}(-)^n\zeta_2^n\gamma_y^n I_n\{\lambda\xi_y(t)\}+4B\zeta_2\sum_{n=1}^{\infty}(-)^n\zeta_1^n\gamma_y^n I_n\{\lambda\xi(t)\}\Big] \\ &\qquad\text{for } t>y/g, \\ i_1=&E\sqrt{\frac{L}{C}}\,\varepsilon^{-\lambda t}\Big[I_0\{\lambda\xi_y(t)\}+\frac{2m}{\sqrt{m^2+1}}\Big\{\sum_{n=1}^{\infty}(-)^n\zeta_1^n\gamma_y^n I_n\{\lambda\xi_y(t)\} \\ &-\sum_{n=1}^{\infty}(-)^n\zeta_2^n\gamma_y^n I_n\{\lambda\xi_y(t)\}\Big\}\Big] \quad\text{for } t>y/g, \end{aligned}\right\} \quad\ldots\ldots(178)$$

where

$$\gamma_y=\left(\frac{t-y/g}{t+y/g}\right)^{\frac{1}{2}} \text{ and } \xi_y(t)=\sqrt{t^2-y^2/g^2}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(179)$$

I_0, I_1, I_2, I_3, I_4 and I_5 are computed and tabulated for small values of the argument, and I_n generally decreases for small arguments as n increases, and has a simple asymtotic expansion for large arguments. It is therefore a simple matter to compute and to express in graphs, a representative set of curves which show the current and potential waves for various values of L, C, G, R, C_e and y. But when $|\zeta_1\gamma|$ is great as compared with unity, the series $\sum_{n=1}^{\infty}(-)^n\zeta_1^n\gamma_y^n I_n\{\lambda\xi_y(t)\}$, which is involved in v_1 and i_1 formulas, will converge slowly, and its numerical calculation will become troublesome. Hence we need some device which will render the calculation of such a series easier. For this purpose we proceed as follows :—

We know the following relation

$$exp\left\{\frac{1}{2}z\left(u-\frac{1}{u}\right)\right\}=\sum_{n=-\infty}^{+\infty}u^n J_n(z).$$

Put $u=j\zeta\gamma_y$ and $z=j\xi_y(t)$, then we get finally

$$exp\left[-\frac{\lambda}{2}\left\{\left(t-\frac{y}{g}\right)\zeta+\left(t+\frac{y}{g}\right)\frac{1}{\zeta}\right]\right.$$

$$=I_0\{\lambda\xi_y(t)\}+\sum_{n=1}^{+\infty}(-)^n(\zeta^n\gamma_y^n+\zeta^{-n}\gamma_y^{-n})I_n\{\lambda\xi_y(t)\},$$

since $I_n(x)=I_{-n}(x)$. Therefore we have the following result:—

$$\sum_{n=1}^{+\infty}(-)^n\zeta^n\gamma_y^n I_n\{\lambda\xi_y(t)\}=exp\left[-\frac{\lambda}{2}\left\{\left(t-\frac{y}{g}\right)\zeta+\left(t+\frac{y}{g}\right)\frac{1}{\zeta}\right\}\right]$$

$$-I_0\{\lambda\xi_y(t)\}-\sum_{n=1}^{+\infty}(-)^n\zeta^{-n}\gamma_y^{-n}I_n\{\lambda\xi_y(t)\}. \quad\ldots\ldots(180)$$

If $|\zeta\gamma_y|>1$, the series $\sum_{n=1}^{+\infty}(-)^n\zeta^{-n}\gamma_y^{-n}I_n\{\lambda\xi_y(t)\}$ will converge rapidly, and will be easily calculable. Hence if we use the term on the right hand side of the equation (180) for the calculation of $\sum_{n=1}^{\infty}(-)^n\zeta^n\gamma_y^n I_n\{\lambda\xi^y(t)\}$, the numerical computation will be done easily when $|\zeta\gamma_y|>1$. We shall take an example.

If the incoming potential wave at the receiving end takes the form of $E\,(t\geq 0)$, and if the condition of the circuit is assumed to be the same as above discussed, we shall get the corresponding first reflected potential wave at the receiving terminal by putting $x=l=0$ in the v_1 formula of (178). Thus we have

$$\left.\begin{aligned}v_1=-E\varepsilon^{-\lambda t}\Big\{I_0(\lambda t)+\left(2-\frac{4}{m}\right)\sum_{n=1}^{+\infty}I_n(\lambda t)+4A\zeta_1\sum_{n=1}^{+\infty}(-)^n\zeta_2^n I_n(\lambda t)\\+4B\zeta_2\sum_{n=1}^{+\infty}(-)^n\zeta_1^n I_n(\lambda t)\Big\}\qquad \text{for}\quad t>0.\end{aligned}\right\}\ldots\ldots(181)$$

We take, for instance, a loaded submarine cable 200 n.m. long, with the constants

$R=3\ \Omega/n.m.$, $C=0{,}4\ \mu.f./n.m.$, $L=0{,}050\ h./n.m.$ and $G=0$.

In this case ζ_1 and ζ_2 are

$\zeta_1=2{,}724$, $\zeta_2=-0{,}368$.

Therfore the series $\sum_{n=1}^{\infty}(\pm)^n\zeta_2^n I_n(\lambda t)$ converges rapidly, while the series $\sum_{n=1}^{\infty}(-)^n\zeta_1^n I_n(\lambda t)$ converges so slowly for suitably chosen t that we shall find it laborious to execute its numerical computation. Putting (180) in (181), we get finally,

$$\left.\begin{aligned} v_1 = E\varepsilon^{-\lambda t}\Big\{&-(1-4B\zeta_2)I_0(\lambda t)-\Big(2-\frac{4}{m}\Big)\sum_{n=1}^{+\infty}I_n(\lambda t)-4A\zeta_1\sum_{n=1}^{+\infty}(-)^n\zeta_2^n I_n(\lambda t)\\ &+4B\zeta_2\sum_{n=1}^{+\infty}\zeta_2^n I_n(\lambda t)-4B\zeta_2 E\,exp\Big\{-\frac{(\zeta_1+1)^2}{2\zeta_1}\lambda t\Big\} \quad \text{for } t>0. \end{aligned}\right\} \quad (182)$$

since we have, from (175), $\zeta_1\zeta_2=-1$.

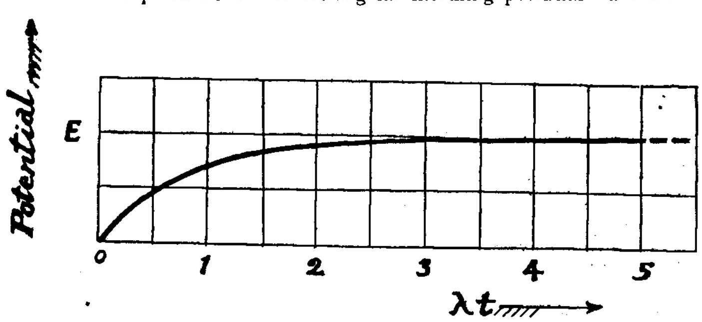
This last result is convenient for numerical calculation. Now the cable in the present case is very long, and therefore all the reflected potential waves except v_1 are so little influential upon the receiving end compared with v_0 for a transient short interval that we may represent the condenser potential v by the sum of the original and the first reflected components, excluding the effects of other reflected waves. Thus we have, for small values of t greater than zero,

$$v \cong v_0 + v_1 = E + v_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(183)$$

Fig. 10 gives a representative curve illustrating the form of the condenser potential in response to the above described constants as well as the incoming potential wave E.

Fig. 10.

Condenser potential due to rectangular incoming potential wave E.



Next we shall consider the current at the condenser terminal due to the direct e.m.f. E at $x=0$. Put $x=l$ in the i_1 formula of (178), then we get the first reflected current wave at the receiving end, and in the case of line leakage being neglected, the original current wave at the same end is given by (119), i.e.,

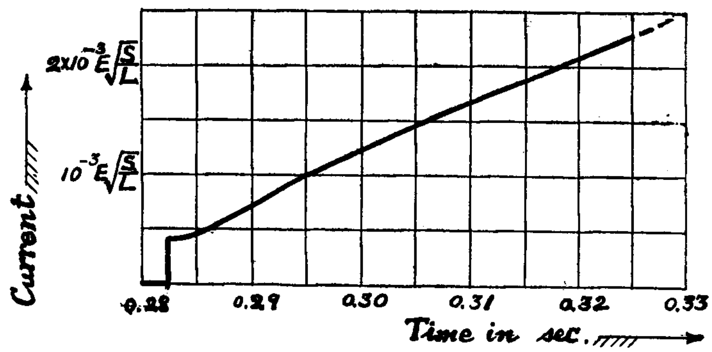
$$i_0 = E\sqrt{\frac{C}{L}}\,\varepsilon^{-\lambda t} I_0\{\lambda \xi_y(t)\} \quad \text{......(184)}$$

For just the same reason as stated above, if the cable is very long, the current of the condenser terminal is given, for a short interval, by $i = i_0 + i_1$. Hence the required value of the current is

$$i \cong E\sqrt{\frac{C}{L}}\,\varepsilon^{-\lambda t}\left[2\,I_0(\lambda t) + \frac{2m}{\sqrt{m^2+1}}\sum_{n=1}^{+\infty}(-)^n(\zeta_1^n - \zeta_2^n)\gamma_l^n I_n\{\lambda \xi_l(t)\}\right], \text{...(185)}$$

Fig. 11.

Received current at condenser terminal due to direct e.m.f. E applied at the beginning of the line.



for small values of t greater than l/g.

In this case we have $|\zeta_1 \gamma_l| < 1$ for a comparatively long duration; therefore we had better use the above expression of i itself than change its form with the aid of (180). Fig. 11 is the graph of the received current due to the direct e.m.f. E applied at the beginning of the cable which possesses the above mentioned constants. From the curve we see that the current is zero until $t = l/g$ at which time it jumps to the value $2E\sqrt{C/L}$ $\varepsilon^{-\lambda l/g}$ suddenly. It then begins to increase.

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