A New Method of Testing Hardness.

By

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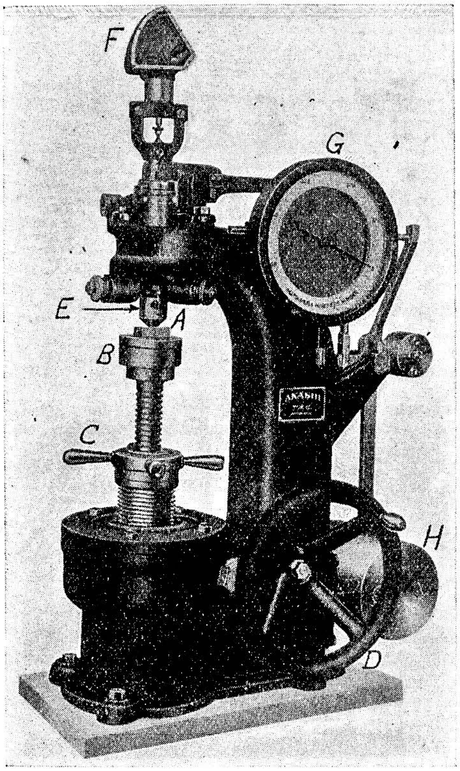
The methods of testing hardness by indentation now in use may be divided into two groups. In the one group a dimension of the indentation caused by the stamp under a constant load is measured, as in Brinell's and Rockwell's tests, and in the other the load required on the stamp to produce an indentation of a constant size is measured, as in Martens' process. By testing with a constant load the size of the indentation varies for different materials, the penetrated depth in a quenched steel being much smaller than that in a soft iron. Thus the tests of these materials are not carried out under the same geometrical conditions. Now as the hardness of a material increases in general with deformation, the consequence of testing with a constant load is that the hardness of different materials is measured in different stages of work-hardening. In this respect a method belonging to the second group appears to be more rational.

The author's method belongs to the second group. A diamond ball of 4 mm diameter is used as the stamp and while the material to be tested is being indented, the loads at two definite depths of indentation are measured.

Fig. 1 shows the author's hardness-meter.[1] A is the test piece, B the stand for it, which is raised or lowered quickly by turning a set of handles C. The slow motion of the stand, that is the motion during testing, is obtained by operating the hand wheel D. E is the vertical spindle, in the lower end of which a diamond stamp is embedded. F is the depth-meter designed by the author, indicating $\frac{1}{1000}$ mm. in 1 mm. It receives motion from three small vertical spindles surrounding the diamond stamp and resting on the surface of the test piece, G is the load-indicating dial,

(1) "Katasameter" made in Akashi Factory, Tokyo.

Fig. 1.



whose needle receives motion from a pendulum H. The pendulum being connected to the vertical spindle E through a link and a lever, is caused to turn out by the upward motion of E.

Experimenting with the machine just described the author found the relations between the depth h and the load P in different materials. Some of them are shown in Fig. 2, where points were plotted with h and the corresponding P as coordinates.

The relationship may be represented by a parabola of the equation :

$$P = ah + bh^2, \quad \dots\dots\dots\dots(1)$$

where a and b are constants.

These constants are determined by the values of two measurements, that is the values P_1 and P_2 for the depths h_1 and h_2 thus,

$$\frac{P_1}{h_1} = a + bh_1, \quad \frac{P_2}{h_2} = a + bh_2,$$

from which we have

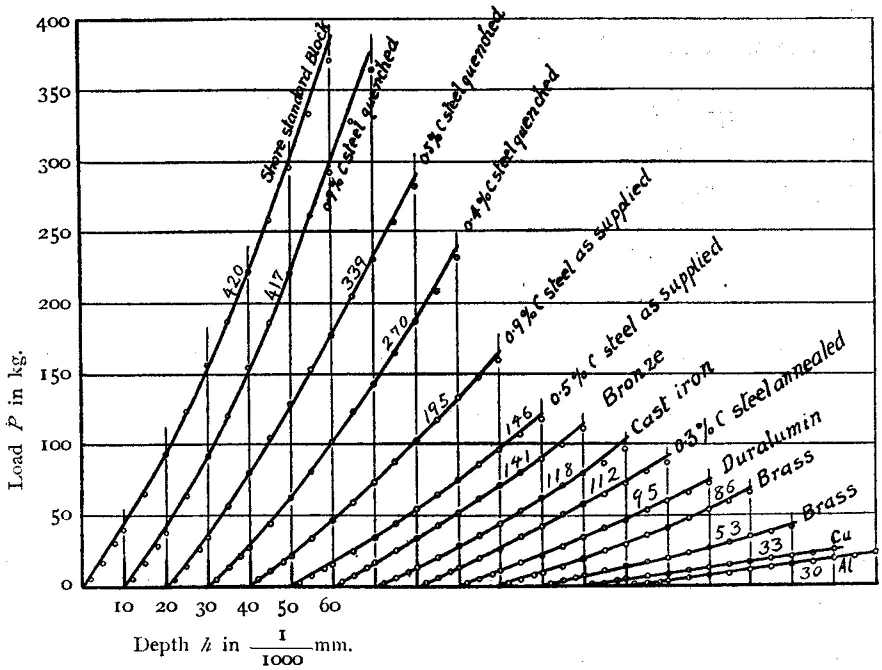
$$a = \frac{1}{h_2 - h_1}\left(-\frac{h_1}{h_2}P_2 + \frac{h_2}{h_1}P_1\right),$$

$$b = \frac{1}{h_2 - h_1}\left(\frac{P_2}{h_2} - \frac{P_1}{h_1}\right).$$

Taking $h_2 = 2h_1$ we have

$$\left.\begin{aligned} a &= \frac{1}{h_1}\left(-\frac{P_2}{2} + 2P_1\right), \\ b &= \frac{1}{2h_1^2}\left(P_2 - 2P_1\right). \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

Fig. 2. Load-depth Diagrams.



Putting these in (1) we get

$$P=\left(-\frac{P_2}{2}+2P_1\right)\frac{h}{h_1}+\left(\frac{P_2}{2}-P_1\right)\frac{h^2}{h_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

In Fig. 2 the parabola of (3) was drawn through the two points for $h_1=0.02$ mm and $h_2=0.04$ mm. It will be seen that the curve traces the other observation points almost exactly as far as $h=0.05$ mm.

To represent the relationship between h and P, Honda and Takahashi took the following exponential function(2):

$$P=khe^{ah},$$

but within $h=0.1$ mm (1) fits the experimental results better.

The author wishes now to make a proposal of taking as the hardness number the work to be done in producing an indentation of $h_3=0.05$ mm,

(2) The Journal of the Iron and Steel Inst. V. CIX, P. 323.

divided by the volume of the latter, a quantity of the dimension of kg per mm^2.

The work to be done is

$$W=\int_0^{h_3} P dh,$$

which with (1) becomes

$$W=\frac{ah_3^2}{2}+\frac{bh_3^3}{3},$$

while the volume of indentation is

$$V=\pi h_3^2\left(r-\frac{h_3}{3}\right)$$

$$\doteqdot \pi h_3^2 r,$$

with an error of only 0.83%, r being the radius of the indenting ball.

The author's hardness number is then

$$H=\frac{W}{V}=\frac{1}{2\pi r}\left(a+\frac{2}{3}bh_3\right). \qquad \text{(4)}$$

Putting in this the values of a and b in (2)

$$H=\frac{1}{2\pi r h_1}\left[\left(\frac{1}{3}\frac{h_3}{h_1}-\frac{1}{2}\right) P_2+\left(2-\frac{2}{3}\frac{h_3}{h_1}\right)P_1\right] \quad \text{...(5)}$$

or with $r=2$, $h_1=0.02$, $h_3=0.05$ mm

$$H=\frac{1}{0.24\pi}(P_1+P_2)\doteqdot \frac{4}{3}(P_1+P_2) \qquad \text{(6)}$$

In the author's machine the load-indicating dial is graduated to indicate $\frac{4}{3}$ times the actual load in kg, hence if Q_1 an Q_2 be the readings for $h_1=0.02$ and $h_2=0.04$ mm respectively, then Q_1+Q_2 will directly give the hardness number.

The numbers written along the P–h curves in Fig. 2 are the hardness numbers found for the respective materials.

Comparisons of the author's hardness number with the Brinell, Shore

Fig. 3. Comparison of the Author's Hardness No. with Brinell No.

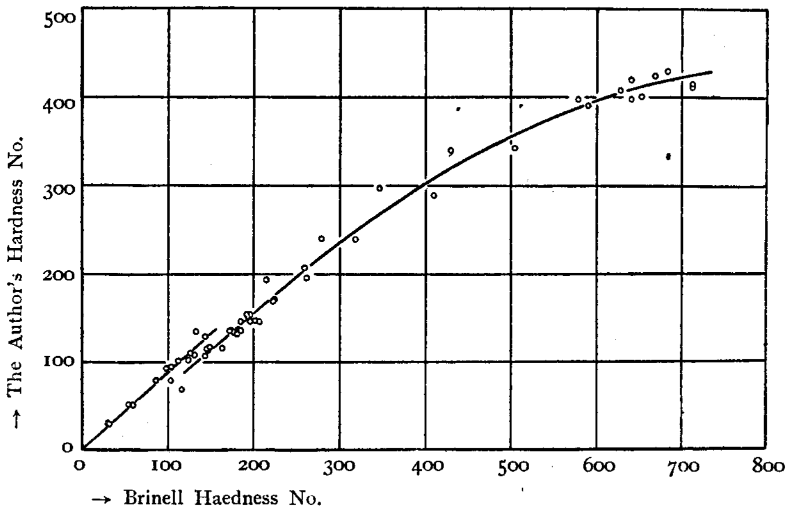
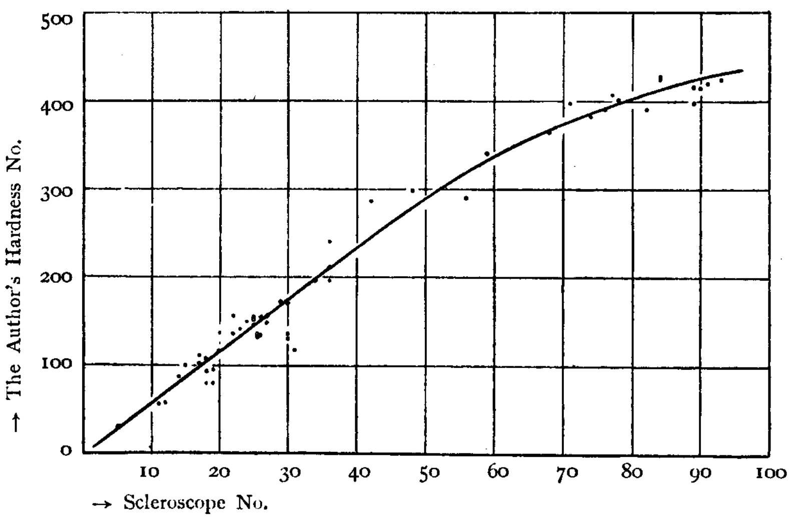


Fig. 4. Comparison of the Author's Hardness No. and Scleroscope No.



and Rockwell numbers are given in Figs. 3, 4 and 5 respectively. The discontinuity of the curve in Fig. 3 is due to the fact that in Brinell's method the softer materials are tested under a 500 kg load, instead of a 3000 kg.

Fig. 5. Comparison of the Author's No. with Rockwell No.

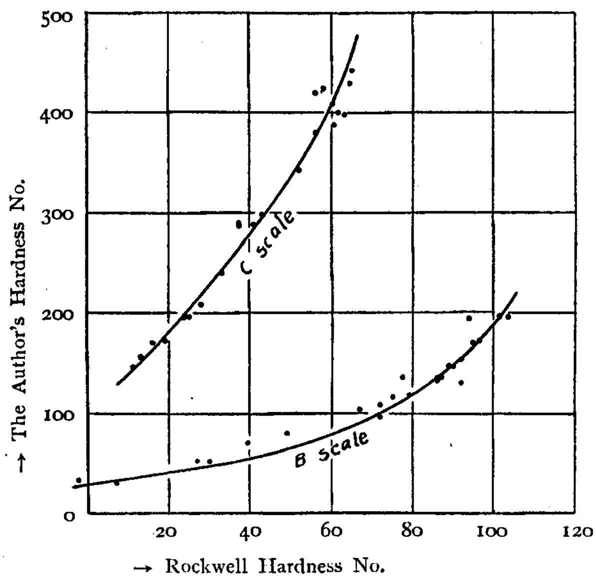
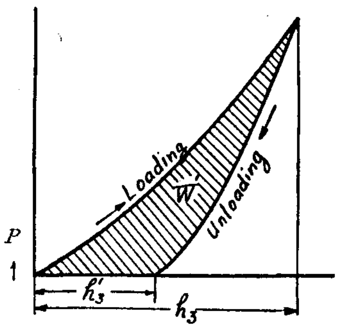


Fig. 6.



Let us now ask a question whether there is not a quantity, which is more appropriate to be taken as the index of hardness.

Let h_3' be the depth of the indentation remaining after the removal of load as shown in Fig. 6. Then the work W' represented by the hatched area is the net work expended in producing a permanent indentation of the depth h_3'. If V' be the volume of such indentation, then the quotient

$$H' = \frac{W'}{V'}$$

will perhaps be a more appropriate index of hardness, because the hardness, as is commonly taken, refers to the permanent indentation.

The scleroscope number is likely a quantity, which is kindred to the work $W - W'$.

Assuming that the permanent indentation of the depth h_3' is also spherical and that its diameter undergoes no change by the removal of load, we have

$$rh_3 = r'h_3', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where r' is the radius of the permanent spherical indentation. The volume of the latter is

$$V' = \pi r' h_3'^2,$$

which with the above relation becomes

$$V' = \pi r h_3 h_3'$$

Next if we take

$$\frac{W'}{W} = \frac{h_3'}{h_3}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$H' = \frac{W'}{V'} = W\frac{h_3'}{h_3}\,\frac{1}{\pi r h_3 h_3'} = \frac{W}{\pi r h_3^2} = H,$$

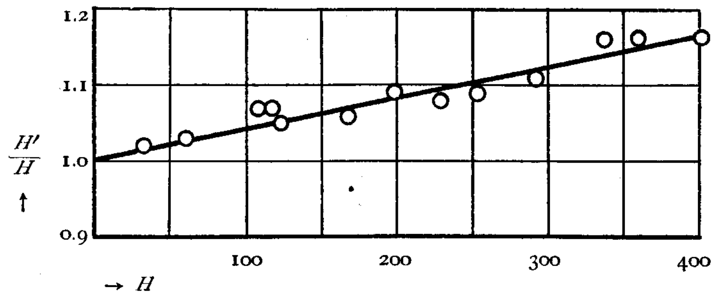
so that the two indices of hardness become identical. By experiments, however, the assumed proportionality (8) does not hold. The results are as given in Table 1.

Table 1.

Test piece	H	W' mmkg.	W mmkg.	$\frac{W'}{W}$	h_3' $\frac{1}{1000}$ mm	h_3 $\frac{1}{1000}$ mm	$\frac{h_3'}{h_3}$	$\frac{H'}{H} = \frac{W'}{W} \div \frac{h_3'}{h_3}$
0.7% C steel Q.	401	1.985	6.227	0.319	13.75	50	0.275	1.16
Ni steel Q.	360	2.893	5.565	0.520	22.5	„	0.45	1.16
0.5% C steel Q.	337	3.199	5.261	0.610	26.25	„	0.525	1.16
Ni-Cr steel Q.	291	2.884	4.529	0.637	28.75	„	0.575	1.11
0.2% C steel Q.	253	2.883	3.952	0.733	33.5	„	0.67	1.09
Cr steel Q.	228	2.688	3.548	0.758	35	„	0.70	1.08
0.9% C steel R.	199	2.372	3.086	0.768	35.25	„	0.705	1.09
Ni-Cr steel R.	167	2.188	2.574	0.850	40	„	0.80	1.06
0.4% C steel A.	123	1.700	1.912	0.888	42.5	„	0.85	1.05
Cast iron	116	1.366	1.803	0.756	35.25	„	0.705	1.07
Brass	107	1.397	1.659	0.842	29.5	„	0.79	1.07
Lautal	62	0.677	0.771	0.878	42.5	„	0.85	1.03
Copper	33	0.502	0.515	0.974	48	„	0.96	1.02

Q. = quenched, A. = annealed, R. = as received.

Fig. 7.



In Fig. 7 points are plotted with H and $\frac{H'}{H}$ in the table as coordinates and a straight line is drawn to represent the mean relation, from which we may write down:

$$H' = \left(1 + \frac{H}{2400}\right) H.$$

Owing, however, to the questionableness of the assumption about the form and the diameter of the permanent indentation, the author hesitates the use of this formula. H' being a quantity not simply determined by experiment, there is no way but to take the author's number H as an approximate substitute for it.

The author's method is useful in measuring the hardness of a material having a cylindrical or a spherical surface, which can not be properly determined by any other method. Lately a question arose about the surface hardness of an electric trolley wire. A wire of 8 mm diameter was taken as the sample. By Rockwell's test the cylindrical surface showed a hardness number (B scale) about 23% smaller than that of the cross section. The wire being manufactured by drawing, the surface ought to have a greater hardness than the interior, but the contrary was the result. By the author's process the result was alike, though the difference between the hardness numbers was only about 14%.

Strange as it may appear, a little consideration will make the matter clear. The indenting stamp penetrates easier into a cylindrical than into a plane surface. In Rockwell's test the stamp penetrates deeper under a constant load and in the author's test the work done for a constant depth of indentation is less, these being respectively the causes of a smaller hardness number apparently resulting.

Now in the author's test if we know the volume of indentation V_c made on the cylindrical surface to the depth h_3, the correction of the apparent hardness number is easily made by multiplying it by $\frac{V}{V_c}$.

The theoretical deduction of V_c will be given in the appendix. Its value is found approximately

$$V_c = \pi h_3^2 r \sqrt{\frac{R}{R+r}} = V \sqrt{\frac{R}{R+r}},$$

where r is the radius of the indenting ball and $=2$ mm and R that of the wire to be tested. The results of a few calculations are given in Table 2.

Table 3 is the result of the comparative test made with trolley wires manufactured by the Fujikura Co.

Table 2.

Dia. of wire $2R$ in mm	$\frac{V}{V_c} = \sqrt{\frac{R+r}{R}}$
8	1.225
10	1.18
12	1.155
15	1.125
20	1.09
30	1.064

Table 3.

Dia. mm	Rockwell No. (B)		Author's No.	
	Cross Section	Cylindrical Surface	Cross Section	Cylindrical Surface
8	57.5	40.4	112	95.5
	58.8	48.0	113	96.0
	58.8	46.5	112.5	99.0
	Mean 58.3	Mean 44.9	Mean 112.5	Mean 96.83 Corrected 118.9
15	51.5	49.6	110	95
	53.5	51.2	105	100
	53.4	44.1	110	99
	Mean 52.8	48.3	Mean 108.3	Mean 98 Corrected 110

Thus by the author's test it is revealed that, as was expected, the surface of the wires is harder than the interior.

Fig. 8 shows the steel rider used in testing wire. This is to provide a plane surface as the seat of the three small spindles transmitting motion to the depth-meter.

For the assertion that the apparent hardness number is corrected simply by multiplying it by the factor $\frac{V}{V_c}$, a proof may be necessary that the cylindrical and plane surfaces cut out from the same material show the equal real hardness numbers. The following test was made :— From a piece of mild steel were cut out the cylinder C and the pieces A and B with plane surface on the side next to the cylinder as shown in Fig. 9, (the depth and the feed of the finishing cut being taken possibly small to avoid any work-hardening of the cut surfaces), and they were tested for hardness along the lines 11 and 22.

The results are recorded in Table 4, from which we may conclude that the assertion about the correction of the apparent hardness number was right.

Fig. 8.

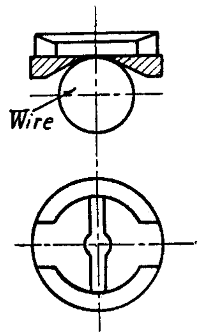
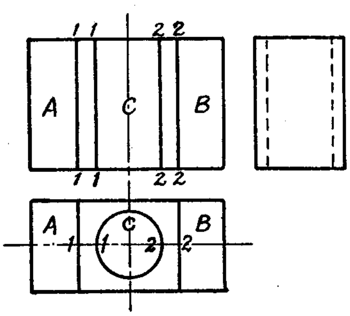


Table 4.

Dia. of C	A_{11}	C_{11}	C_{22}	B_{22}
mm	126	98	98	123
	123	98	101	119
	123	102	96	128
6.75	121	95	100	124
	Mean 123.3	98.3	98.8	123.5
	Corrected	124	124.7	
	123	107	104	125
	129	108	102	129
	125	104	107	129
10.85	126	109	110	123
	Mean 125.2	107	105.8	126 8
	Corrected	125.2	123.8	
	125	114	113	119
	122	108	116	127
	129	113	110	126
15.05	125	114	111	124
	Mean 125.3	112.3	112.5	124
	Corrected	126·3	126.6	
	128	112	116	126
	129	110	114	122
	126	115	111	128
18.95	122	116	114	125
	Mean 126.3	113.3	113.8	125.3
	Corrected	124.5	125	

Fig. 9.



In case of testing hardness of a sphericul surface the correction factor, by which the apparent hardness number is to be multiplied, is found as

$$\frac{V}{V_s}=\frac{R+r}{R},$$

where R is the radius of the spherical surface.

A further application of the author's method is the measurement of hardness of a thin sheet metal. When a thin sheet is pressed by the indenting ball, it tends to bend more or less concave towards the upside. This bending disturbs the measurement of the depth of indentation. Moreover when the thickness is very thin, the mensurement of load is affected by the hardness of the seat, on which the specimen is placed. The first disturbance may be avoided by placing the specimen on a spherical seat and making it to contact with the seat at a small area just beneath the indenting ball as shown in Fig. 10.

Fig. 10.

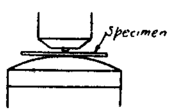
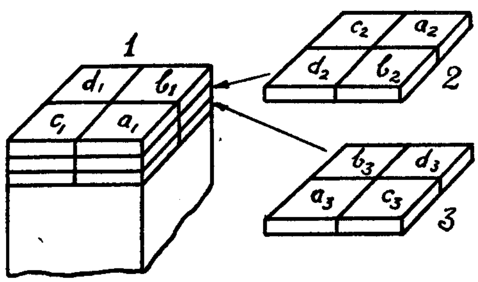


Fig. 11.



About the second disturbance the following test was made :— From a mild steel square bar were cut out the three slices 1, 2 and 3 and each slice was cut into four pieces *a*, *b*, *c* and *d* as shown in Fig. 11. All *a* pieces were finished to 3 mm, *b* pieces to 2 mm, *c* pieces to 1.5 mm and *d* pieces to 1 mm in thickness. They were tested for hardness, being placed on a steel spherical seat of 80 mm radius. The results are recorded in Table 5.

From the table it will be seen that down to 1 mm thickness there appears no influence of the seat.

Table 5.

Thickness mm	Piece	Reading Q_1	Reading Q_2	H	Piece	Reading Q_1	Reading Q_2	H	Piece	Reading Q_1	Reading Q_2	H
3	a_1	39	86	125	a_2	40	90	130	a_3	41	90	131
		39	87	126		40	88.5	128.5		39	87	126
		39.5	88	127.5		39.5	89	128.5		40	87	127
		Mean		126.2				129				128
2	b_1	40	89	129	b_2	39	88	127	b_3	39	89	128
		39	87	126		40	90	130		39	88	127
		38	86	124		38	88	126		40	90	130
		Mean		126.3				127.7				128.3
1.5	c_1	38	87	125	c_2	39	89	128	c_3	39	87.5	126.5
		39	88	127		39	89	128		39	88	127
		40.5	90	130.5		40	90	130		40	89	129
		Mean		127.5				128.7				127.5
1.0	d_1	40	85	125	d_2	39	86	125	d_3	40	87	127
		38.5	87	125.5		40	88	128		40	88	128
		39.5	87	126.5		41	88	129		40	86	126
		Mean		125.7				127.3				127

Table 6.

Sheet No.	0.3 mm Q_1	0.3 mm Q_2	2.5 mm Q_1	2.5 mm Q_2
1	28	74	30	67
	29	76	30	67
	29	75	32	69
	31	80	32	70
2	29	75	31	68
	29	76	31	69
	30	75	31	68
	30	80	29	64
3	31	78	32	70
	31	76	31	69
	30	75	30	67
	31	74	31	68

Further for the sheets of a thickness less than 1 mm tests were made with sheet duralumin especially manufactured for the test purpose by Sumitomo Shindo Kokwan Co. The sheets were of the thicknesses 2.5, 1.0, 0.5 and 0.3 mm. All sheets, after being rolled, were heat-treated in the same process, so as to obtain practically the same hardness in them.

Prelimiuarily Q_1 and Q_2 for $h_1=0.02$ and $h_2=0.04$ mm were obtained for the sheets of 0.3 and those of 2.5 mm thickness as in Table 6. Here the influence of the seat on the values of Q_2 of the 0.3 mm sheet is clearly seen.

To avoid the influence we have to take the depth of indentation smaller.

Table 7.

Sheet No.	0.3 mm			0.5 mm			1.0 mm				2.5 mm			
	Q_5	Q_{10}	Q_{20}	Q_5	Q_{10}	Q_{20}	Q_5	Q_{10}	Q_{20}	Q_{40}	Q_5	Q_{10}	Q_{20}	Q_{40}
1	7	13	28	7	14	31	7	14	30	68	7	14	30	67
	7	14	29	7	14	29	7	14	30.5	69	7	14	30	67
	6	13	29	6.5	13.5	30	7	14.5	30.5	68	7	15	32	69
	6.5	14	31	7	14	29	6.5	13.5	30	69	7	14.5	32	70
2	7	14	29	7	14	30	6.5	13.5	30	66	7	14	31	68
	6.5	13	29	6	13	29	6.5	14	29	65	7	14.5	31	69
	7	14	30	6	13	29	6.5	13.5	30	67	7	14	31	68
	6	14	30	7	14	30	7	14	31	69	6.5	13	29	64
3	7	14	31	7	14	31	6.5	13	29	65	7.5	15	32	70
	7	14	31	7	14	29	6.5	13	30	67	7	14.5	31	69
	7	14	30	6.5	13	28	7	14	30	66	7	14	30	67
	7	14	31	7	14	30	7	14	30	65	7	15	31	68
4	7	14	31	6.5	14	30	7	14	31	69	7	15	31	68
	6	13	29	6.5	13.5	30	7	14	31	70	7	15	32	69
	6	13.5	30	6	13	30	7	15	32	71	6.5	14	29	66
	6.5	14	30	6	13	30	7	15	33	72	7	14	31	68
5	6.5	13	30	7	14	31	7	15	32	70	7.5	16	34	70
	6.5	13	29	7	14	32	6.5	14	31	68	7	14	31	67
	6.5	13	29	6.5	13	30	7	15	32	71	7	15	33	70
	7	13	29	6.5	14	30	7	15	31	70	7	15	33	69
6	6.5	12.5	28	6	13	30	6.5	14	31	68	7	14.5	31	67
	6.5	13	29	7	14	31	6.5	14	30	66	7	14	31	67
	6.5	13.5	30	6.5	13	30	6.5	14	29	66	7	14	31	67
	7	14	31	6	12.5	29	7	14	30	67	7.5	15	32	68
7	6.5	13	29	7	13.5	30	7	14	31	69	7	14	31	67
	6.5	13	30	6.5	13	30	6.5	14	30	68	7.5	15	32	69
	7	14	31	6	13	29	6.5	13	30	66	7	14	31	67
	6.5	13	29	7	14	31	7	14	30	67	6.5	14	30	66
8	7	14	30	7	14	31	7	14	31	68	7	14	31	67
	6.5	13.5	30	7	14	31	7	15	31	68	6.5	14	31	67
	6.5	13	28	7	14.5	31	7	14	30	67	7	14	31	66
	6.5	13	29	7	14	31	7	15	31	68	6.5	13	29	64
9	6	13	27	6.5	13	29	7	14	30	65	7	14	30	66
	6.5	13.5	30	6	13	29	7	14	29	65	6	13	29	64
	7	14	32	6	13	28	7	14	30	64	7	14	31	67
	7	14	31	6	13	30	7	15	30	66	7	14.5	31	67
10	7	13.5	30	7	14	30	7	14	30	64	7	14	31	68
	6	13	30	6.5	13	29	7	14	30	65	6.5	13	30	66
	6	12.5	29	6	13	29	7	14	30	64	7	14	31	68
	7	14	31	7	14	31	7	14	30	65	6	13	29	65

In Table 7 are given the readings of the load indicator corresponding to $h=0.005$, 0.01, 0.02 and 0.04 mm for the sheets of different thicknesses. They are denoted here by Q_5, Q_{10}, Q_{20} and Q_{40} respectively.

By (5), if the ratios $h_1 : h_2 : h_3$ remain unchanged, the hardness number referring to the indentation of a 0.0125 mm depth is found as

$$H_{12.5}=4(Q_5+Q_{10})$$

and that referring to the indentation of a 0.025 mm depth as

$$H_{25}=2(Q_{10}+Q_{20}).$$

In Table 8 are given the hardness numbers as the mean of four tests in Table 7.

Table 8.

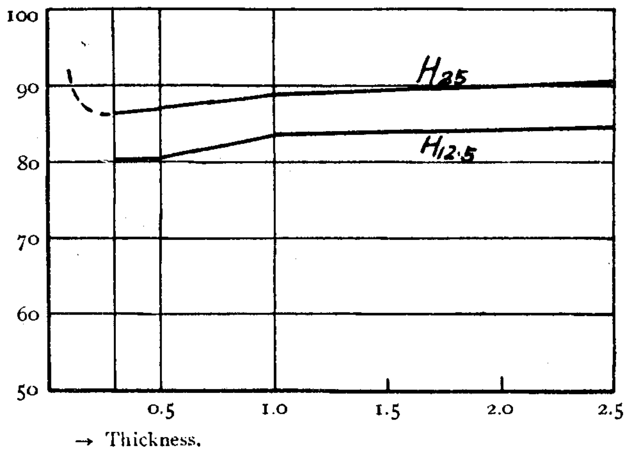
Sheet No.	0.3 mm		0.5 mm		1.0 mm			2.5 mm		
	$H_{12.5}$	H_{25}	$H_{12.5}$	H_{25}	$H_{12.5}$	H_{25}	H_{50}	$H_{12.5}$	H_{25}	H_{50}
1	80.5	85.5	83	87.3	83.5	88.5	98.8	85.5	90.8	99.3
2	81.5	86.5	80	86	81.5	87.5	96.8	83	88.8	97.8
3	84	89.5	82.5	86.5	81	86.5	95.5	87	91.3	99.5
4	80	87.3	78.5	86.8	86	92.5	102.3	85.5	90.5	98.5
5	78.5	84.5	82	89	86.5	92.5	101.3	88.5	95.5	101.8
6	79.5	85.5	78	86.3	82.5	88	96.8	86	91.3	98.5
7	79.5	86	80	86.8	82	88	97.8	85	90.5	98.3
8	80	85.3	84.5	90.3	86	90.5	98.5	82	88.5	96.5
9	81	88.3	76.5	84	85	88	94.8	82.5	88.3	96.3
10	79	86.5	80.5	86.5	84	88	94.5	80.5	87.5	97
Mean	80.4	86.5	80.6	87	83.8	89	97.7	84.6	90.3	98.4

H_{50} is the standard hardness number hitherto denoted by H.

In Fig. 12 points are plotted taking the thickness as abscissa and the hardness numbers $H_{12.5}$ and H_{25} as ordinates. From the figure it will be seen that the sheets were not made to equal hardness, thicker one being a little harder.

If there be any influence of the seat on the measurement of load, the left hand end of the H_{25} curve should bend upwards as shown by the dotted line. Down to 0.3 mm thickness this tendency is not observed. Therefore we may conclude that the measurement H_{25} for 0.3 mm sheet is,

Fig. 12.



as all the other measurements, free from an influence of the seat and it may be taken as the number for the comparison of hardness of duralumin sheets of different, down to 0.3 mm thicknesses.

It is possible to find the standard hardness number H_{50} in terms of Q_{10} and Q_{20}. Putting in (5) $r=2$, $h_1=0.01$ and $h_3=0.05$ mm we get

$$H_{50}=\frac{1}{0.24\pi}(7P_2-8P_1)$$

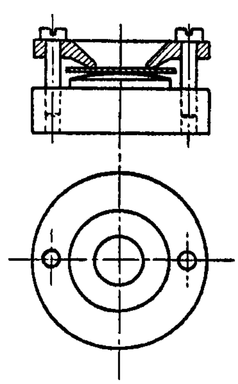
$$\doteqdot \frac{4}{3}(7P_2-8P_1)=7Q_{20}-8Q_{10}.$$

This formula, being based on the assumption that the P–h curve is a parabola, may lead to an erroneous result, because the curve is only approximately parabolic and h_3 is too great in comparison with h_2.

As seen from Table 8, H_{50} is about 10% greater than H_{25} in duralumin sheets.

Fig. 13 shows the spherical seat used in testing the duralumin sheets. It is provided with a ring, by which the specimen is pressed lightly against the seat.

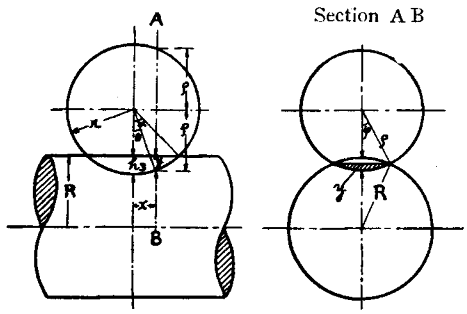
Fig. 13.



APPENDIX:

Deduction of the Volume of Indentation made on a Cylindrical Piece by a Spherical Stamp.

First we will find the area of overlap of the two circles on a section AB.



$$(R+\rho-y)^2+\rho^2-2(R+\rho-y)\rho\cos\varphi=R^2$$

or $$\cos\varphi=\frac{\rho^2+\frac{y^2}{2}+R\rho-Ry-\rho y}{\rho(R+\rho-y)}$$

As y is, in the present problem, very small compared with the other dimensions, neglecting y^2 we have

$$\cos\varphi=\frac{\rho-y}{\rho\left(1-\frac{y}{R+\rho}\right)}\quad\cdots\cdots\cdots\cdots(1)$$

The hatched area $=\rho^2(\varphi-\sin\varphi\cos\varphi)$.

In an indentation of $h_3=0.05$ mm, φ is very small, so that we may put

$$\sin\varphi\cos\varphi=\varphi\left(1-\frac{\varphi^2}{2}\right)\left(1-\frac{\varphi^2}{6}\right)\doteqdot\varphi\left(1-\frac{2}{3}\varphi^2\right).$$

Then $$\varphi-\sin\varphi\cos\varphi=\frac{2}{3}\varphi^3$$

Again to express this in terms of $\cos\varphi$

$$\cos\varphi=1-\frac{\varphi^2}{3}\qquad\therefore\quad\varphi^3=2\sqrt{2}\,(1-\cos\varphi)^{\frac{3}{2}}$$

Therefore $$\varphi-\sin\varphi\cos\varphi=\frac{4\sqrt{2}}{3}(1-\cos\varphi)^{\frac{3}{2}},$$

which with (1)

$$=\frac{4\sqrt{2}}{3}\left[1-\frac{\rho-y}{\rho\left(1-\frac{y}{R+\rho}\right)}\right]^{\frac{3}{2}}=\frac{4\sqrt{2}}{3}\left(\frac{R}{R+\rho-y}\cdot\frac{y}{\rho}\right)^{\frac{3}{2}}.$$

Therefore the hatched area

$$=\frac{4\sqrt{2}}{3}\left(\frac{Ry}{R+\rho-y}\right)^{\frac{3}{2}}\rho^{\frac{1}{2}}$$

The total area of overlap is then

$$F=\frac{4\sqrt{2}}{3}y^{\frac{3}{2}}\left[\left(\frac{R}{R+\rho-y}\right)^{\frac{3}{2}}\rho^{\frac{1}{2}}+\left(\frac{\rho}{R+\rho-y}\right)^{\frac{3}{2}}R^{\frac{1}{2}}\right]$$

$$=\frac{4\sqrt{2}}{3}(R+\rho)y^{\frac{3}{2}}\rho^{\frac{1}{2}}\sqrt{\frac{R}{(R+\rho-y)^3}}$$

$$=\frac{4\sqrt{2}}{3}\sqrt{\frac{R}{(R+r-h_3)^3}}(R+\rho)y^{\frac{3}{2}}\rho^{\frac{1}{2}}$$

or if

$$K=\frac{4\sqrt{2}}{3}\sqrt{\frac{R}{(R+r-h_3)^3}},\quad F=K(R+\rho)y^{\frac{3}{2}}\rho^{\frac{1}{2}}.$$

The volume of the indentation in question is

$$V_c=\int F dx=K\int(R+\rho)y^{\frac{3}{2}}\rho^{\frac{1}{2}}dx,$$

but

$$x=r\sin\theta,\qquad dx=r\cos\theta d\theta,$$

$$y=r(\cos\theta-\cos\alpha),\qquad \rho=r\cos\theta,$$

therefore

$$V_c=2K\left[Rr^3\int_0^\alpha(\cos\theta-\cos\alpha)^{\frac{3}{2}}(\cos\theta)^{\frac{3}{2}}d\theta\right.$$
$$\left.+\,r^4\int_0^\alpha(\cos\theta-\cos\alpha)^{\frac{3}{2}}(\cos\theta)^{\frac{5}{2}}d\theta\right].$$

Putting

$$(\cos\theta-\cos\alpha)^{\frac{3}{2}}=\left(\frac{\alpha^2-\theta^2}{2}\right)^{\frac{3}{2}},\quad (\cos\theta)^{\frac{3}{2}}=\left(1-\frac{\theta^2}{2}\right)^{\frac{3}{2}}\doteq 1-\frac{3}{4}\theta^2,$$

$$(\cos\theta)^{\frac{5}{2}}=\left(1-\frac{\theta^2}{2}\right)^{\frac{5}{2}}\doteq 1-\frac{5}{4}\theta^2$$

and integrating

$$V_c=\frac{K\pi}{16\sqrt{2}}\left[Rr_3\left(3\alpha^4-\frac{3}{8}\alpha^6\right)+r^4\left(3\alpha^4-\frac{5}{8}\alpha^6\right)\right].$$

Putting the value of K

$$V_c=\frac{\pi}{4}\left[Rr^3\left(\alpha^4-\frac{\alpha^6}{8}\right)+r^4\left(\alpha^4-\frac{5}{24}\alpha^6\right)\right]\sqrt{\frac{R}{(R+r-h_3)^3}}.$$

Again as $\quad \dfrac{h_3}{r}=(1-\cos\alpha)=\dfrac{\alpha^2}{2},\qquad \therefore\ \alpha^4=4\dfrac{h_3^2}{r^2},\quad \alpha^6=8\dfrac{h_3^3}{r^3}$

$$V_c = \pi h_3^2 r\left[1 - \frac{3R+5r}{12(R+r)}\,\frac{h_3}{r}\right]\sqrt{\frac{R(R+r)^2}{(R+r-h_3)^3}}$$

$$\doteqdot \pi h_3^2 r\left[1 - \frac{h_3}{R+r}\,\frac{3R-13r}{12r}\right]\sqrt{\frac{R}{R+r.}}.$$

Neglecting $\dfrac{h_3}{R+r}\,\dfrac{3R-12r}{12r}$ against 1 we finally get

$$V_c = \pi h_3^2 r\sqrt{\frac{R}{R+r}}.$$

The error arising from the approximation is so small as

-0.49% for $R = 4$ mm,

-0.33 ,, ,, $R = 5$,, ,

$+0.06$,, ,, $R = 10$,, .