On the Conformal Transformations of Circles, Circular Arcs, and Straight Lines.*

By Busuke Hudimoto.

In this paper, a few different methods of the conformal transformations of a circular arc and flap wing section from a circle will be discussed. And as further applications of these methods the problems of biplanes and special latticed wings will be treated.

In hydrodynamics and especially in aerodynamics it is common to use the conformal transformations to solve the problems of two-dimensional flow of a perfect fluid. In aerodynamics the most remarkable and commonly used method to calculate the lift of the aerofoil is the transformation used in the case of the Joukowski section. A circle in the z-plane is transformed into a portion of a straight line or circular arc in the ζ-plane by the relation of the following form:

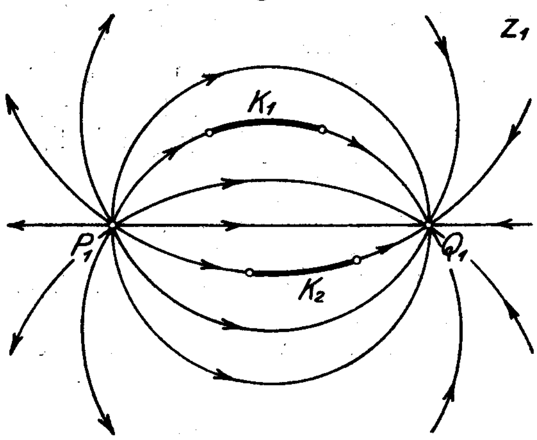
$$\zeta = z + \frac{r^2}{z}.$$

A more generalized form of this is the v. Kármán and Trefftz's[1] transformation; also, the transformations for arbitrary aerofoil sections were attempted by W. Müller,[2] F. Höhndorf[3] and Th. Theodorsen.[4] So there is nothing new to be solved, but in the present paper the author will discuss the problems of the transformation of the circle into a circular arc, straight line or their combinations by a somewhat different interpretation with the purpose of solving the problems of the arbitrary biplane**[5] with sections of straight lines or circular arcs and also the problems of the latticed wings with unequal pitches and unequal chords.

1. General Principles.

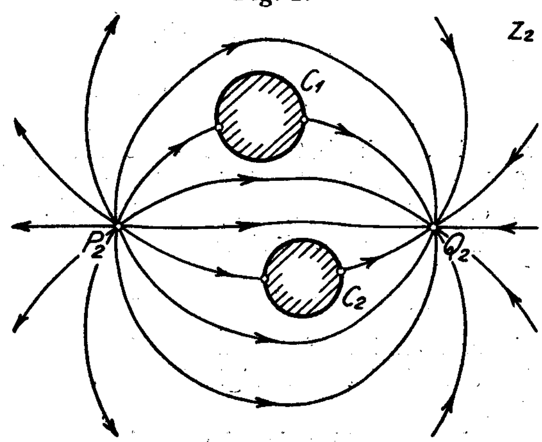
The problems of the two-dimensional irrotational and steady flow of a perfect fluid around a circle can be solved rather easily and also the flow around two circles can be solved exactly by introducing the elliptic functions[6] and we make use of these flow patterns. And let us consider in the z_1-plane a source and a sink of the same strength, as in Fig. 1. The source placed at P_1 and the sink at Q_1. As the fluid flows out from the point P_1 towards the point Q_1, the stream lines of this flow are all circles passing through both P_1 and Q_1. On the other hand we consider a circle or two circles in the z_2-plane as in Fig. 2, in which the case of two circles is shown. We denote these circles by C_1 and C_2,

Fig. 1.



and we consider a flow outside of these circles due to a source at the point P_2 and a sink at the point Q_2 of the same strength as in the z_1-plane. In Fig. 2 the probable stream lines are shown. Let the complex velocity potentials of the flow in the z_1-plane and z_2-plane be denoted by W_1

Fig. 2.



and W_2 respectively, then the conjugate complexes of the velocities are expressed by $\frac{dW_1}{dz_1}$ and $\frac{dW_2}{dz_2}$. If this flow pattern in the z_2-plane is conformally transformed into that of the z_1-plane, then

$$\frac{dW_1}{dz_1} = \frac{dW_2}{dz_2} \frac{dz_2}{dz_1}$$

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** C. Ferrari has already treated the same problems of the biplanes with and without stagger of both straight line section and circular arc section, but it seems that the author's methods are somewhat different from those of Ferrari so they are published here again.

or

$$\frac{dz_1}{dz_2}=\frac{dW_2}{dz_2}\,\frac{dz_1}{dW_1}.$$

Integrating the above relation we get the functional relation between z_1 and z_2. And then by this relation the circles C_1 and C_2 in the z_2-plane which form the portions of the stream lines are transformed into two circular arcs K_1 and K_2 which also correspond to the portions of the stream lines in the z_1-plane. In the case of two circles we may more conveniently transform the outside region of these circles into a rectangular region of the s-plane following Lagally, and consider the flow by a source and a sink in this region. If the complex velocity potential in the s-plane be denoted by W_s then

$$\frac{dW_1}{dz_1}=\frac{dW_s}{ds}\,\frac{ds}{dz_1}.$$

Integrating the above relation we get the functional relation between z_1 and s.

In the above lines we treated the case of a source and a sink, but in the same manner we can find the functional relations between the z_1 and z_2 or s-plane by a properly chosen pair of irrotational vortices or doublets.

2. Cases of one Circle.

At first we investigate the problems of transforming a circle into a straight line or their combination.

(a) We consider in the z_2-plane a circle with radius 1 — in the following lines we take the radius of the circle to be 1 for simplicity — with its centre at the origin, and place two sources with equal strength of q at the points $z_2=i$ and $-i$, and a sink of the same strength at the origin, Fig. 3, then W_2 is expressed as follows:

$$W_2=\frac{q}{2\pi}\log(z_2-i)+\frac{q}{2\pi}\log(z_2+i)-\frac{q}{2\pi}\log z_2$$

or

$$\frac{dW_2}{dz_2}=\frac{q}{2\pi}\left\{\frac{1}{z_2-i}+\frac{1}{z_2+i}-\frac{1}{z_2}\right\}.$$

And in this case the circle of radius 1 formes one of the stream lines.

Now if in the z_1-plane a source of strength q be placed at the origin, then W_1 is expressed by

$$W_1=\frac{q}{2\pi}\log z_1,$$

and in this case the real axis of the z_1-plane is a stream line of the flow and

$$\frac{dW_1}{dz_1}=\frac{q}{2\pi}\,\frac{1}{z_1}.$$

Combining these velocities we get

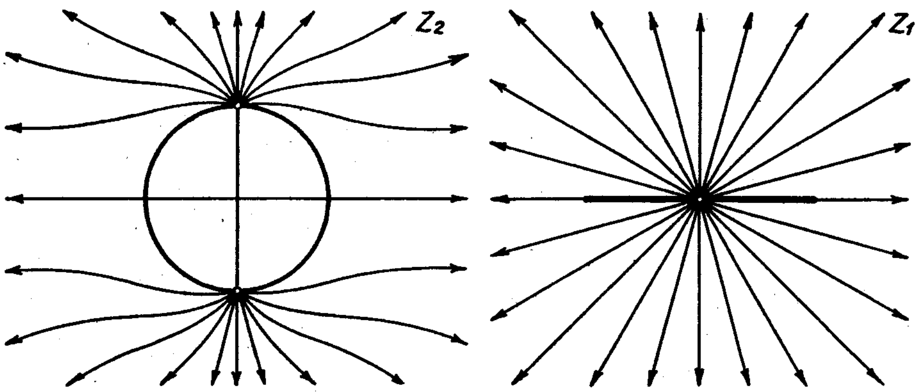
$$\frac{dz_1}{z_1}=\left\{\frac{1}{z_2-i}+\frac{1}{z_2+i}-\frac{1}{z_2}\right\}dz_2,$$

$$\therefore\quad \log z_1=\log c\,\frac{(z_2-i)(z_2+i)}{z_2},$$

$$\therefore\quad z_1=c\left(z_2+\frac{1}{z_2}\right).$$

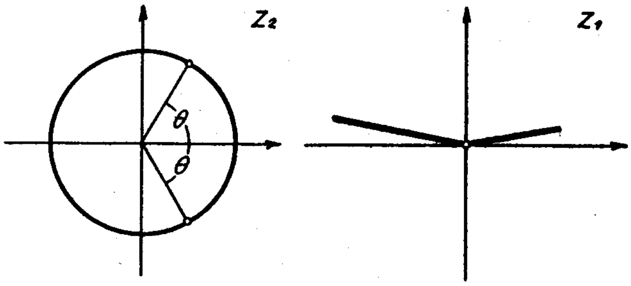
Here c s the integration constant, which determines the ratio of the sizes of the figures in these planes, and this is exactly the same as the usual transformation applied in aerodynamics.

Fig. 3.



(b) In the same way as in the previous case we place two sources q_1 and q_2 on the circle at the points $z_2=e^{i\theta}$ and $e^{-i\theta}$, but with unequal strengths, Fig. 4. Then

Fig. 4.



$$W_2=\frac{q_1}{2\pi}\log(z_2-e^{i\theta})+\frac{q_2}{2\pi}\log(z_2-e^{-i\theta})-\frac{q_1+q_2}{4\pi}\log z_2$$

$$=\frac{q_1+q_2}{4\pi}\log\frac{(z_2-e^{i\theta})^k(z_2-e^{-i\theta})^{2-k}}{z_2},$$

where $k=\dfrac{2q_1}{q_1+q_2}$.

This circle corresponds to a stream line.

In the z_1-plane place a source of strength $\dfrac{q_1+q_2}{2}$ on the origin, then

$$W_1=\frac{q_1+q_2}{4\pi}\log z_1,$$

from these equations we get

$$z_1=c\,\frac{(z_2-e^{i\theta})^k(z_2-e^{-i\theta})^{2-k}}{z_2}.$$

Again c is the integration constant and by this relation the circle in the z_2-plane is transformed into a straight line aerofoil with hinged flap. And this is exactly the same as that deduced by the method of Schwarz-Christoffel's transformation.[7]

and the circle of radius 1 is a stream line.

In the z_1-plane we place a source and a sink at the points $z_1=-a$ and a with strength $\dfrac{q}{2}$, then

$$W_1=\frac{q}{4\pi}\log(z_1+a)-\frac{q}{4\pi}\log(z_1-a).$$

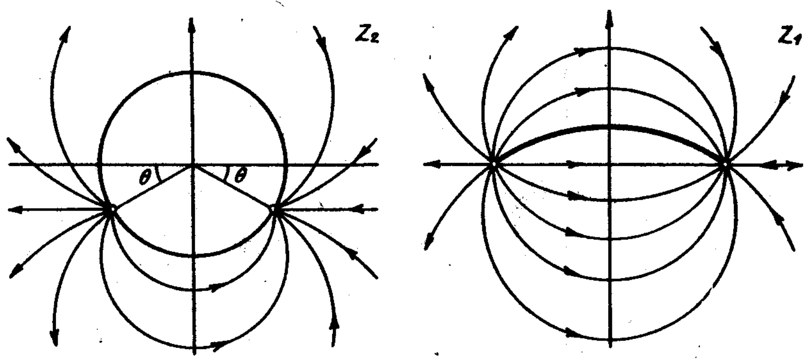
From above two equations we get

$$\frac{z_1+a}{z_1-a}=\left(\frac{z_2+e^{i\theta}}{z_2-e^{-i\theta}}\right)^2$$

By this relation the circle in the z_2-plane is transformed into a segment of a circular arc from the point $z_1=-a$ to the point $z_1=a$.

If instead of the strength $\dfrac{q}{2}$, we place a source and a sink of strength $\dfrac{q}{n}$ at the points $z_1=-a$ and a, we will get the following relation

Fig. 5.



$$\frac{z_1+a}{z_1-a}=\left(\frac{z_2+e^{i\theta}}{z_2-e^{-i\theta}}\right)^n.$$

This is the generalized transformation by v. Kármán and Trefftz.[1] And when $n=1$, the circle in the z_2-plane is transformed into a circle in the z_1-plane.

In the same way if we take a source and a sink of the strength q outside of the circle in the z_2-plane, as in Fig. 6, then

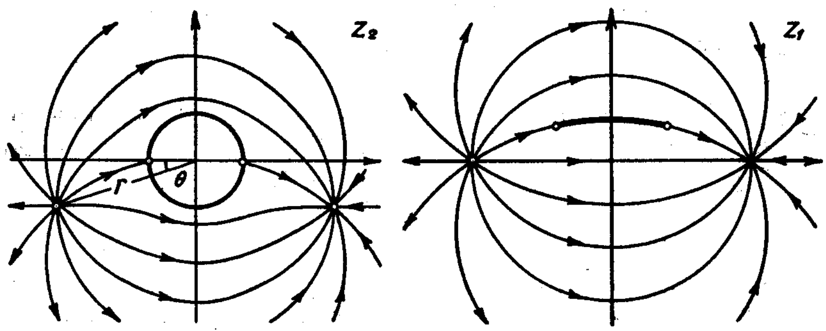
3. Cases of one Circle.

We shall now treat the problems of transforming a circle into circular arc or their combination.

(a) Consider a circle in the z_2-plane with its centre at the origin, and a source and a sink with same strength q at the points $z_2=-e^{i\theta}$ and $e^{-i\theta}$, as in Fig. 5. Then

$$W_2=\frac{q}{2\pi}\log(z_2+e^{i\theta})-\frac{q}{2\pi}\log(z_2-e^{-i\theta}),$$

Fig. 6.



$$W_2=\frac{q}{2\pi}\log(z_2+re^{i\theta})+\frac{q}{2\pi}\log\left(z_2+\frac{e^{i\theta}}{r}\right)$$
$$-\frac{q}{2\pi}\log(z_2-re^{-i\theta})-\frac{q}{2\pi}\log\left(z_2-\frac{e^{-i\theta}}{r}\right),$$

where the positions of the source and sink are represented by $-re^{i\theta}$ and $re^{-i\theta}$ respectively. Then

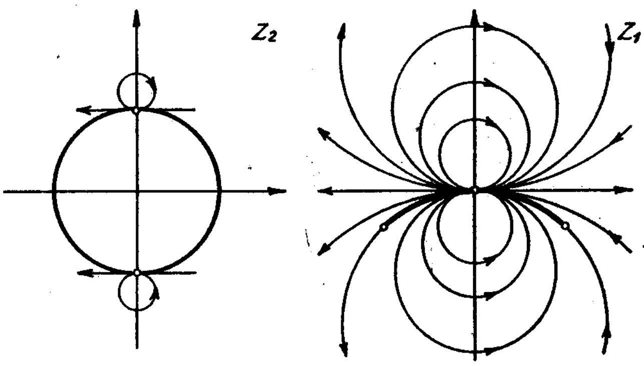
$$\frac{z_1+a}{z_1-a}=\frac{z_2+\dfrac{e^{2i\theta}}{z_2}+e^{i\theta}\left(r+\dfrac{1}{r}\right)}{z_2+\dfrac{e^{-2i\theta}}{z_2}-e^{-i\theta}\left(r+\dfrac{1}{r}\right)}.$$

By this relation the circle in the z_2-plane is transformed into a portion of a circular arc passing through the points $z_1=-a$ and a.

(b) Next we place two doublets of unequal strength m_1 and m_2 but with their axes lying in the same direction at the points $z_2=i$ and $-i$ on the circle, as in Fig. 7a; then

$$W_2=\frac{m_1}{z_2-i}+\frac{m_2}{z_2+i};$$

Fig. 7a.



and in the z_1-plane a doublet of strength $\dfrac{m_1+m_2}{2}$ at the origin would give us

$$W_1=\frac{m_1+m_2}{2z_1}.$$

From these we get

$$\frac{1}{z_1}=\frac{k}{z_2-i}+\frac{2-k}{z_2+i}+c,$$

where c is the integration constant and

$$c=0, \qquad k=\frac{2m_1}{m_1+m_2}.$$

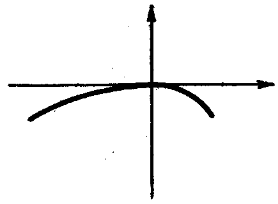
By this relation the circle in the z_2-plane is transformed into a portion of a circular arc passing through the origin of the z_1-plane.

When $k=1$ and $c=0$, the circle is transformed into a straight line and

$$z_1=\frac{1}{2}\left(z_2+\frac{1}{z_2}\right).$$

The above transformation is the simplest case. More generally by using the doublets, source and sink, properly placed on the circumference of the circle we can transform the circle into an aerofoil section composed of two segments of circular arc with different radius but with common tangent at their connection as shown in Fig. 7b, which has already been treated by Sonnefeld.[8]

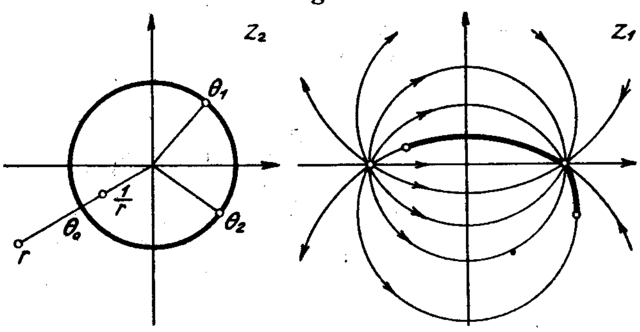
Fig. 7b.



(c) Next we place a source of strength $\dfrac{q_1+q_2}{2}$ at the point $re^{i\theta_0}$ and two sinks of strengths q_1 and q_2 at the points $e^{i\theta_1}$ and $e^{i\theta_2}$ respectively on the circle of the z_2-plane, Fig. 8, then

$$W_2=\frac{q_1+q_2}{4\pi}\log(z_2-re^{i\theta_0})+\frac{q_1+q_2}{4\pi}\log\left(z_2-\frac{e^{i\theta_0}}{r}\right)$$
$$-\frac{q_1}{2\pi}\log(z_2-e^{i\theta_1})-\frac{q_2}{2\pi}\log(z_2-e^{i\theta_2}).$$

Fig. 8.



In the z_1-plane we may place a source and a sink of strength $\dfrac{q_1+q_2}{2}$ on the points $z_1=-a$ and a, then

$$W_1=\frac{q_1+q_2}{4\pi}\log\frac{z_1+a}{z_1-a}.$$

Hence

$$\frac{z_1+a}{z_1-a}=\frac{\left(z_2-re^{i\theta_0}\right)\left(z_2-\dfrac{e^{i\theta_0}}{r}\right)}{(z_2-e^{i\theta_1})^k(z_2-e^{i\theta_2})^{2-k}},$$

where

$$k=\frac{2q_1}{q_1+q_2}.$$

By this relation the circle in the z_2-plane is transformed into an aerofoil section of circular arc with hinged flap.

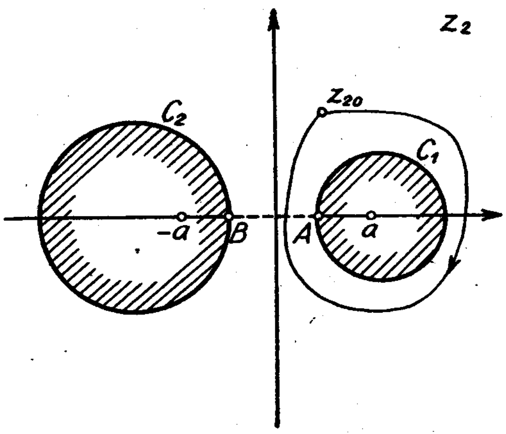
When the source goes to infinity in the z_1 and z_2-planes, by taking $a=0$ we get an aerofoil section of straight line with hinged flap as already treated in the previous section of this paper.

In the preceding transformations we used mainly the source and the sink, but we can also conveniently apply the irrotational vortices.

4. Cases of two Circles.

Let us consider two circles C_1 and C_2 in the z_2-plane with their centres on the real axis as shown in Fig. 9, and assuming that the position of the origin was properly chosen—the method is

Fig. 9.

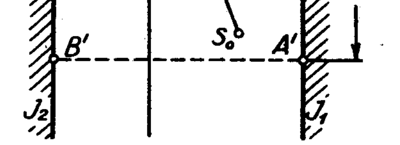


found in the paper of Lagally—then by making a cut between these circles along AB we can transform this outside region of two circles into a rectangular region in the s-plane, Fig. 10, by the following relation

$$s=\log\frac{z_2+a}{z_2-a},$$

where a is a real constant determined by Lagally's method.

Fig. 10.

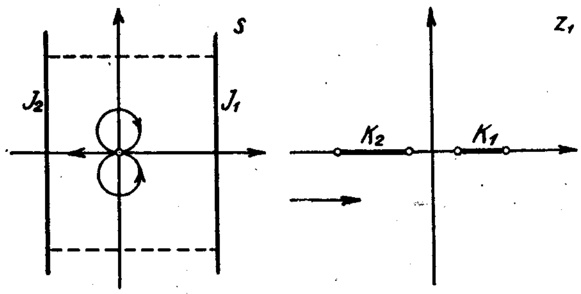


In the s-plane the height of the rectangle is equal to 2π; the circumferences of two circles C_1 and C_2 are transformed into two straight lines J_1 and J_2 parallel to the imaginary axis passing through $s=\alpha$ and $s=-\beta$ respectively; and the cut AB is transformed into two straight lines AB and $A'B'$ parallel to the real axis as shown in Fig. 10. The point of infinity of the z_2-plane is transformed to the origin of the s-plane, i.e. $s=0$ or more generally to $s=\pm 2n\pi i$ where $n=0, 1, 2, 3, \ldots\ldots$, so s is a periodic function with period $2\pi i$. Hence if we start from a point s_0 and reach to the point $s_0+2\pi i$, the corresponding point in the z_2-plane returns to its initial position.

(a) For our first example we shall discuss the transformation of the s-plane into the outside region of two segments of a straight line, i.e. the case of a tandem biplane, Fig. 11.

The parallel flow with velocity u along the real axis in the z_2-plane is equivalent to the flow due to a doublet placed at the origin with its axis along the real axis in the negative direction in the s-plane. This flow is represented by an elliptic function, and the conjugate complex of the velocity in the s-plane is expressed as follows:

Fig. 11.



$$\frac{dW_s}{ds}=-u\wp(s)+u\wp(s+2\beta)+ik,$$

where $\wp$ is the elliptic function of Weierstrass with periods $2\omega_1=2(\alpha+\beta)$ and $2\omega_2=2\pi i$ and k is an unknown real constant.

In the z_1-plane the flow along the real axis with the velocity u is represented by

$$W_1=uz_1.$$

From these we get

$$z_1=\zeta(s)-\zeta(s+2\beta)+ic_1s+c_2,$$

where ζ is the ζ-function, c_2 is an integration constant, and $c_1=\dfrac{k}{u}$.

Now the condition to be satisfied by the above relation is that when we start from a point s_0 and reach to the point $s_0+2\omega_2$, z_1 must return to the initial value, in other words z_1 must have a period of $2\omega_2$. By this condition c_1 is determined, namely

$$2\eta_2-2\eta_2+ic_12\omega_2=0,$$

where $\quad \eta_2=\zeta(\omega_2),$

$\therefore \quad c_1=0.$

$\therefore \quad z_1=\zeta(s)-\zeta(s+2\beta)+c_2.$

By this relation the s-plane is transformed into the outside region of a tandem biplane in the z_1-plane, as in Fig. 11, and the last term determines the position of the aerofoils in the z_1-plane.

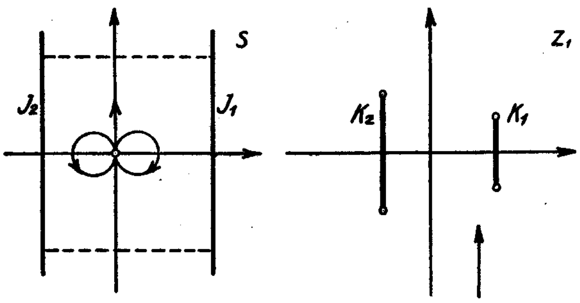
(b) Another case is that in which the flow is along the imaginary axis in the z_2-plane. In this case

$$\frac{dW_s}{ds} = iv\wp(s) + iv\wp(s+2\beta) + ik,$$

where v is the magnitude of the velocity at infinity in the z_2-plane. Hence we get the relation between z_1 and s;

$$z_1 = \zeta(s) + \zeta(s+2\beta) - ic_1 s + c_2,$$

Fig. 12.



where c_1 and c_2 are integration constants and c_1 can be determined as before, and we get

$$c_1 = -\frac{2\eta_2}{\omega_2} i$$

and

$$z_1 = \zeta(s) + \zeta(s+2\beta) - \frac{2\eta_2 s}{\omega_2} + c_2.$$

By this relation the s-plane is transformed into the outside region of a biplane with parallel chords without stagger in the z_1-plane, as shown in Fig. 12.

(c) Combining the above two cases as in Fig. 13, we can represent the flow in the s plane by

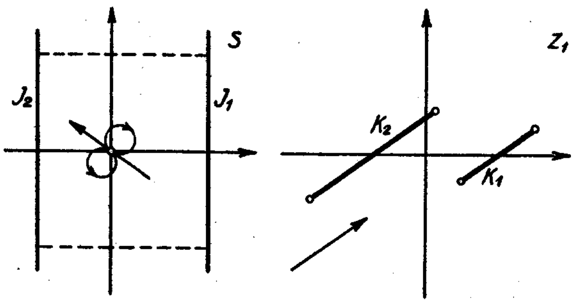
$$\frac{dW_s}{ds} = -w\wp(s) + \overline{w}\wp(s+2\beta) + ik,$$

where $w = u - iv$ and $\overline{w} = u + iv$.

And in the z_1-plane we take

$$\frac{dW_1}{dz_1} = u - iv.$$

Fig. 13.



From these we get

$$z_1 = \zeta(s) - \frac{\overline{w}}{w}\zeta(s+2\beta) + i\frac{ks}{w} + c,$$

where c is the integration constant. Let k be determined as before, then

$$k = \frac{2v\eta_2}{\omega_2},$$

$$\therefore\ z_1 = \zeta(s) - \frac{u+iv}{u-iv}\zeta(s+2\beta) + \frac{2v\eta_2 i}{\omega_2(u-iv)}s + c.$$

By this relation we get in the z_1-plane a biplane with parallel chords and with stagger.

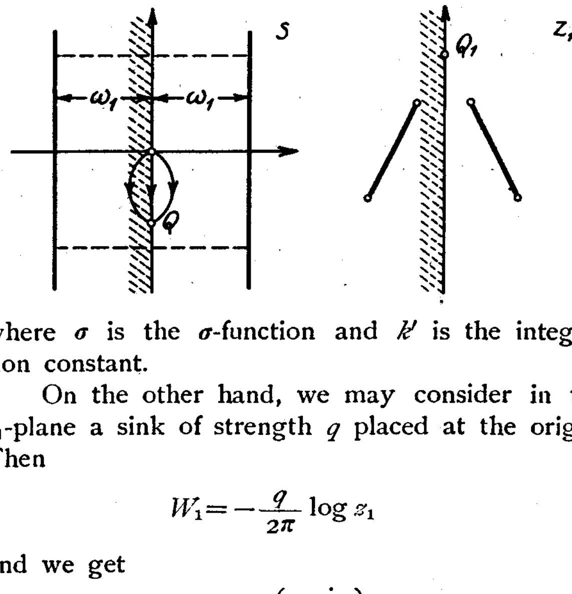
(d) Next we consider the case when the radii of the two circles are equal, i.e. $\alpha = \beta$. In the s-plane we place a source of strength q at the origin and a sink of the same strength at the point Q on the imaginary axis, Fig. 14. The flow due to these source and sink is represented by

$$\frac{dW_s}{ds} = \frac{q}{2\pi}\{\zeta(s) - \zeta(s-im)\} + ik,$$

where $s = im$ for point Q, $\omega_1 = \alpha = \beta$, and $\omega_2 = \pi i$. Hence

$$W_s = \frac{q}{2\pi}\log\frac{\sigma(s)}{\sigma(s-im)} + iks + k',$$

Fig. 14.



where σ is the σ-function and k' is the integration constant.

On the other hand, we may consider in the z_1-plane a sink of strength q placed at the origin. Then

$$W_1 = -\frac{q}{2\pi}\log z_1$$

and we get

$$\log z_1 = \log\frac{\sigma(s-im)}{\sigma(s)} - ic_1 s + c_2,$$

or

$$z_1 = c'_2 e^{-ic_1 s}\frac{\sigma(s-im)}{\sigma(s)},$$

where c_1 and c'_2 are constants and c_1 is determined by the condition that z_1 must be a periodic function of s with period $2\omega_2$, hence

$$z_1 = c'_2 e^{i\frac{m\eta_2}{\omega_2}s}\frac{\sigma(s-im)}{\sigma(s)}.$$

By this relation the two circles are transformed into a biplane with decalage in the z_1-plane. This transformation is also useful to investigate the interference effect of the ground upon the flat plate.

In the case $\alpha \neq \beta$, a sink is placed at the point $s=m$ instead of at a point on the imaginary axis, then

$$\frac{dW_s}{ds}=\frac{q}{2\pi}\{\zeta(s)+\zeta(s+2\beta)-\zeta(s-m) -\zeta(s+2\beta+\overline{m})\}+ik,$$

where m is a complex number, $\overline{m}$ is its conjugate complex, $\omega_1=\alpha+\beta$, and $\omega_2=\pi i$.
Combining this and

$$W_1=-\frac{q}{2\pi}\log z_1$$

we get

$$z_1=ce^{\frac{\eta_2(m-\overline{m})}{\omega_2}s}\frac{\sigma(s-m)\sigma(s+2\beta+\overline{m})}{\sigma(s)\sigma(s+2\beta)},$$

where c is the integration constant.
By this generalized relation we get a biplane of unequal chords with stagger and decalage in the z_1-plane.

5. Cases of two Circles.

In the preceding section we investigated the problems of biplanes with sections of straight lines. Now we proceed to the cases of circular arcs.

(a) For the sake of simplicity let $\alpha=\beta$, and place a source and a sink with same strength q at the points P and Q on the imaginary axis of the s-plane, Fig. 15. Then

$$\frac{dW_s}{ds}=\frac{q}{2\pi}\{\zeta(s-im)-\zeta(s+in)\}+ik$$

or

$$W_s=\frac{q}{2\pi}\log\frac{\sigma(s-in)}{\sigma(s+in)}+iks+k',$$

where $s=im$ for point P,

$s=-in$ for point Q,

and k and k' are constants.

In the z_1-plane we consider a source and a sink of the same strength placed at the points P_1 and Q_1, then the flow due to this source and sink is represented by

$$W_1=\frac{q}{2\pi}\log\frac{z_1+ib}{z_1-ib},$$

where $z_1=-ib$ for point P_1,

$z_1=ib$ " " Q_1.

Hence

$$\log\frac{z_1+ib}{z_1-ib}=\log\frac{\sigma(s-im)}{\sigma(s+in)}+ic_1s+c_2$$

or

$$\frac{z_1+ib}{z_1-ib}=c'_2e^{ic_1s}\frac{\sigma(s-im)}{\sigma(s+in)}$$

where c_1 and c'_2 are constants. c_1 can be determined as before and equal to $\frac{\eta_2(m+n)}{\omega_2}$. c'_2 is determined by the condition that $s=0$ corresponds to $z_1=\infty$, and we get

$$\frac{z_1+ib}{z_1-ib}=e^{i\frac{\eta_2(m+n)}{\omega_2}s}\frac{\sigma(s-im)\sigma(in)}{\sigma(s+in)\sigma(-im)}.$$

By this relation we get two segments of circular arcs symmetrically situated with respect to the imaginary axis in the z_1-plane, whose circles cut each other at two points on this axis. This relation may be used for calculating the interference effect of the ground upon an aerofoil with section of circular arc which is placed fairly near the ground surface.[9]

(b) Instead of a source and a sink we now place a pair of irrotational vortices with circulation of Γ at the points P and Q, Fig. 16, then

$$\frac{dW_s}{ds}=\frac{i\Gamma}{2\pi}\zeta(s-m)-\frac{i\Gamma}{2\pi}\zeta(s+\overline{m})+ik,$$

where $s=-\overline{m}$ for point P,

$s=m$ " " Q,

$\overline{m}$ is the conjugate complex of m, and k is a real constant.

Hence

$$W_s=\frac{i\Gamma}{2\pi}\log\frac{\sigma(s-m)}{\sigma(s+\overline{m})}+iks+k'.$$

Fig. 15.

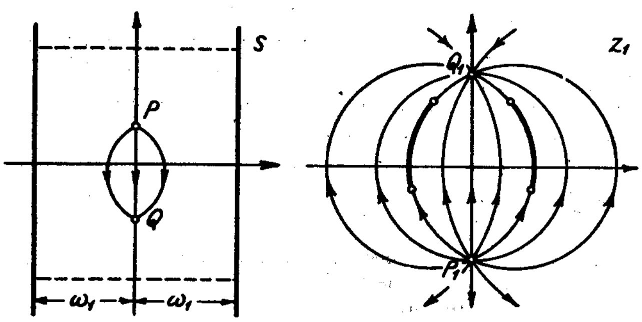
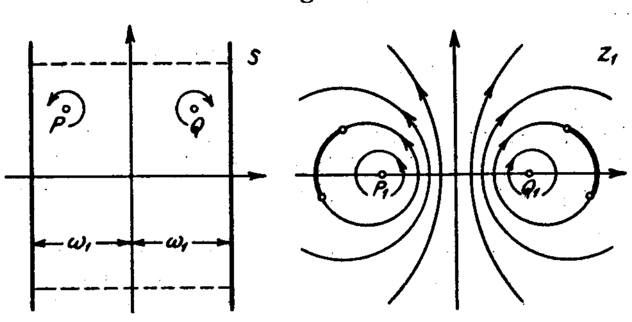


Fig. 16.



On the other hand we consider a pair of vortices placed at the points P_1 and Q_1 in the z_1-plane, then

$$W_1=\frac{i\Gamma}{2\pi}\log\frac{z_1-c}{z_1+c}$$

where $z_1=-c$ for point P_1,

$z_1=c$ „ „ Q_1.

From these we get

$$\log\frac{z_1-c}{z_1+c}=\log\frac{\sigma(s-m)}{\sigma(s+\overline{m})}+c_1 s+c_2.$$

Determining the constants c_1 and c_2 as before, we get

$$\frac{z_1-c}{z_1+c}=e^{\frac{\eta_2(m+\overline{m})}{\omega_2}s}\,\frac{\sigma(s-m)\sigma(\overline{m})}{\sigma(s+\overline{m})\sigma(-m)}\,.$$

By this relation we get two segments of circular arcs symmetrically situated with respect to the imaginary axis and whose circles do not intersect anywhere in the z_1-plane. This transformation is also useful for the investigation of the ground effect.*

(c) More generally we consider the case $\alpha\neq\beta$ and place a source and a sink of the same strength q at the points P and Q in the s-plane, Fig. 17, then

$$\frac{dW_s}{ds}=\frac{q}{2\pi}\{\zeta(s-m)-\zeta(s-n)+\zeta(s+\overline{m}+2\beta)-\zeta(s+\overline{n}+2\beta)\}+ik,$$

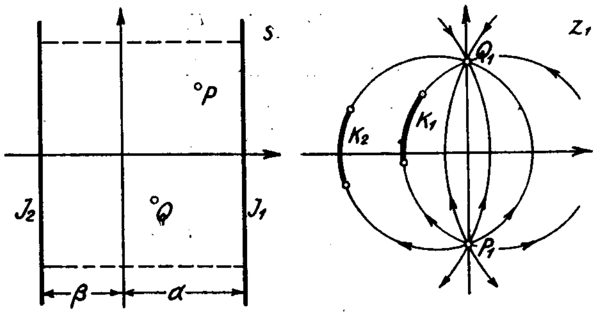
where $\omega_1=\alpha+\beta$, $\omega_2=\pi i$,

$s=m$ for point P,

$s=n$ „ „ Q,

and $\overline{m}$, $\overline{n}$ are the conjugate complexes of m and n respectively.

Fig. 17.



Hence

$$W_s=\frac{q}{2\pi}\log\frac{\sigma(s-m)\sigma(s+\overline{m}+2\beta)}{\sigma(s-n)\sigma(s+\overline{n}+2\beta)}+iks+k'.$$

As in the preceding cases, again we may consider a source and a sink at the points P_1 and Q_1 in the z_1-plane, then

$$W_1=\frac{q}{2\pi}\log\frac{z_1+ia}{z_1-ia}\,,$$

where $z_1=ia$ for point Q_1 and

$z_1=-ia$ „ „ P_1.

From these we get

$$\frac{z_1+ia}{z_1-ia}=e^{-\frac{\eta_2}{\omega_2}(n-m-\overline{n}+\overline{m})s}\times\frac{\sigma(s-m)\sigma(s+\overline{m}+2\beta)\sigma(n)\sigma(\overline{n}+2\beta)}{\sigma(s-n)\sigma(s+\overline{n}+2\beta)\sigma(m)\sigma(\overline{m}+2\beta)}$$

By this relation we get a general biplane of aerofoils of circular arc section in the z_1-plane.

(d) When the circles of K_1 and K_2 do not intersect anywhere in the z_1-plane, we take a pair of properly chosen irrotational vortices instead of a source and a sink in the same way as the case (b) of this section.

6. Cases of two Circles.

(a) In the cases described in sections 4 and 5 we placed source and sink in the interior of the rectangular region of the s-plane; now if we place them on the vertical side of the rectangle we have a circle and a circular arc in the z_1-plane, Fig. 18. And if we transform this circle into the

Fig. 18.

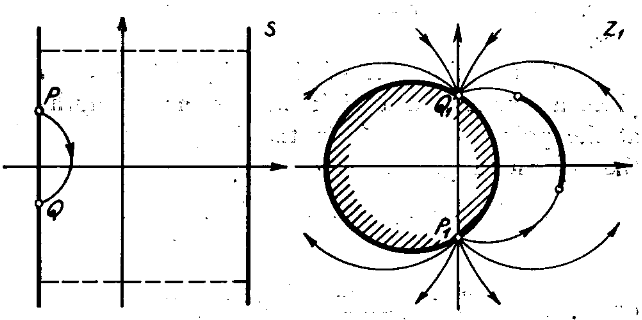
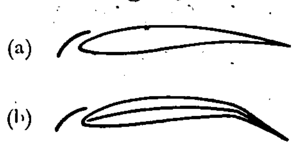


Fig. 19.



Joukowski section by applying the usual transformation, we get a Joukowski section with an auxiliary aerofoil[10], i.e. a slotted wing section as shown in Fig. 19a. And it is also possible to investigate the effect of the slot and flap, as shown in Fig. 19b, by applying suitable transformation as mentioned in section 3 of this paper.

(b) As a further application of the method, let us investigate the two-dimensional flow through

* The lift and its moment acting on the aerofoil can be calculated by the same method as described in the author's previous paper "The lift on an aerofoil with a circular arc section placed near the ground."[9] It needs only to replace a in the eq. (1) by a pure imaginary number ia and ν in the eq. (2) by $i\nu$ which is also a pure imaginary number.

the latticed wings. In the z_2-plane, we consider a source and a sink placed at the points $z_2=-a$ and a and transform this into the z_1-plane by the relation

$$z_1=\log\frac{z_2+a}{z_2-a}.$$

Then the stream lines in the z_2-plane are transformed into the parallel lines in the direction of the real axis and the whole plane of z_2 is transformed into a strip domain of height 2π in the z_1-plane.

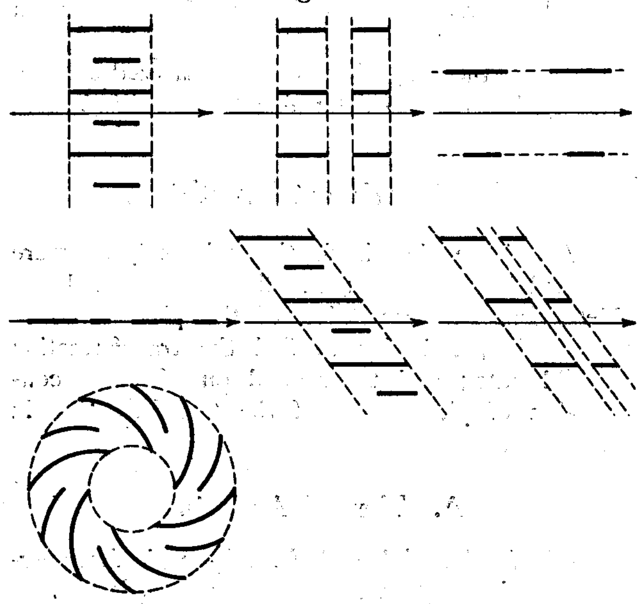
If instead of a source and a sink we place a pair of irrotational vortices at the points $z_2=-a$ and a, and transform the z_2-plane into the z_1-plane by the following relation

$$z_1=i\log\frac{z_2+a}{z_2-a};$$

then the stream lines in the z_2-plane are transformed into parallel straight lines in the direction of the real axis, and the whole plane of z_2 is transformed into a strip domain of breadth 2π in the z_1-plane.

Applying such transformations to the flow

Fig. 20.



around a circle whose centre is at the origin, a flow due to the source and sink at the points $z_2=-a$ and a respectively or to a pair of irrotational vortices, we have latticed wings of equal pitch and equal chord whose axes are parallel or perpendicular to the chord of each wing[11]. Also if we apply the same method to the flow around two circles we get latticed wings of unequal pitches and unequal chords in the z_1-plane, as shown in Fig. 20.*

Further more, combining these transformations we get latticed wings with inclined axis to the chord of each wing. By applying to this latticed wing further transformation as $z_1=\log t$, we get a circular latticed wing with sections of logarithmic spiral in the t-plane as shown in Fig. 20, which has applications to the theory of water turbines, centrifugal pumps, and turbo-blowers.

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* The velocity of the flow around these types of latticed wings but with equal pitch can be calaulated without difficulty by the same method used in the book of Grammel-Die hydrodynamischen Grundlagen des Fluges. page 93.