Mathematical Theories of Bourdon Pressure Tubes and Bending of Curved Pipes.

By Masasuke Tueda.

First Report: Mathematical Analysis.

Mathematically strict solutions for the theories of Bourdon pressure tubes and bending of curved pipes are obtained simultaneously in order to give a theoretical standard to the various approximate theories relating to the same problems. The first report contains only the mathematical analysis of the problem; the examples of numerical calculations and applications to various modified forms of cross-section will be given in further reports.

I. Introduction.

Up to now, there have been published several papers treating either the theory of bending of curved circular pipes(1) or the theory of Bourdon pressure tubes;(2) all of which, however, so far as the author is aware, only afford more or less approximate solutions to each problem independently. There seems to exist no theoretical standard to verify the accuracy of those approximate theories. The present paper gives a mathematically more accurate solution of the problems common both to the bending of curved pipes and of Bourdon pressure tubes, by means of an application of Prof. Meissner's method of solving the differential equations.(3)

In order to apply Meissner's method of solution to the case of constant wall thickness, the shape of the cross-section must be assumed to be built up of circular arcs, unless the Poisson's ratio $(1/m)$ be neglected. In every previous theory of Bourdon pressure tubes it has been assumed as an ellipse, but the cross-section of any actual Bourdon tube far more resembles the shape built up of two pairs of circular arcs than an ellipse. Because of this fact, the following also may be assumed:

(i) The center-line of the pipe forms a part of a circular arc in a plane, which shall be called here "the plane of symmetry."

(ii) The cross-sectional form of the middle surface of the pipe wall is constant along the center-line, and is built up of a circle or of two pairs of circular arcs symmetrical with respect to two axes in and perpendicular to the plane of symmetry.

(iii) The thickness of the wall of the pipe is constant.

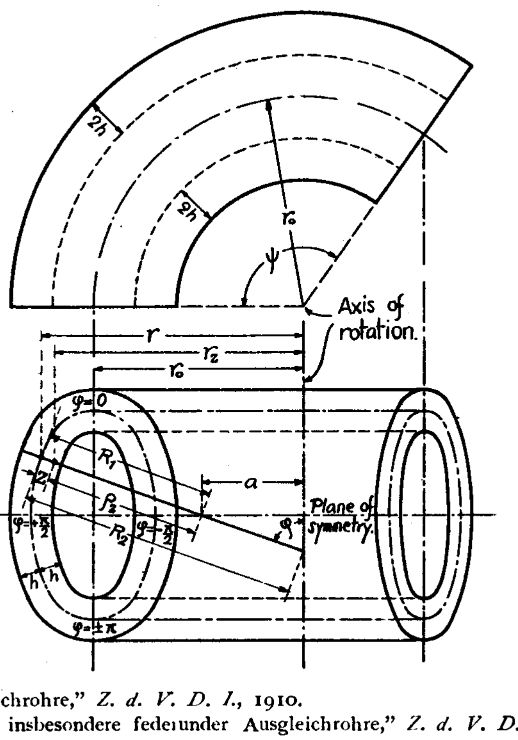
(iv) The external force applied is either the internal pressure p or the uniform bending moment M in the plane of symmetry, both of which, of course, may be applied simultaneously. In the present case, if we consider the free end of the pipe to be closed, the internal pressure p never exhibits the action of the bending moment.

(v) The material of the pipe is homogeneous and isotropic, and Hook's law is necessarily applicable.

II. Notations of Position and Displacement.

The surface of the curved pipe to be treated

Fig. 1.



(1) A. Bantlin, "Formänderung und Beanspruchung federnder Ausgleichrohre," *Z. d. V. D. I.*, 1910.
Th. v. Kármán, "Ueber die Formänderung dünnwandiger Rohre, insbesondere federunder Ausgleichrohre," *Z. d. V. D. I.*, 1911.
H. Lorenz, *Techinische Elastizitätslehre*, 1913.
W. Hovgaard, "Deflection and stresses in pipe bends," *Proc. of the W. E. C.*, Tokyo, 1928, Vol. III.
T. Matumura, "Deformation of curved pipes due to bending," read before the general meeting of the Society of Mechanical Engineers, Kyoto, 1931.

(2) H. Lorenz, "Theorie der Röhrenfedermanometer," *Z. d. V. D. I.*, 1910.
T. Sunatani, "The theory of a Bourdon tube pressure gauge," *J. of the Soc. of Mech. Engrs.*, 1924.

(3) E. Meissner, "Das Elastizitätsproblem für dünne Schalen von Ringflächen-, Kugel- oder Kegelform," *Physik. Zeitschr.*, 1913.

here forms a part of a surface of revolution, and the position of the "axis of rotation" of the surface is assumed here to be fixed.

In Fig. 1, let

ψ be an angle in the plane of symmetry measured around the axis of rotation; that is the angle of rotation,

z be the perpendicular distance from a point to the original undeformed middle surface of the wall, and

φ be an angle between the normal of the same surface and the axis of rotation;

where z for points inside the middle surface, and the direction of the inward drawn normal are assumed to be positive. Then the surfaces ψ=constant (a meridian plane), z=constant ($z=0$ being the original undeformed middle surface of the wall) and φ=constant (generally a conical surface) intersect each other at right angles, and ψ, z and φ constitute an orthogonal curvilinear co-ordinate system, which we now employ to represent the position of a point in the wall.

Further, the following general notations may be used: (see Fig. 1.)

R_1......The first principal radius of curvature of the original undeformed middle surface of the wall in a meridian section, which is assumed to be a positive constant.

R_2......The second principal radius of curvature of the same surface, that is the length of a normal of the original undeformed middle surface between the axis of rotation and the surface; R_2 is a function of φ alone.

r......The radius of rotation of a point on the same surface, that is the perpendicular distance from that point to the axis of rotation. r, which is also a function of φ alone, is assumed always to be positive.

r_0......The radius of the center-line of the pipe before deformation, that is a positive constant.

a......The radius of rotation of the center of curvature of the radius R_1, which is a positive or negative constant. $a=r_0$ for circular cross-sections.

$2h$......The constant thickness of the wall.

$$\left.\begin{aligned} \rho_z &= R_1 - z, \\ r_z &= r - z \cdot \sin\varphi. \end{aligned}\right\} \qquad (1)$$

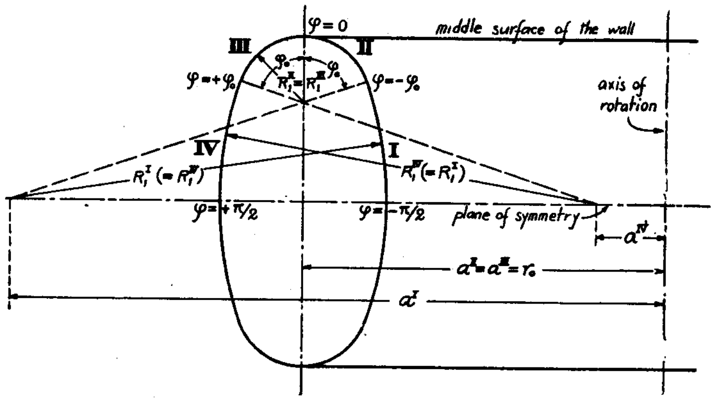
Then we have

$$\left.\begin{aligned} r &= R_1 \sin\varphi + a = R_2 \sin\varphi, \\ r_z &= \rho_z \sin\varphi + a = (R_2 - z)\sin\varphi. \end{aligned}\right\} \qquad (2)$$

These general notations stand for the properties of each circular arc which constructs the cross-section, and, if necessary, we make the distinction as shown in Fig. 2.

In the theory of curved pipes the change of the curvature of the center-line of the pipe cannot

Fig. 2



be considered very small, even when the deformation of the cross-section be confined to the very small amount usual in the mathematical theory of elasticity. But, in the present case, it will easily be seen that the centre-line of the pipe, after deformation, still keeps the shape of a circular arc. Owing to this fact, it is convenient to consider the displacement in separate two parts: the first part contains the change of ψ and r_0 into $\psi+\omega$ and $r_0+\delta$ keeping the shape of the cross-section unaltered; and the second part contains only the deformation of the cross-section relative to two axes of symmetry. The displacement δ becomes a constant in the present. To represent the displacement of the second part, that is the deformation of the cross-section, we use the following notations:

χ......The change of φ; $\chi=0$ when $\varphi=0$.

ζ......The change of z.

III. Fundamental Differential Equations.

At any point in the wall, let

σ_ψ, σ_φ and σ_z be normal stresses,

ε_ψ, ε_φ " ε_z " " strains,

$\tau_{\varphi z}$, $\tau_{z\psi}$ " $\tau_{\psi\varphi}$ " shearing stresses and

$\gamma_{\varphi z}$, $\gamma_{z\psi}$ " $\gamma_{\psi\varphi}$ " " strains;

where the suffix ψ indicates the direction tangential to the equatorial line, the suffix φ indicates the direction tangential to the meridian line and the suffix z indicates the direction normal to the middle surface.

The relations between strains and displacements, and the conditions of equilibrium of stresses at a point with respect to the above curvilinear coordinate system are expressed generally as follows.[4]

$$\left.\begin{aligned}
\epsilon_\psi &= \frac{\partial \omega}{\partial \psi} + \frac{1}{r_z}\left[\frac{\partial r_z}{\partial \psi}\omega + \frac{\partial r_z}{\partial \varphi}\chi + \frac{\partial r_z}{\partial z}\zeta + \delta\right],\\
\epsilon_\varphi &= \frac{\partial \chi}{\partial \varphi} + \frac{1}{\rho_z}\left[\frac{\partial \rho_z}{\partial \psi}\omega + \frac{\partial \rho_z}{\partial \varphi}\chi + \frac{\partial \rho_z}{\partial z}\zeta\right],\\
\epsilon_z &= \frac{\partial \zeta}{\partial z}.
\end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned}
\gamma_{\varphi z} &= \rho_z \frac{\partial \chi}{\partial z} + \frac{1}{\rho_z}\frac{\partial \zeta}{\partial \varphi},\\
\gamma_{z\psi} &= \frac{1}{r_z}\frac{\partial \zeta}{\partial \psi} + r_z \frac{\partial \omega}{\partial z},\\
\gamma_{\psi\varphi} &= \frac{r_z}{\rho_z}\frac{\partial \omega}{\partial \varphi} + \frac{\rho_z}{r_z}\frac{\partial \chi}{\partial \psi}.
\end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial \psi}(r_z\rho_z\sigma_\psi) - \rho_z\sigma_\psi\frac{\partial r_z}{\partial \psi} - r_z\sigma_\varphi\frac{\partial \rho_z}{\partial \psi}\\
&\qquad + \frac{\partial}{\partial \varphi}(r_z^2\tau_{\psi\varphi}) + \frac{\partial}{\partial z}(\rho_z r_z^2\tau_{z\psi}) = 0,\\
&\frac{\partial}{\partial \varphi}(r_z\rho_z\sigma_\varphi) - \rho_z\sigma_\psi\frac{\partial r_z}{\partial \rho} - r_z\sigma_\varphi\frac{\partial \rho_z}{\partial \varphi}\\
&\qquad + \frac{\partial}{\partial z}(r_z\rho_z^2\tau_{\varphi z}) + \frac{\partial}{\partial \psi}(\rho_z^2\tau_{\psi\varphi}) = 0,\\
&\frac{\partial}{\partial z}(r_z\rho_z\sigma_z) - \rho_z\sigma_\psi\frac{\partial r_z}{\partial z} - r_z\sigma_\varphi\frac{\partial \rho_z}{\partial z}\\
&\qquad + \frac{\partial}{\partial \psi}(\rho_z\tau_{z\psi}) + \frac{\partial}{\partial \varphi}(r_z\tau_{\varphi z}) = 0.
\end{aligned}\right\} \quad (5)$$

From the assumptions for the shape of the pipe and the external forces applied, we can recognise immediately that stresses, strains, χ and ζ are all independent of ψ, and hence we can put

$$\left.\begin{aligned}
&\tau_{\psi\varphi} = \tau_{z\psi} = \gamma_{\psi\varphi} = \gamma_{z\psi} = 0,\\
&\frac{\partial \omega}{\partial \psi} = \text{constant} \equiv \omega_0.
\end{aligned}\right\} \quad (6)$$

Substituting equations (2) and (6) in equations (3), (4) and (5), and putting, for brevity, $\tau \equiv \tau_{\varphi z}$ and $\gamma \equiv \gamma_{\varphi z}$, we have

$$\left.\begin{aligned}
\epsilon_\psi &= \omega_0 + \frac{1}{r_z}\left[\chi\rho_z\cos\varphi - \zeta\sin\varphi + \delta\right],\\
\epsilon_\varphi &= \frac{1}{\rho_z}\left[\frac{\partial}{\partial \varphi}(\chi\rho_z) - \zeta\right],\\
\epsilon_z &= \frac{\partial \zeta}{\partial z},\\
\gamma &= \rho_z\frac{\partial \chi}{\partial z} + \frac{1}{\rho_z}\frac{\partial \zeta}{\partial \varphi};
\end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial \varphi}(\sigma_\varphi r_z) - \sigma_\psi\rho_z\cos\varphi + \frac{\partial}{\partial z}(\tau r_z\rho_z) - \tau r_z = 0,\\
&\frac{\partial}{\partial z}(\sigma_z r_z\rho_z) + \sigma_\psi\rho_z\sin\varphi + \sigma_\varphi r_z + \frac{\partial}{\partial \varphi}(\tau r_z) = 0.
\end{aligned}\right\} \quad (8)$$

(4) Geiger u. Scheel, *Handbuch der Physik*, Bd. 6. 1928. Love, *Mathematical theory of elasticity*. 1927.

Similarly as Love, Meissner and others, we follow the fundamental assumptions of Kirchhoff's theory of plate, that is

$$\epsilon_z = \sigma_z = \gamma = 0. \quad (9)$$

Then, from the equations (7), we get

$$\frac{\partial \zeta}{\partial z} = 0 \quad \text{and} \quad \rho_z\frac{\partial \chi}{\partial z} + \frac{1}{\rho_z}\frac{\partial \zeta}{\partial \varphi} = 0,$$

or

$$\left.\begin{aligned}
\zeta &= fn.(\varphi),\\
\chi &= \chi_0 - \frac{z}{R_1(R_1 - z)}\frac{d\zeta}{d\varphi} \approx \chi_0 - \frac{z}{R_1^2}\frac{d\zeta}{d\varphi},
\end{aligned}\right\} \quad (10)$$

where χ_0 represents the value of χ at a point $z=0$, that is on the middle surface of the wall.

Now we define, further,

$$\left.\begin{aligned}
u &= \chi_0 R_1 + \delta\cos\varphi,\\
w &= \zeta - \delta\sin\varphi,
\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}
\theta &= \frac{1}{R_1}\left(u + \frac{dw}{d\varphi}\right),\\
\epsilon_1 &= \frac{1}{R_1}\left(\frac{du}{d\varphi} - w\right),\\
\epsilon_2 &= \omega_0 + \frac{u\cos\varphi - w\sin\varphi}{r}.
\end{aligned}\right\} \quad (12)$$

The geometrical meanings of these values are:

u, the total displacement of a point on the middle surface in the direction tangential to the meridian line,

w, do. in the normal direction,

ϵ_1, the strain at the same point tangential to the meridian line,

ϵ_2, do. tangential to the equatorial line, and

θ, the change of the inclination of the normal to the middle surface due to deformation.

Putting equations (9), (10), (11) and (12) into equations (7), and neglecting the terms of the second and the higher power of $\left(\frac{z}{R_1}\right)$ and $\left(\frac{z}{R_2}\right)$, we have

$$\left.\begin{aligned}
\epsilon_\varphi &= \epsilon_1 - \frac{z}{R_1}\left(\frac{d\theta}{d\rho} - \epsilon_1\right),\\
\epsilon_\psi &= \epsilon_2 - \frac{z}{R_2}(\theta\cot\varphi - \epsilon_2 + \omega_0).
\end{aligned}\right\} \quad (13)$$

In each of the equations written above the coefficient of z represents the change of the curvature $\frac{1}{R_1}$ and $\frac{1}{R_2}$ respectively due to deformation.

Relations between stresses and strains for the case $\sigma_z = \epsilon_z = 0$ are:

$$\left.\begin{aligned}\sigma_\varphi&=\frac{E}{1-\mu^2}(\varepsilon_\varphi+\mu\varepsilon_\psi),\\ \sigma_\psi&=\frac{E}{1-\mu^2}(\varepsilon_\psi+\mu\varepsilon_\varphi),\end{aligned}\right\}\tag{14}$$

where μ is Poisson's ratio $(1/m)$, and E is the modulous of elasticity of the material.

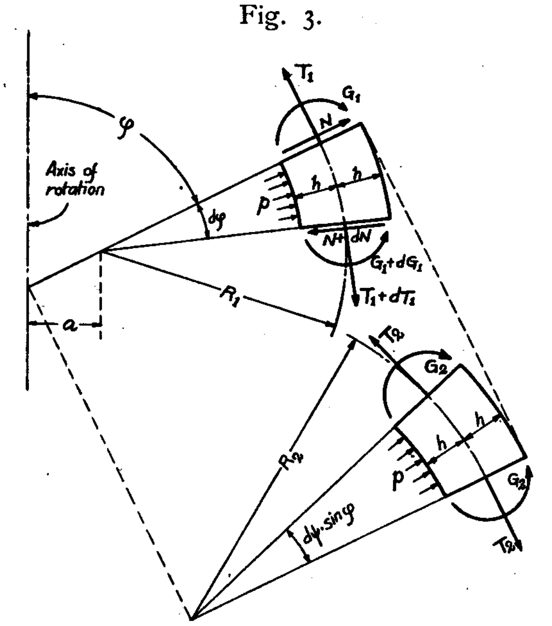
Solving the last two equations (12), we get

$$\left.\begin{aligned}u&=\sin\varphi\int\frac{R_1\varepsilon_1-R_2\varepsilon_2+R_2\omega_0}{\sin\varphi}d\varphi\\&\qquad+u_c\cdot\sin\varphi,\\ w&=\cos\varphi\int\frac{R_1\varepsilon_1-R_2\varepsilon_2+R_2\omega_0}{\sin\varphi}d\varphi\\&\qquad+u_c\cdot\cos\varphi-R_2\varepsilon_2+R_2w_0.\end{aligned}\right\}\tag{15}$$

If we start the integration from $\varphi=\pm\frac{\pi}{2}$, the integration constant u_c becomes zero on account of the condition of symmetry. Substitution of equations (15) into the first of equations (12) gives

$$\frac{r}{R_1}\frac{d\varepsilon_2}{d\varphi}-(\varepsilon_1-\varepsilon_2)\cos\varphi=-\theta\sin\varphi+\omega_0\cos\varphi.\tag{16}$$

Fig. 3.



Further, as shown in Fig. 3, let.

T_1 and T_2 be the total normal forces acting on the cross-sections φ=constant and ψ=constant respectively per unit length of the middle layers of the wall,

G_1 and G_2 be the bending moments acting on the same cross-sections per unit length and

N be the total shearing force acting on the cross-section φ=constant per unit length;

that is

$$\left.\begin{aligned}T_1&=\int_{-h}^{+h}\sigma_\varphi\frac{r_z}{r}dz=\int_{-h}^{+h}\sigma_\varphi\frac{R_2-z}{R_2}dz,\\ T_2&=\int_{-h}^{+h}\sigma_\psi\frac{\rho_z}{R_1}dz=\int_{-h}^{+h}\sigma_\psi\frac{R_1-z}{R_1}dz,\\ G_1&=\int_{-h}^{+h}\sigma_\varphi\frac{r_z}{r}zdz=\int_{-h}^{+h}\sigma_\varphi\frac{R_2-z}{R_2}zdz,\\ G_2&=\int_{-h}^{+h}\sigma_\psi\frac{\rho_z}{R_1}zdz=\int_{-h}^{+h}\sigma_\psi\frac{R_1-z}{R_1}zdz,\\ N&=\int_{-h}^{+h}\tau\frac{r_z}{r}dz=\int_{-h}^{+h}\tau\frac{R_2-z}{R_2}dz.\end{aligned}\right\}\tag{17}$$

Substituting equations (13) and (14) in equations (17), and neglecting the terms of the second and the higher power of $\left(\frac{z}{R_1}\right)$ and $\left(\frac{z}{R_2}\right)$ as before, we get

$$\left.\begin{aligned}T_1&=\frac{2hE}{1-\mu^2}(\varepsilon_1+\mu\varepsilon_2),\\ T_2&=\frac{2hE}{1-\mu^2}(\varepsilon_2+\mu\varepsilon_1),\end{aligned}\right\}\tag{18}$$

$$\left.\begin{aligned}G_1&=-\frac{2h^3E}{3(1-\mu^2)}\left\{\frac{1}{R_1}\frac{d\theta}{d\varphi}+\mu\frac{\theta\cot\varphi}{R_2}\right.\\&\qquad\left.-\varepsilon_1\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\mu\frac{\omega_0}{R_2}\right\},\\ G_2&=-\frac{2h^3E}{3(1-\mu^2)}\left\{\frac{\theta\cot\varphi}{R_2}+\frac{\mu}{R_1}\frac{d\theta}{d\varphi}\right.\\&\qquad\left.+\varepsilon_2\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\frac{\omega_0}{R_2}\right\}.\end{aligned}\right\}\tag{19}$$

In order to obtain the conditions of equilibrium of these forces and moments, multiply both equations (8) by dz and the first of them by zdz, and integrate them between the limits $\pm h$. Then, considering the surface conditions of the wall

$$\left.\begin{aligned}&[\tau]_{z=\pm h}=[\sigma_z]_{z=-h}=0,\\ &[\sigma_z]_{z=+h}=-p,\end{aligned}\right\}\tag{20}$$

we can obtain the following conditions of equilibrium.

$$\left.\begin{aligned}&\frac{d}{d\varphi}(T_1r)-T_2R_1\cos\varphi-Nr=0,\\ &\frac{d}{d\varphi}(Nr)+T_1r+T_2R_1\sin\varphi\\&\qquad=p(R_1-h)(r-h\sin\varphi),\\ &\frac{d}{d\varphi}(G_1r)-G_2R_1\cos\varphi-NrR_1=0.\end{aligned}\right\}\tag{21}$$

Eliminating T_2 from the first two equations (21), and then integrating with respect to φ, we get

$$r(T_1\sin\varphi+N\cos\varphi)=\frac{1}{2}p(r-h\sin\varphi)^2+C.$$

If we put $\qquad [N]_{\varphi=0}\equiv 2h\tau_0, \qquad (22)$

the integration constant C becomes

$$C=2h\tau_0r_0-\frac{pr_0^2}{2},$$

and T_1 and T_2 can be expressed by the following:

$$\left.\begin{aligned} T_1&=-N\cot\varphi+\frac{p}{2}\frac{(r-h\sin\varphi)^2-r_0^2}{r\sin\varphi}+\frac{2h\tau_0 r_0}{r\sin\varphi},\\ T_2&=-\frac{1}{R_1}\frac{d}{d\varphi}\left(\frac{Nr}{\sin\varphi}\right)+\frac{pR_1}{2}\left[\left(1-\frac{h}{R_1}\right)^2\right.\\ &\qquad\left.-\frac{a^2-r_0^2}{R_1^2\sin^2\varphi}\right]-\frac{2h\tau_0 r_0}{R_1\sin^2\varphi},\end{aligned}\right\}\tag{23}$$

or, introducing a new variable V such as

$$V\equiv NR_2=\frac{Nr}{\sin\varphi},\tag{24}$$

they become

$$\left.\begin{aligned} T_1&=-\frac{V\cos\varphi}{r}+\frac{p}{2}\frac{(r-h\sin\varphi)^2-r_0^2}{r\sin\varphi}+\frac{2h\tau_0 r_0}{r\sin\varphi},\\ T_2&=-\frac{1}{R_1}\frac{dV}{d\varphi}+\frac{pR_1}{2}\left[\left(1-\frac{h}{R_1}\right)^2\right.\\ &\qquad\left.-\frac{a^2-r_0^2}{R_1^2\sin^2\varphi}\right]-\frac{2h\tau_0 r_0}{R_1\sin^2\varphi}.\end{aligned}\right\}\tag{25}$$

From equations (16), (18), (19), (21) and (25) we can obtain two simultaneous differential equations of the variables θ and V, which, however, appear impossible of solution; therefore, we must make some suitable approximations effecting no appreciable errors to the results. For this purpose, we now investigate the following two cases separately:

Case I. The theory of Bourdon pressure tubes and of thin walled curved pipes in general.

Case II. The bending of comparatively thick walled curved pipes with the circular cross-section.

In Case I, it will easily be seen that ε_1 and ε_2 become very small compared to the amount of deformation, that is to the values of θ and $\frac{d\theta}{d\varphi}$; and, therefore, we can neglect the terms of ε_1 and ε_2 compared to those of θ and $\frac{d\theta}{d\varphi}$ in equations (19).

In Case II, however, the cross-section is difficult to deform, and the neglect of ε_1 and ε_2 will cause considerable errors in the results. But, even in this case, it is possible to estimate from the previous approximate theories that ε_1 and ε_2 still take considerably smaller values than θ and $\frac{d\theta}{d\varphi}$; and, therefore, we can substitute, with sufficient accuracy, the values of ε_1 and ε_2 obtained by some of the previous approximate theories in place of those in equations (19). We take here the results of Kármán's first approximate theory,[5] that is

$$\left.\begin{aligned} \varepsilon_1&=0,\\ \varepsilon_2&=\frac{\omega_0 R_1}{r_0}(1-k\sin^2\varphi)\sin\varphi,\end{aligned}\right\}\tag{26}$$

or

$$T_2=2hE\omega_0\frac{R_1}{r_0}(1-k\sin^2\varphi)\sin\varphi,\tag{27}$$

where

$$k=\frac{6}{5+24\left(\frac{hr_0}{R_1^2}\right)^2}.\tag{28}$$

In order to estimate both ε_1 and ε_2, in the present case, we make further the following approximate estimations based on the equation (27). Putting $p=0$ and $\tau_0\approx 0$ in equations (23), and eliminating N from them, we get

$$T_1=\frac{\cos\varphi}{r}\int_{\varphi=-\frac{\pi}{2}}^{\varphi}T_2R_1d\varphi\approx\frac{\cos\varphi}{r_0}\int_{\varphi=-\frac{\pi}{2}}^{\varphi}T_2R_1d\varphi.$$

By substituting the value of T_2 given by the equation (27) in the above,

$$\begin{aligned} T_1&=-2hE\omega_0\left(\frac{R_1}{r_0}\right)^2\left(1-k+\frac{k}{3}\cos^2\varphi\right)\cos^2\varphi\\ &\approx-2hE\omega_0\left(\frac{R_1}{r_0}\right)^2\left(1-\frac{2}{3}k\right)(1-\sin^2\varphi).\end{aligned}$$

Then from the equations (18), we have approximately

$$\left.\begin{aligned} \varepsilon_1&=-\omega_0\frac{R_1}{r_0}\left[\frac{R_1}{r_0}\left(1-\frac{2}{3}k\right)(1-\sin^2\varphi)\right.\\ &\qquad\left.+\mu(1-k\sin^2\varphi)\sin\varphi\right],\\ \varepsilon_2&=\omega_0\frac{R_1}{r_0}\left[\mu\frac{R_1}{r_0}\left(1-\frac{2}{3}k\right)(1-\sin^2\varphi)\right.\\ &\qquad\left.+(1-k\sin^2\varphi)\sin\varphi\right].\end{aligned}\right\}\tag{29}$$

By the above approximation, equations (19) become:

For Case I,

$$\left.\begin{aligned} G_1&=-B\left[\frac{1}{R_1}\frac{d\theta}{d\varphi}+\mu\frac{\theta\cos\varphi}{r}+\mu\frac{\omega_0\sin\varphi}{r}\right],\\ G_2&=-B\left[\frac{\theta\cos\varphi}{r}+\frac{\mu}{R_1}\frac{d\theta}{d\varphi}+\frac{\omega_0\sin\varphi}{r}\right].\end{aligned}\right\}\tag{30$_{\rm I}$}$$

For Case II,

$$\left.\begin{aligned} G_1&=-B\left[\frac{1}{R_1}\frac{d\theta}{d\varphi}+\mu\frac{\theta\cos\varphi}{r}+\frac{\omega_0}{r}\times\right.\\ &\qquad\left\{\frac{R_1}{r_0}\left(1-\frac{2}{3}k\right)(1-\sin^2\varphi)\right.\\ &\qquad\left.\left.+\mu(2-k\sin^2\varphi)\sin\varphi\right\}\right],\\ G_2&=-B\left[\frac{\theta\cos\varphi}{r}+\frac{\mu}{R_1}\frac{d\theta}{d\varphi}+\frac{\omega_0}{r}\times\right.\\ &\qquad\left\{\mu\frac{R_1}{r_0}\left(1-\frac{2}{3}k\right)(1-\sin^2\varphi)\right.\\ &\qquad\left.\left.+(2-k\sin^2\varphi)\sin\varphi\right\}\right],\end{aligned}\right\}\tag{30$_{\rm II}$}$$

(5) See foot-note (1).

where $$B=\frac{2h^3E}{3(1-\mu^2)}. \tag{31}$$

Substitute equations (25) in the equation (16) by the relations (18), and also substitute equations $(30)_I$ or $(30)_{II}$ in the third of equations (21), then we have the fundamental simultaneous differential equations of the variables θ and V in the following form.

$$\left.\begin{aligned} L_{(V)}+\alpha\mu V&=\alpha\lambda_1\theta+\Phi_1,\\ L_{(\theta)}-\alpha\mu\theta&=-\alpha\lambda_2 V+\Phi_2, \end{aligned}\right\} \tag{32}$$

where $$\alpha=\frac{R_1}{a}, \tag{33}$$

$$\lambda_1=2hER_1, \qquad \lambda_2=\frac{R_1}{B}, \tag{34}$$

$$L_{(x)}=\frac{1+\alpha\sin\varphi}{\sin\varphi}\frac{d^2x}{d\varphi^2}+\alpha\cot\varphi\frac{dx}{d\varphi}-\frac{\alpha^2\cdot\cos^2\varphi}{\sin\varphi(1+\alpha\sin\varphi)}x. \tag{35}$$

And, for Case I,

$$\left.\begin{aligned} \Phi_1&=\frac{\cos\varphi}{\sin^4\varphi(1+\alpha\sin\varphi)}\Big[\Big\{\frac{p}{2}(a^2-r_0^2)+2h\tau_0r_0\Big\}\times\\ &\quad(2+3\alpha\sin\varphi)-\frac{pa^2}{2}\Big(1-\frac{h^2}{R_1^2}\Big)\alpha^3\sin^3\varphi\Big]\\ &\quad-\alpha\lambda_1\omega_0\cot\varphi,\\ \Phi_2&=-\frac{\alpha\omega_0\cot\varphi}{1+\alpha\sin\varphi}\Big\{\mu-(1-\mu)\alpha\sin\varphi\Big\}; \end{aligned}\right\} \tag{36$_I$}$$

for Case II,

$$\left.\begin{aligned} \Phi_1&=\frac{\cos\varphi}{\sin^4\varphi(1+\alpha\sin\varphi)}2h\tau_0r_0(2+3\alpha\sin\varphi)\\ &\quad-\alpha\lambda_1\omega_0\cot\varphi,\\ \Phi_2&=-\frac{\alpha\omega_0\cot\varphi}{1+\alpha\sin\varphi}\Big\{\sum_{\nu=0}^{3}A_\nu\sin^\nu\varphi\Big\}, \end{aligned}\right\} \tag{36$_{II}$}$$

$$\left.\begin{aligned} \text{where}\quad A_0&=\mu\Big(2-\alpha^2+\frac{2}{3}\alpha^2k\Big),\\ A_1&=-2\alpha\Big(2-\mu-\frac{2}{3}k\Big),\\ A_2&=-\Big\{3\mu k+\alpha^2(2-\mu)\Big(1-\frac{2}{3}k\Big)\Big\},\\ A_3&=\alpha(1-3\mu)k. \end{aligned}\right\} \tag{37}$$

Now, let

θ_1 and V_1 be the complementary functions of the general solutions of the differential equations (32), and

θ_2 and V_2 be the particular integrals of the same differential equations.

Then the required general solutions of equations (32) will be given by

$$\left.\begin{aligned} \theta&=\theta_1+\theta_2,\\ V&=V_1+V_2. \end{aligned}\right\} \tag{38}$$

IV. The Complementary Functions of the General Solutions.

Eliminating θ and V respectively from equations (32), and putting

$$\Phi_1=\Phi_2=0,\quad \theta=\theta_1 \text{ and } V=V_1,$$

we get $$\left.\begin{aligned} LL_{(\theta_1)}+\alpha^2n^2\theta_1&=0,\\ LL_{(V_1)}+\alpha^2n^2V_1&=0, \end{aligned}\right\} \tag{39}$$

where $$n^2=\lambda_1\lambda_2-\mu^2. \tag{40}$$

Two differential equations (39) are of quite the same form to each other, and both θ_1 and V_1 must be given by the same fundamental system of integrals, differing only by the values of the integration constants. We need, therefore, to solve only one of the two equations (39), which we determine here as the first, that is the equation of θ_1.

According to Meissner, the above differential equation of the fourth order can be divided into two equations of the second order as follows:

$$\left.\begin{aligned} L_{(\theta_1)}-i\alpha n\theta_1&=0,\\ L_{(\theta_1)}+i\alpha n\theta_1&=0, \end{aligned}\right\} \tag{41}$$

where $i=\sqrt{-1}$.

Obviously the integrals of the above two differential equations are conjugate imaginaries to each other, and, if we know the integral of one of the two equations, the integral of the other can be obtained by changing the sign of i. It is, therefore, sufficient to solve only one, the first for example, of the two equations (41), that is

$$L_{(\theta_1)}-i\alpha n\theta_1=0,$$

or $$\frac{1+\alpha\sin\varphi}{\sin\varphi}\frac{d^2\theta_1}{d\varphi^2}+\alpha\cot\varphi\frac{d\theta_1}{d\varphi}-\Big\{\frac{\alpha^2\cos^2\varphi}{\sin\varphi(1+\alpha\sin\varphi)}+i\alpha n\Big\}\theta_1=0. \tag{42}$$

Putting $$\sin\varphi=x, \tag{43}$$

the equation (42) becomes

$$(1-x^2)(1+\alpha x)^2\frac{d^2\theta_1}{dx^2}+(1+\alpha x)(\alpha-x-2\alpha x^2)\frac{d\theta_1}{dx}-\{\alpha^2+i\alpha nx+\alpha^2(in-1)x^2\}\theta_1=0, \tag{44}$$

which has the form

$$(x-b)^2\Big[\sum\beta_{2q}(x-b)^q\Big]\frac{d^2\theta_1}{dx^2}+(x-b)\times\Big[\sum\beta_{1q}(x-b)^q\Big]\frac{d\theta_1}{dx}+\Big[\sum\beta_{0q}(x-b)^q\Big]\theta_1=0, \tag{45}$$

that is a differential equation of Fuchs' type, integral of which can be given by

$$\theta_1=(x-b)^\rho\sum_{v=0}^{\infty}C_v(x-b)^v, \tag{46}$$

an infinite power series expanded around a pole $x=b$.

The domain of convergency of the series (46) is inside a circle having the point $x=b$ as centre, and passing a singular point nearest to this centre. The singular points of the differential equation (44) in the finite region are

$$x=+1,\ x=-1 \text{ and } x=-\frac{1}{a}:$$

the points $x=\pm 1$, poles of the first order, are the intersection of the neutral layer of the wall and the plane of symmetry in a meridian plane, and the points $x=-\frac{1}{a}$, poles of the second order, lie always on the axis of rotation, which in the present case can never be attained.

In order to obtain as large a range of convergency as possible, at the same time making the convergency in the useful range as good as possible, we now assume the value of b as follows:

$$\text{for } -\pi \leqq \varphi \leqq 0, \qquad b=-1;$$
$$\text{and for } 0 \leqq \varphi \leqq \pi, \qquad b=+1.$$

Then the ranges of convergency on the real axis become:

for $-\pi \leqq \varphi \leqq 0$, that is for $b=-1$,

$$-3<x<+1 \text{ when } \frac{1}{a}\geqq 3 \text{ and } \frac{1}{a}=1,$$

$$-\frac{1}{a}<x<\frac{1}{a}-2 \text{ when } 1<\frac{1}{a}<3,$$

$$\frac{1}{a}-2<x<-\frac{1}{a} \qquad \text{,,} \qquad 0<\frac{1}{a}<1;$$

and for $0 \leqq \varphi \leqq \pi$, that is for $b=+1$,

$$-1<x<+3 \qquad \text{when} \quad \frac{1}{a}\geqq 1,$$

$$-\frac{1}{a}<x<\frac{1}{a}+2 \qquad \text{,,} \qquad \frac{1}{a}<1.$$

To avoid the difficulties of convergency for a circular cross-section, we make here the following limitation to the value of a, that is

$$\frac{1}{a}=\frac{r_0}{R_1}>2, \text{ or more preferably } \frac{1}{a}\geqq 3.$$

In practice, most oval cross-section, Bourdon pressure tubes for example, seem to give no difficulties of convergency in the useful ranges of x.

Because of this, we give the solutions separately for each range of φ greater and smaller than zero.

(a) $0 \leqq \varphi \leqq \pi, \quad b=+1$

From the equation (44),

$$(x-1)^2(1+x)(1+ax)^2\frac{d^2\theta_1}{dx^2}-(x-1)(1+ax)\times$$
$$(a-x-2ax^2)\frac{d\theta_1}{dx}+(x-1)\{a^2+ianx$$
$$+a^2(in-1)x^2\}\theta_1=0,$$

or

$$(x-1)^2\Big[\sum_{q=0}^{3}\beta_{2q}(x-1)^q\Big]\frac{d^2\theta_1}{dx^2}+(x-1)\times$$
$$\Big[\sum_{q=0}^{3}\beta_{1q}(x-1)^q\Big]\frac{d\theta_1}{dx}+\Big[\sum_{q=0}^{3}\beta_{0q}(x-1)^q\Big]\theta_1=0, \quad (47)$$

where

$$\left.\begin{aligned}
&\beta_{20}=2(1+a)^2, && \beta_{10}=(1+a)^2,\\
&\beta_{21}=(1+a)(1+5a), && \beta_{11}=(1+a)(1+5a),\\
&\beta_{22}=2a(1+2a), && \beta_{12}=3a(1+2a),\\
&\beta_{23}=a^2, && \beta_{13}=2a^2,\\
&\quad\beta_{00}=0,\\
&\quad\beta_{01}=ian(1+a),\\
&\quad\beta_{02}=ian(1+2a)-2a^2,\\
&\quad\beta_{03}=(in-1)a^2.
\end{aligned}\right\} \quad (48)$$

Putting $b=+1$, the equation (46) becomes

$$\theta_1=(x-1)^\rho\sum_{v=0}^{\infty}C_v(x-1)^v. \qquad (49)$$

Substituting equations (48) and (49) in the equation (47), and equating the coefficient of the term of the lowest power to zero, we get

$$\rho(2\rho-1)=0,$$

that is $\rho=0$, or $\rho=\frac{1}{2}$.

The general integral of θ_1, therefore, can be given by

$$\theta_1=\sum_{v=0}^{\infty}C_v(x-1)^v+(x-1)^{\frac{1}{2}}\sum_{v=0}^{\infty}C'_v(x-1)^v. \qquad (50)$$

In order to get the relations between the coefficients C_v of the first integral, put

$$\theta_1=\sum_{v=0}^{\infty}C_v(x-1)^v \qquad (51)$$

in the equation (47), and equate the coefficient of the term $(x-1)^v$ to zero; then we have

$$\begin{aligned}
&C_v\{v(v-1)\beta_{20}+v\beta_{10}+\beta_{00}\}\\
&+C_{v-1}\{(v-1)(v-2)\beta_{21}+(v-1)\beta_{11}+\beta_{01}\}\\
&+C_{v-2}\{(v-2)(v-3)\beta_{22}+(v-2)\beta_{12}+\beta_{02}\}\\
&+C_{v-3}\{(v-3)(v-4)\beta_{23}+(v-3)\beta_{13}+\beta_{03}\}=0. \quad (52)
\end{aligned}$$

By the relation (52), all the coefficients C_v can be expressed by C_0 and other known constants, where C_0 is an arbitrary integration constant.

Next we are to determine the relation between the coefficients C'_v of the second integral, which, however, becomes so complicated as to be difficult to calculate. Therefore, it is better to secure the second integral by the following different way.

Putting $$\theta_1=\cos\varphi\cdot\theta_1^* \tag{53}$$

in the equation (42), and then employing the transformation of the independent variable, (43), we have

$$(1-x^2)(1+ax)^2\frac{d^2\theta_1^*}{dx^2}+\{a+(a^2-3)x-7ax^2-4a^2x^3\}\frac{d\theta_1^*}{dx}-\{(1+a^2)+a(3+in)x+a^2(1+in)x^2\}\theta_1^*=0, \tag{54}$$

or

$$(x-1)^2\Big[\sum_{q=0}^{3}\beta_{2q}^*(x-1)^q\Big]\frac{d^2\theta_1^*}{dx^2}+(x-1)\Big[\sum_{q=0}^{3}\beta_{1q}^*\times(x-1)^q\Big]\frac{d\theta_1^*}{dx}+\Big[\sum_{q=0}^{3}\beta_{0q}^*(x-1)^q\Big]\theta_1^*=0, \tag{55}$$

$$\left.\begin{aligned}
&\text{where}\quad \beta_{2q}^*=\beta_{2q}, \quad (q=0,\ 1,\ 2,\ 3.)\\
&\beta_{10}^*=3(1+a)^2, \qquad \beta_{00}^*=0,\\
&\beta_{11}^*=(3+11a)(1+a),\quad \beta_{01}^*=(1+a)(1+2a+ian),\\
&\beta_{12}^*=a(7+12a), \qquad \beta_{02}^*=a(3+2a)+ian(1+2a),\\
&\beta_{13}^*=4a^2, \qquad \beta_{03}^*=a^2(1+in).
\end{aligned}\right\} \tag{56}$$

The differential equation (54) or (55) thus obtained is also Fuchs' type, and its integral can be given similarly as before, that is

$$\theta_1^*=(x-1)^{\rho^*}\sum_{v=0}^{\infty}C_v^*(x-1)^v. \tag{57}$$

The values of ρ^* can be determined by the same way as before:

$$\rho^*=0, \text{ and } \rho^*=-\frac{1}{2}.$$

Hereupon, we employ the former value $\rho^*=0$ for the purpose of getting the required second integral of the original differential equation (47). The relation between the coefficients C_v^* can be obtained also by the same way as before, that is

$$\begin{aligned}
&C_v^*\{v(v-1)\beta_{20}^*+v\beta_{10}^*+\beta_{00}^*\}\\
&+C_{v-1}^*\{(v-1)(v-2)\beta_{21}^*+(v-1)\beta_{11}^*+\beta_{01}^*\}\\
&+C_{v-2}^*\{(v-2)(v-3)\beta_{22}^*+(v-2)\beta_{12}^*+\beta_{02}^*\}\\
&+C_{v-3}^*\{(v-3)(v-4)\beta_{23}^*+(v-3)\beta_{13}^*+\beta_{03}^*\}=0,
\end{aligned} \tag{58}$$

and C_0^* is also an arbitrary integration constant.

If we put $$C_0=C_0^*=1, \tag{59}$$

all the coefficients C_v and C_v^* become constructed by known constants only, and the two fundamental systems of integral of the differential equation (47) are given by the following expressions.

$$\sum_{v=0}^{\infty}C_v(x-1)^v=\sum_{v=0}^{\infty}C_v(\sin\varphi-1)^v, \tag{60}$$

$$\cos\varphi\sum_{v=0}^{\infty}C_v^*(x-1)^v=\cos\varphi\sum_{v=0}^{\infty}C_v^*(\sin\varphi-1)^v. \tag{61}$$

Coefficients C_v and C_v^* are complex functions, and if we express conjugate imaginaries by $\overline{C_v}$ and $\overline{C_v^*}$ respectively, the infinite series

$$\sum_{v=0}^{\infty}\overline{C_v}(\sin\varphi-1)^v \tag{62}$$

and $$\cos\varphi\sum_{v=0}^{\infty}\overline{C_v^*}(\sin\varphi-1)^v \tag{63}$$

become the integrals of the second differential equation of (41), and, therefore, all four expressions (60)~(63) are the integrals of the first differential equation of (39). Consequently, any proper sum of these values must also be integrals of the same equation, and we take here the following four expressions as the fundamental system of integrals of the first differential equation of (39).

$$\left.\begin{aligned}
\theta_{1.1}&=\frac{1}{2}\Big[\sum_{v=0}^{\infty}C_v(\sin\varphi-1)^v+\sum_{v=0}^{\infty}\overline{C_v}(\sin\varphi-1)^v\Big]=\sum_{v=0}^{\infty}\frac{C_v+\overline{C_v}}{2}(\sin\varphi-1)^v,\\
\theta_{1.2}&=\frac{1}{2i}\Big[\sum_{v=0}^{\infty}C_v(\sin\varphi-1)^v-\sum_{v=0}^{\infty}\overline{C_v}(\sin\varphi-1)^v\Big]=\sum_{v=0}^{\infty}\frac{C_v-\overline{C_v}}{2i}(\sin\varphi-1)^v,\\
\theta_{1.3}&=\frac{\cos\varphi}{2}\Big[\sum_{v=0}^{\infty}C_v^*(\sin\varphi-1)^v+\sum_{v=0}^{\infty}\overline{C_v^*}(\sin\varphi-1)^v\Big]=\cos\varphi\sum_{v=0}^{\infty}\frac{C_v^*+\overline{C_v^*}}{2}(\sin\varphi-1)^v,\\
\theta_{1.4}&=\frac{\cos\varphi}{2i}\Big[\sum_{v=0}^{\infty}C_v^*(\sin\varphi-1)^v-\sum_{v=0}^{\infty}\overline{C_v^*}(\sin\varphi-1)^v\Big]=\cos\varphi\sum_{v=0}^{\infty}\frac{C_v^*-\overline{C_v^*}}{2i}(\sin\varphi-1)^v.
\end{aligned}\right\} \tag{64}$$

Or putting

$$\left.\begin{aligned}
\frac{C_v+\overline{C_v}}{2}&=\text{Real part of } C_v\equiv k_v,\\
\frac{C_v-\overline{C_v}}{2i}&=\text{Imaginary part of } C_v\equiv j_v,\\
\frac{C_v^*+\overline{C_v^*}}{2}&=\text{Real part of } C_v^*\equiv k_v^*,\\
\frac{C_v^*-\overline{C_v^*}}{2i}&=\text{Imaginary part of } C_v^*\equiv j_v^*,
\end{aligned}\right\} \tag{65}$$

that is
$$\left.\begin{aligned}
C_v&=k_v+ij_v, & \overline{C_v}&=k_v-ij_v,\\
C_v^*&=k_v^*+ij_v^*, & \overline{C_v^*}&=k_v^*-ij_v^*,
\end{aligned}\right\} \tag{65$_a$}$$

in which $$k_0=k_0^*=1 \text{ and } j_0=j_0^*=0, \tag{66}$$

we have $$\theta_{1.1}=\sum_{v=0}^{\infty}k_v(\sin\varphi-1)^v,$$

$$\left.\begin{aligned}\theta_{1.2}&=\sum_{v=0}^{\infty}j_v(\sin\varphi-1)^v,\\ \theta_{1.3}&=\cos\varphi\sum_{v=0}^{\infty}k_v^*(\sin\varphi-1)^v,\\ \theta_{1.4}&=\cos\varphi\sum_{v=0}^{\infty}j_v^*(\sin\varphi-1)^v.\end{aligned}\right\}\qquad(67)$$

Then the general solution of the first differential equation of (39) is given by

$$\theta_1=B_1\theta_{1.1}+B_2\theta_{1.2}+B_3\theta_{1.3}+B_4\theta_{1.4},\qquad(68)$$

where $B_1\sim B_4$ are four integration constants.

Next, in order to obtain the general solution of V_1, that is the second differential equation of (39), we proceed as follows.

From equations (64) we have

$$\left.\begin{aligned}\sum_{v=0}^{\infty}C_v(\sin\varphi-1)^v&=\theta_{1.1}+i\theta_{1.2},\\ \sum_{v=0}^{\infty}\overline{C_v}(\sin\varphi-1)^v&=\theta_{1.1}-i\theta_{1.2},\\ \cos\rho\cdot\sum_{v=0}^{\infty}C_v^*(\sin\varphi-1)^v&=\theta_{1.3}+i\theta_{1.4},\\ \cos\varphi\cdot\sum_{v=0}^{\infty}\overline{C_v^*}(\sin\varphi-1)^v&=\theta_{1.3}-i\theta_{1.4}.\end{aligned}\right\}$$

Putting these values into equations (41), and separating them into the real and the imaginary parts, we get

$$\left.\begin{aligned}L_{(\theta_{1\cdot1})}&=-an\theta_{1.2},\\ L_{(\theta_{1\cdot2})}&=an\theta_{1.1},\\ L_{(\theta_{1\cdot3})}&=-an\theta_{1.4},\\ L_{(\theta_{1\cdot4})}&=an\theta_{1.3}.\end{aligned}\right\}\qquad(69)$$

On the other hand, from the second equation of (32), putting $\Phi_2=0$, we have

$$V_1=-\frac{1}{a\lambda_2}\{L_{(\theta_1)}-a\mu\theta_1\},$$

which becomes, by equations (68) and (69),

$$V_1=\frac{B_1}{\lambda_2}(n\theta_{1.2}+\mu\theta_{1.1})+\frac{B_2}{\lambda_2}(\mu\theta_{1.2}-n\theta_{1.1})+\frac{B_3}{\lambda_2}(n\theta_{1.4}+\mu\theta_{1.3})+\frac{B_4}{\lambda_2}(\mu\theta_{1.4}-n\theta_{1.3}).$$

Or putting $\frac{B_q}{\lambda_2}\equiv D_q$, $(q=1,\ 2,\ 3,\ 4)$, (70)

and

$$\left.\begin{aligned}V_{1.1}&=n\theta_{1.2}+\mu\theta_{1.1}\\ &=\sum_{v=0}^{\infty}(nj_v+\mu k_v)(\sin\varphi-1)^v,\\ V_{1.2}&=\mu\theta_{1.2}-n\theta_{1.1}\\ &=\sum_{v=0}^{\infty}(\mu j_v-nk_v)(\sin\varphi-1)^v,\\ V_{1.3}&=n\theta_{1.4}+\mu\theta_{1.3}\\ &=\cos\varphi\sum_{v=0}^{\infty}(nj_v^*+\mu k_v^*)(\sin\varphi-1)^v,\\ V_{1.4}&=\mu\theta_{1.4}-n\theta_{1.3}\\ &=\cos\varphi\sum_{v=0}^{\infty}(\mu j_v^*-nk_v^*)(\sin\varphi-1)^v,\end{aligned}\right\}\qquad(71)$$

we have

$$V_1=D_1V_{1.1}+D_2V_{1.2}+D_3V_{1.3}+D_4V_{1.4},\qquad(72)$$

which is the required general solution of V_1.

(b) $-\pi\leqq\varphi\leqq0,\quad b=-1.$

Since the process of solution in this case is quite the same as before, it will be sufficient to give only the important expressions necessary for the numerical calculations.

Instead of the equations (47) and (55) we get in this case the following differential equations:

$$(x+1)^2\Big[\sum_{q=0}^{3}\beta_{2q}(x+1)^q\Big]\frac{d^2\theta_1}{dx^2}+(x+1)\Big[\sum_{q=0}^{3}\beta_{1q}(x+1)^q\Big]\frac{d\theta_1}{dx}+\Big[\sum_{q=0}^{3}\beta_{0q}(x+1)^q\Big]\theta_1=0,\qquad(73)$$

where

$$\left.\begin{aligned}&\beta_{20}=2(1-a)^2, &&\beta_{10}=(1-a)^2,\\ &\beta_{21}=-(1-a)(1-5a), &&\beta_{11}=-(1-a)(1-5a),\\ &\beta_{22}=-2a(1-2a), &&\beta_{12}=-3a(1-2a),\\ &\beta_{23}=-a^2, &&\beta_{13}=-2a^2,\\ &\beta_{00}=0, &&\\ &\beta_{01}=ian(1-a), &&\\ &\beta_{02}=-ian(1-2a)-2a^2, &&\\ &\beta_{03}=-(in-1)a^2. &&\end{aligned}\right\}\qquad(74)$$

$$(x+1)^2\Big[\sum_{q=0}^{3}\beta_{2q}^*(x+1)^q\Big]\frac{d^2\theta_1^*}{dx^2}+(x+1)\Big[\sum_{q=0}^{3}\beta_{1q}^*(x+1)^q\Big]\frac{d\theta_1^*}{dx}+\Big[\sum_{q=0}^{3}\beta_{0q}^*(x+1)^q\Big]\theta_1^*=0,\qquad(75)$$

where $\beta_{2q}^*=\beta_{2q}$, $(q=0,\ 1,\ 2,\ 3.)$

$$\underbrace{\begin{aligned}&\beta_{10}^*=3(1-a)^2, &&\beta_{00}^*=0,\\ &\beta_{11}^*=-(1-a)(3-11a), &&\beta_{01}^*=-(1-a)(1-2a-ian),\\ &\beta_{12}^*=-a(7-12a), &&\beta_{02}^*=-a(3-2a)-ian(1-2a),\\ &\beta_{13}^*=-4a^2, &&\beta_{03}^*=-a^2(1+in).\end{aligned}}_{(76)}$$

The fundamental system of integrals of the equation (73) is expressed similarly by

$$\sum_{v=0}^{\infty}C_v(x+1)^v=\sum_{v=0}^{\infty}C_v(\sin\varphi+1)^v,$$

and

$$\cos\varphi\sum_{v=0}^{\infty}C_v^*(x+1)^v=\cos\varphi\sum_{v=0}^{\infty}C_v^*(\sin\varphi+1)^v. \quad (77)$$

Equations (52) and (58) hold good in this case also; and equations (62)~(72) can be used here if we write $(\sin\varphi+1)$ in place of $(\sin\varphi-1)$.

V. Particular Integrals.

Eliminating V from equations (32), and writing θ_2 instead of θ, we have

$$LL_{(\theta_2)}+a^2n^2\theta_2=\Phi, \quad (78)$$

where $\Phi=L_{(\Phi_2)}+a\mu\Phi_2-a\lambda_2\Phi_1.$ (79)

If we put the values of Φ_1 and Φ_2 in the equation (79), Φ becomes

$$\Phi=\frac{\cos\varphi}{\sin^4\varphi(1+a\sin\varphi)}\Big[H_0(2+3a\sin\varphi)+\sum_{v=1}^{4}H_v\sin^{v+2}\varphi\Big], \quad (80)$$

where, for Case I,

$$\left.\begin{aligned}
H_0&=-a\mu\omega_0-a\lambda_2\Big\{\frac{p}{2}(a^2-r_0^2)+2h\tau_0r_0\Big\},\\
H_1&=a^2(n^2-1)\omega_0+a^4\lambda_2\frac{pa^2}{2}\Big(1-\frac{h^2}{R_1^2}\Big),\\
H_2&=a^3(n^2+1)\omega_0,\\
H_3&=H_4=0;
\end{aligned}\right\}(81)_{\text{I}}$$

and for Case II,

$$\left.\begin{aligned}
H_0&=-a\mu\omega_0\Big(2-a^2+\frac{2}{3}a^2k\Big)-2a\lambda_2h\tau_0r_0,\\
H_1&=a^2\omega_0\Big[n^2-4-\mu^2-a^2(2-2\mu-\mu^2)\\
&\qquad+\frac{1}{3}k\{9\mu-2+2a^2(2-2\mu-\mu^2)\}\Big],\\
H_2&=a\omega_0\Big[a^2(n^2-4+6\mu-\mu^2)-4k(3\mu+a^2\mu-a^2)\Big],\\
H_3&=-a^2\omega_0\Big[a^2(2-3\mu+\mu^2)-\frac{1}{3}k(27-90\mu\\
&\qquad+9\mu^2+4a^2-6\mu a^2+2\mu^2a^2)\Big],\\
H_4&=a^3\omega_0k(5-\mu)(1-3\mu).
\end{aligned}\right\}(81)_{\text{II}}$$

Now, according to Wissler,[6] let us consider the following differential equation of the second order:

$$L_{(t)}-iant=\frac{1}{an}\Phi, \quad (82)$$

and assume $t=P_{(\varphi)}+iQ_{(\varphi)}$ (83)

to be an integral of it. Putting the equation (83) into the equation (82), and separating them into the real and the imaginary parts, we have

$$\left.\begin{aligned}
L_{(P)}+anQ&=\frac{1}{an}\Phi,\\
L_{(Q)}-anP&=0.
\end{aligned}\right\}\quad (84)$$

By further eliminatieg P from equations (84), we get

$$LL_{(Q)}+a^2n^2Q=\Phi.$$

We know, therefore, that a particular integral of the differential equation (78) is given by the imaginary part of an integral of the differential equation (82), that is

$$\theta_2=Q_{(\varphi)}. \quad (85)$$

Then, from the second equation of (32), employing the second relation of (84), we get a particular integral of V_2 as follows:

$$\begin{aligned}
V_2&=-\frac{1}{a\lambda_2}\{L_{(\theta_2)}-a\mu\theta_2-\Phi_2\}\\
&=-\frac{1}{a\lambda_2}\{L_{(Q)}-a\mu Q-\Phi_2\}\\
&=-\frac{1}{\lambda_2}\Big\{nP-\mu Q-\frac{1}{a}\Phi_2\Big\}.
\end{aligned}\quad (86)$$

We need, therefore, only to solve the differential equation (82), that is

$$\frac{1+a\sin\varphi}{\sin\varphi}\Big[\frac{d^2t}{d\varphi^2}+\frac{a\cos\varphi}{1+a\sin\varphi}\frac{dt}{d\varphi}-\frac{a^2\cos^2\varphi}{(1+a\sin\varphi)^2}t\Big]-iant=\frac{1}{an}\frac{\cos\varphi}{\sin^4\varphi(1+a\sin\varphi)}\times\Big[H_0(2+3a\sin\varphi)+\sum_{v=1}^{4}H_v\sin^{v+2}\varphi\Big]. \quad (87)$$

Although Wissler has solved a similar differential equation in his paper,[7] since his solution seems somewhat unskilful in regard to the convergency of the series, here we do not follow him. In order to give a solution by infinite series having quite the same range of convergency as those in the complementary functions, we make the transformation of the variables t and φ as follows:

$$\left.\begin{aligned}
t&=\frac{H_0}{an}\cot\varphi+\cos\varphi\cdot t^*,\\
\sin\varphi&=x.
\end{aligned}\right\}\quad (88)$$

Then the equation (87) becomes

$$(1-x^2)(1+ax)^2\frac{d^2t^*}{dx^2}+\{a+(a^2-3)x-7ax^2-4a^2x^3\}\frac{dt^*}{dx}-\{(1+a^2)+a(3+in)x$$

(6) H. Wissler, *Festigkeitsberechnung von Ringflächenschalen*, Promotionsarbeit, Zürich, 1916.

(7) The subject of Wissler's paper is an axial symmetrical ring shell, which is the case when the surface of revolution is completed, but the shape of a meridian section is not closed; a practical example of such a case is the corner part of the end plate of a cylindrical boiler.

$$+a^2(1+in)x^2\}t^* = \left(\frac{H_1}{an}+iH_0\right)$$

$$+\left(\frac{H_2}{an}-\frac{a}{n}H_0+iaH_0\right)x+\frac{H_3}{an}x^2+\frac{H_4}{an}x^3. \quad (89)$$

The equation (89) has only the same singular points as (54), and the left hand sides of these two equations have quite the same form; therefore, the same conditions of convergency must hold here for the solution of t^*, which will also be given by the same form as before, that is

$$t^* = \sum_{v=0}^{\infty} C_v^{**}(x-b)^v. \quad (90)$$

Similarly as before, we assume

$$b=-1 \quad \text{for} \quad -\pi \leqq \varphi \leqq 0,$$

and $$b=+1 \quad \text{for} \quad 0 \leqq \varphi \leqq \pi.$$

(a) $0 \leqq \varphi \leqq \pi, \quad b=+1.$

The equations (89) and (90) become in this case:

$$(x-1)^2\left[\sum_{q=0}^{3}\beta_{2q}^*(x-1)^q\right]\frac{d^2t^*}{dx^2}$$

$$+(x-1)\left[\sum_{q=0}^{3}\beta_{1q}^*(x-1)^q\right]\frac{dt^*}{dx}$$

$$+\left[\sum_{q=0}^{3}\beta_{0q}^*(x-1)^q\right]t^* = \sum_{q=0}^{4}\delta_q^*(x-1)^q, \quad (91)$$

$$t^* = \sum_{v=0}^{\infty} C_v^{**}(x-1)^v = \sum_{v=0}^{\infty} C_v^{**}(\sin\varphi-1)^v. \quad (92)$$

The coefficients β_{vq}^* ($\substack{v=0,\,1,\,2,\\ q=0,\,1,\,2,\,3.}$) are given by (56), and δ_q^* by the following.

$$\left.\begin{aligned}
\delta_0^* &= 0,\\
\delta_1^* &= -\left\{\frac{1}{an}(H_1+H_2+H_3+H_4) - \frac{a}{n}H_0 + i(1+a)H_0\right\},\\
\delta_2^* &= -\left\{\frac{1}{an}(H_2+2H_3+3H_4) - \frac{a}{n}H_0 + iaH_0\right\},\\
\delta_3^* &= -\frac{1}{an}(H_3+3H_4),\\
\delta_4^* &= -\frac{1}{an}H_4.
\end{aligned}\right\} \quad (93)$$

The relation between the coefficients C_v^{**} is given by:

$$\begin{aligned}
&C_v^{**}\{v(v-1)\beta_{20}^*+v\beta_{10}^*+\beta_{00}^*\}\\
&+C_{v-1}^{**}\{(v-1)(v-2)\beta_{21}^*+(v-1)\beta_{11}^*+\beta_{01}^*\}\\
&+C_{v-2}^{**}\{(v-2)(v-3)\beta_{22}^*+(v-2)\beta_{12}^*+\beta_{02}^*\}\\
&+C_{v-3}^{**}\{(v-3)(v-4)\beta_{23}^*+(v-3)\beta_{13}^*+\beta_{03}^*\}=\delta_v^*,
\end{aligned} \quad (94)$$

where $\delta_v^*=0$ for $v \geqq 5$; and, for the sake of simplicity, we put the arbitrary constant C_0^{**} as

$$C_0^{**}=0. \quad (95)$$

Separating, as before, the coefficients C_v^{**} into the real and the imaginary parts, that is, putting

$$C_v^{**}=k_v^{**}+ij_v^{**}, \quad (96)$$

we have the expressions for P and Q as follows:

$$\left.\begin{aligned}
P &= \frac{H_0}{an}\cot\varphi + \cos\varphi\sum_{v=0}^{\infty}k_v^{**}(\sin\varphi-1)^v,\\
Q &= \cos\varphi\sum_{v=0}^{\infty}j_v^{**}(\sin\varphi-1)^v.
\end{aligned}\right\} \quad (97)$$

Then the required particular integrals of the differential equations are:

$$\left.\begin{aligned}
\theta_2 &= \cos\varphi\sum_{v=0}^{\infty}j_v^{**}(\sin\varphi-1)^v,\\
V_2 &= \frac{1}{\lambda_2}\Big[\cos\varphi\sum_{v=0}^{\infty}(\mu j_v^{**}-nk_v^{**})(\sin\varphi-1)^v\\
&\qquad + \frac{1}{a}(\Phi_2 - H_0\cot\varphi)\Big].
\end{aligned}\right\} \quad (98)$$

(b) $-\pi \leqq \varphi \leqq 0, \quad b=-1$

The equations (89) and (90) become in this case:

$$(x+1)^2\left[\sum_{q=0}^{3}\beta_{2q}^*(x+1)^q\right]\frac{d^2t^*}{dx^2}$$

$$+(x+1)\left[\sum_{q=0}^{3}\beta_{1q}^*(x+1)^q\right]\frac{dt^*}{dx}$$

$$+\left[\sum_{q=0}^{3}\beta_{0q}^*(x+1)^q\right]t^* = \sum_{q=0}^{4}\delta_q^*(x+1)^q, \quad (99)$$

$$t^* = \sum_{v=0}^{\infty} C_v^{**}(x+1)^v = \sum_{v=0}^{\infty} C_v^{**}(\sin\varphi+1)^v. \quad (100)$$

The coefficients β_{vq}^* ($\substack{v=0,\,1,\,2,\\ q=0,\,1,\,2,\,3.}$) are given by (76), and δ_q^* by the following.

$$\left.\begin{aligned}
\delta_0^* &= 0,\\
\delta_1^* &= \frac{1}{an}(H_1-H_2+H_3-H_4)+\frac{a}{n}H_0 + i(1-a)H_0,\\
\delta_2^* &= \frac{1}{an}(H_2-2H_3+3H_4)-\frac{a}{n}H_0+iaH_0,\\
\delta_3^* &= \frac{1}{an}(H_3-3H_4),\\
\delta_4^* &= \frac{1}{an}H_4.
\end{aligned}\right\} \quad (101)$$

Equations (94), (95) and (96) hold good in this case also, and equations (97) and (98) can be used here if we write $(\sin\varphi+1)$ in place of $(\sin\varphi-1)$.

VI. Summary of Results.

Here we collect the results obtained above for the expressions of stresses, strains and displacements using the following simplified notations (102)~(105) for the infinite series.

$$\left.\begin{aligned}\theta_{1.1}&=\sum_{v=0}^{\infty}k_v(\sin\varphi\mp 1)^v,\\ \theta_{1.2}&=\sum_{v=0}^{\infty}j_v(\sin\varphi\mp 1)^v,\\ \theta_{1.3}&=\cos\varphi\sum_{v=0}^{\infty}k_v^*(\sin\varphi\mp 1)^v,\\ \theta_{1.4}&=\cos\varphi\sum_{v=0}^{\infty}j_v^*(\sin\varphi\mp 1)^v,\\ \theta_2&=\cos\varphi\sum_{v=0}^{\infty}j_v^{**}(\sin\varphi\mp 1)^v.\end{aligned}\right\}\quad(102)$$

$$\left.\begin{aligned}V_{1.1}&=\sum_{v=0}^{\infty}(nj_v+\mu k_v)(\sin\varphi\mp 1)^v,\\ V_{1.2}&=\sum_{v=0}^{\infty}(\mu j_v-nk_v)(\sin\varphi\mp 1)^v,\\ V_{1.3}&=\cos\varphi\sum_{v=0}^{\infty}(nj_v^*+\mu k_v^*)(\sin\varphi\mp 1)^v,\\ V_{1.4}&=\cos\varphi\sum_{v=0}^{\infty}(\mu j_v^*-nk_v^*)(\sin\varphi\mp 1)^v,\\ V_{2.0}&=\cos\varphi\sum_{v=0}^{\infty}(\mu j_v^{**}-nk_v^{**})(\sin\varphi\mp 1)^v.\end{aligned}\right\}\quad(103)$$

$$\theta=B_1\theta_{1.1}+B_2\theta_{1.2}+B_3\theta_{1.3}+B_4\theta_{1.4}+\theta_2. \quad(104)$$

$$V_s=B_1V_{1.1}+B_2V_{1.2}+B_3V_{1.3}+B_4V_{1.4}+V_{2.0}. \quad(105)$$

$$V=\frac{2h^3E}{3(1-\mu^2)}\left(\frac{V_s}{R_1}+\frac{\omega_0\cos\varphi}{r}\right)+\left\{\frac{p}{2}(a^2-r_0^2)+2h\tau_0r_0\right\}\cot\varphi \quad \text{for Case I. } (106)_{\text{I}}$$

$$V=\frac{2h^3E}{3(1-\mu^2)}\left(\frac{V_s}{R_1}+\frac{\omega_0\cos\varphi}{r}\Phi_0\right)+2h\tau_0r_0\cot\varphi \quad \text{for Case II, } (106)_{\text{II}}$$

where

$$\left.\begin{aligned}\Phi_0&=2+\left(1-\frac{2}{3}k\right)\left(2-\mu\frac{R_1^2}{r_0^2}\right)+\left\{3\mu k\frac{r_0}{R_1}+\frac{R_1}{r_0}(2-\mu)\left(1-\frac{2}{3}k\right)\right\}\sin\varphi\\ &\qquad-(1-3\mu)k\sin^2\varphi,\\ k&=\frac{6\left(\frac{R_1}{r_0}\right)^2}{5\left(\frac{R_1}{r_0}\right)^2+24\left(\frac{h}{R_1}\right)^2}.\end{aligned}\right\}\quad(107)$$

$$T_1=-\frac{2h^3E}{3(1-\mu^2)}\left(\frac{V_s}{R_1}+\frac{\omega_0\cos\varphi}{r}\right)\frac{\cos\varphi}{r}+2h\tau_0r_0\frac{\sin\varphi}{r}+\frac{p}{2r}\left[\{a^2-r_0^2+(R_1-h)^2\}\sin\varphi+2a(R_1-h)\right]$$

for Case I. $(108)_{\text{I}}$

$$T_1=-\frac{2h^3E}{3(1-\mu^2)}\left(\frac{V_s}{R_1}+\frac{\omega_0\cos\varphi}{r}\Phi_0\right)\frac{\cos\varphi}{r}+2h\tau_0r_0\frac{\sin\varphi}{r}$$

for Case II. $(108)_{\text{II}}$

$$T_2=-\frac{2h^3E}{3(1-\mu^2)}\frac{1}{R_1}\frac{d}{d\varphi}\left(\frac{V_s}{R_1}+\frac{\omega_0\cos\varphi}{r}\right)+\frac{pR_1}{2}\left(1-\frac{h}{R_1}\right)^2 \quad \text{for Case I. } (109)_{\text{I}}$$

$$T_2=-\frac{2h^3E}{3(1-\mu^2)}\frac{1}{R_1}\frac{d}{d\varphi}\left(\frac{V_s}{R_1}+\frac{\omega_0\cos\varphi}{r}\Phi_0\right)$$

for Case II. $(109)_{\text{II}}$

$$N=\frac{2h^3E}{3(1-\mu^2)}\left(\frac{V_s}{R_1}+\frac{\omega_0\cos\varphi}{r}\right)\frac{\sin\varphi}{r}+\left\{\frac{p}{2}(a^2-r_0^2)+2h\tau_0r_0\right\}\frac{\cos\varphi}{r} \quad \text{for Case I. } (110)_{\text{I}}$$

$$N=\frac{2h^3E}{3(1-\mu^2)}\left(\frac{V_s}{R_1}+\frac{\omega_0\cos\varphi}{r}\Phi_0\right)\frac{\sin\varphi}{r}+2h\tau_0r_0\frac{\cos\varphi}{r}$$

for Case II. $(110)_{\text{II}}$

$$\left.\begin{aligned}\varepsilon_1&=\frac{1}{2hE}(T_1-\mu T_2),\\ \varepsilon_2&=\frac{1}{2hE}(T_2-\mu T_1).\end{aligned}\right\}\quad(111)$$

$$\left.\begin{aligned}G_1&=-\frac{2h^3E}{3(1-\mu^2)}\left\{\frac{1}{R_1}\frac{d\theta}{d\varphi}+\mu\frac{\theta\cos\varphi}{r}-\varepsilon_1\left(\frac{1}{R_1}-\frac{\sin\varphi}{r}\right)+\mu\frac{\omega_0\sin\varphi}{r}\right\},\\ G_2&=-\frac{2h^3E}{3(1-\mu^2)}\left\{\frac{\theta\cos\varphi}{r}+\frac{\mu}{R_1}\frac{d\theta}{d\varphi}+\varepsilon_2\left(\frac{1}{R_1}-\frac{\sin\varphi}{r}\right)+\frac{\omega_0\sin\varphi}{r}\right\}.\end{aligned}\right\}\quad(112)$$

$$\left.\begin{aligned}u&=\sin\varphi\int_{\pm\frac{\pi}{2}}^{\varphi}\left[\frac{R_1\varepsilon_1}{\sin\varphi}-\frac{r(\varepsilon_2-\omega_0)}{\sin^2\varphi}\right]d\varphi,\\ w&=\cos\varphi\int_{\pm\frac{\pi}{2}}^{\varphi}\left[\frac{R_1\varepsilon_1}{\sin\varphi}-\frac{r(\varepsilon_2-\omega_0)}{\sin^2\varphi}\right]d\varphi\\ &\qquad-\frac{r(\varepsilon_2-\omega_0)}{\sin\varphi}.\end{aligned}\right\}\quad(113)$$

Since the values of u and w at $\varphi=0$ are difficult to calculate from the expressions (113), we consider the displacements parallel and perpendicular to the plane of symmetry, and represent them by η and ν respectively. Then

$$\left.\begin{aligned}\eta&=u\cos\varphi-w\sin\varphi,\\ \nu&=u\sin\varphi+w\cos\varphi,\end{aligned}\right\}\quad(114)$$

or

$$\left.\begin{aligned}\frac{d\eta}{d\varphi}&=R_1(\varepsilon_1\cos\varphi-\theta\sin\varphi),\\ \frac{d\nu}{d\varphi}&=R_1(\varepsilon_1\sin\varphi+\theta\cos\varphi),\end{aligned}\right\}$$

and, therefore,

$$\left.\begin{aligned}\eta&=R_1\int(\varepsilon_1\cos\varphi-\theta\sin\varphi)\,d\varphi+\eta_0,\\ \nu&=R_1\int(\varepsilon_1\sin\varphi+\theta\cos\varphi)\,d\varphi+\nu_0.\end{aligned}\right\}\quad(115)$$

If we start the integrations from $\varphi=\pm\frac{\pi}{2}$ respectively, the integration constants η_0 and ν_0 take the following values because of the condition of symmetry.

$$\left.\begin{aligned}\eta_0&=(w)_{\varphi=\pm\frac{\pi}{2}},\\ \nu_0&=0.\end{aligned}\right\}\quad(116)$$

Then, at $\varphi=0$, we have

$$\left.\begin{aligned}\delta=u_{\varphi=0}=\eta_{\varphi=0}&=R_1\int_{\pm\frac{\pi}{2}}^{0}(\varepsilon_1\cos\varphi-\theta\sin\varphi)\,d\varphi+\eta_0,\\ w_{\varphi=0}=\nu_{\varphi=0}&=R_1\int_{\pm\frac{\pi}{2}}^{0}(\varepsilon_1\sin\varphi+\theta\cos\varphi)\,d\varphi.\end{aligned}\right\}\quad(117)$$

VII. Boundary Conditions and the Determination of Unknown Constants.

In the present paper, as defined previously in the introduction, we are treating the cross-sections built up of two pairs of circular arcs as shown in Fig. 2, or of a circle; and, according to the solutions obtained above, we have to treat them separately: the first must be treated in four parts, that is

$$\begin{aligned}&\text{I}\quad -\frac{\pi}{2}\leqq\varphi\leqq-\varphi_0,\\ &\text{II}\quad -\varphi_0\leqq\varphi\leqq 0,\\ &\text{III}\quad 0\leqq\varphi\leqq+\varphi_0,\\ &\text{IV}\quad +\varphi_0\leqq\varphi\leqq+\frac{\pi}{2};\end{aligned}$$

and the second, a circular cross-section, must be treated in two parts, namely

$$\begin{aligned}&\text{I}\quad -\frac{\pi}{2}\leqq\varphi\leqq 0,\\ &\text{II}\quad 0\leqq\varphi\leqq+\frac{\pi}{2}.\end{aligned}$$

Owing to the condition of symmetry, we need not consider the range of φ outside the above. The solution for each part gives four integration constants, and, moreover, there are two common unknown constants ω_0 and τ_0; an oval cross-section, therefore, gives 18 unknown constants, and a circular cross-section 10 of them.

In order to describe the boundary conditions to be satisfied, it is necessary to make a distinction between each part by adding the index I~IV respectively, and the values at the beginning and the end of each part are distinguished by the suffix A and B, thus, for example,

$$\left.\begin{aligned}&\varphi^{\rm I}_A=-\frac{\pi}{2},\quad \varphi^{\rm I}_B=\varphi^{\rm II}_A=-\varphi_0,\quad \varphi^{\rm II}_B=\varphi^{\rm III}_A=0,\\ &\varphi^{\rm III}_B=\varphi^{\rm IV}_A=+\varphi_0,\quad \varphi^{\rm IV}_B=+\frac{\pi}{2}.\end{aligned}\right\}\quad(118)$$

(a) Boundary conditions at $\varphi=\pm\frac{\pi}{2}$, ($^{\rm I}_A$ and $^{\rm IV}_B$).

From the condition of symmetry, we have

$$u^{\rm I}_A=\left(\frac{dw}{d\varphi}\right)^{\rm I}_A=\theta^{\rm I}_A=N^{\rm I}_A=0,$$
$$u^{\rm IV}_B=\left(\frac{dw}{d\varphi}\right)^{\rm IV}_B=\theta^{\rm IV}_B=N^{\rm IV}_B=0.$$

The conditions $u=0$ are satisfied by beginning the integration from $\varphi=\pm\frac{\pi}{2}$ respectively, and all other conditions are reducible to

$$\theta^{\rm I}_A=\theta^{\rm IV}_B=(V_s)^{\rm I}_A=(V_s)^{\rm IV}_B=0,$$

which leads to the results

$$B^{\rm I}_1=B^{\rm I}_2=B^{\rm IV}_1=B^{\rm IV}_2=0.\quad(119)$$

(b) Boundary conditions at $\varphi=\pm\varphi_0$, ($^{\rm I}_B$, $^{\rm II}_A$ and $^{\rm III}_B$, $^{\rm IV}_A$). (for Case I only.)

At both sides of these sections we must have equal values of T_1, T_2, N, G_1, G_2, θ, η ane ν (or u and w), among which the conditions for η and ν or u and w are satisfied by continuing the integration successively. All other conditions are reducible to the following:

$$\left.\begin{aligned}&\theta^{\rm I}_B=\theta^{\rm II}_A,\quad V^{\rm I}_B=V^{\rm II}_A,\\ &\left(\frac{1}{R_1}\frac{d\theta}{d\varphi}\right)^{\rm I}_B=\left(\frac{1}{R_1}\frac{d\theta}{d\varphi}\right)^{\rm II}_A,\quad (T_2)^{\rm I}_B=(T_2)^{\rm II}_A;\\ &\theta^{\rm III}_B=\theta^{\rm IV}_A,\quad V^{\rm III}_B=V^{\rm IV}_A,\\ &\left(\frac{1}{R_1}\frac{d\theta}{d\varphi}\right)^{\rm III}_B=\left(\frac{1}{R_1}\frac{d\theta}{d\varphi}\right)^{\rm IV}_A,\quad (T_2)^{\rm III}_B=(T_2)^{\rm VI}_A.\end{aligned}\right\}\quad(120)$$

(c) Boundary conditions at $\varphi=0$, ($^{\rm II}_B$, $^{\rm III}_A$).

T_1, T_2, N, G_1, G_2, θ, η and ν (or u and w) at both sides of this section must be equal, among which the condition for N being satisfied already, since $N_{\varphi=0}=2h\tau_0$ for both sides of the section. All other conditions can be reduced, in this case, to the following five conditions:

$$\left.\begin{aligned}&\theta^{\rm II}_B=\theta^{\rm III}_A,\quad \left(\frac{d\theta}{d\varphi}\right)^{\rm II}_B=\left(\frac{d\theta}{d\varphi}\right)^{\rm III}_A,\\ &(V_s)^{\rm II}_B=(V_s)^{\rm III}_A,\quad \left(\frac{dV_s}{d\varphi}\right)^{\rm II}_B=\left(\frac{dV_s}{d\varphi}\right)^{\rm III}_A,\\ \text{and}\quad &w^{\rm II}_B=w^{\rm III}_A\quad(\text{or } \nu^{\rm II}_B=\nu^{\rm III}_A);\end{aligned}\right\}\quad(121)$$

the conditions for u and η being satisfied by those for T_1 and T_2 of itself.

(d) Boundary conditions for the entire meridian cross-section.

Besides these boundary conditions mentioned above, there are still two more conditions to be satisfied in the entire meridian cross-section, namely

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{T_2(r-r_0)-G_2\sin\varphi\}R_1d\varphi=M, \qquad (122)$$

and

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}T_2R_1d\varphi=pF, \qquad (123)$$

where F represents the inner cross-sectional area of the pipe. The condition (123), however, is satisfied by itself in the solution obtained above, as will be shown briefly in the following.

From the second equation of (25), we have

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}T_2R_1d\varphi=-2\Big[V\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+p\Big[(R_1-h)^2\varphi$$
$$+(a^2-r_0^2)\cot\varphi\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+4h\tau_0r_0\Big[\cot\varphi\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}},$$

in which the first and the last terms in the right are zero, and the coefficient of p always represents the inner cross-sectional area F.

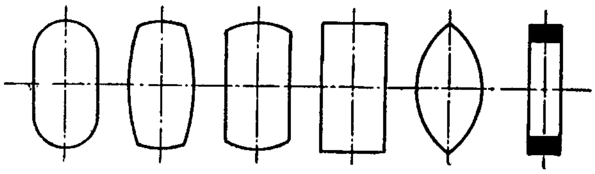
Equations (119), (120), (121) and (122) offer 18 conditions for a oval, and 10 for a circular cross-section, a number sufficient for the determination of all the unknown constants.

VIII. Conclusion.

The foregoing article, which gives a mathematically strict solution of the present problem, may be regarded as a theoretical standard for various approximate theories relating to the same problem.

The present theory, moreover, can be applied to various modified forms of cross-section, such as given in Fig. 4 for example, if we employ suitable infinite series and boundary conditions with the proper independent variable.

Fig. 4.



In the further reports to follow, the author will give some examples of numerical calculations along with some applications to the modified forms of cross-section.