Stability of a Pair of Vortices between Two Cylinders.

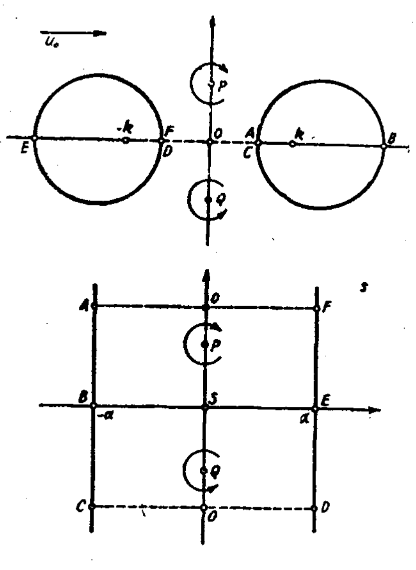
By B. Hudimoto.

The mutual interference between bodies which move in fluid is very important and several experiments were carried out hitherto. Even the simplest case such as two bodies placed side by side or one after another in the flow of fluid has applications not only in the field of aerodynamics but also in other fields of engineering, for example, wind effect on buildings and mutual interference of bridge piers. Results of experiments* on the drag measurement of two circular cylinders placed one after another show that the drag changes with the distance between two cylinders while another simple experiment carried out by the author shows that the change of drag seems closely related with the stability of vortices. Hence, in this paper, the author tries to investigate the stability in a simple case of two circular cylinders.

1. Flow around two circular cylinders.

For the sake of simplicity, two-dimensional flow of perfect fluid is assumed and restricted to the case of two circular cylinders with same radii. The steady flow around two circles was solved by Lagally** using elliptic functions, and in the following we proceed in the same way. In the present case we take the straight line passing through the two centers of the cylinders as x-axis and take the origin on the mid-point of these centers as shown in Fig 1. We transform z-plane into s-plane by the following relation

Fig. 1.



$$s=\log\frac{z-k}{z+k}, \qquad (1)$$

where k is a positive real number. If we choose k properly, the outside region of two circles is transformed into a rectangular region $ACDF$ in s-plane as shown in Fig. 1, corresponding points in both planes are denoted by the same letters. The origin S of s-plane corresponds to the point of infinity of z-plane. The lengths of AB and BC are both equal to π and the lengths of BS and SE are also equal and we denote them by a.

The flow which we consider in this paper consists of a parallel flow of velocity u_0 in the direction of x-axis and a pair of vortices of circulation Γ situated at P and Q on y-axis and symmetrical with respect to x-axis. If we denote two positions of the pair of vortices to be $s=p$ and q and their conjugate complexes by $\bar{p}$ and $\bar{q}$, the complex velocity in s-plane is expressed as follows,

$$\frac{dw}{ds}=\bar{v}_s=2ku_0\{\wp(s)-\wp(s+2a)\}$$
$$+\frac{i\Gamma}{2\pi}\{\zeta(s-p)-\zeta(s+2a+\bar{p})$$
$$-\zeta(s-q)+\zeta(s+2a+\bar{q})\}, \qquad (2)$$

where $\wp$ and ζ are the functions of Weierstrass with periods of $4a$ and 2π.

Hence the velocity in z-plane is

$$\bar{v}_z=\bar{v}_s\frac{ds}{dz}. \qquad (3)$$

2. Equilibrium positions of the vortices.

In this paragraph we discuss the equilibrium positions. We express the point P in z-plane by $z=ih$ conseqnently the point Q is at $z=-ih$ and the corresponding points in s-plane by $p=im$ and $q=-im$ respectively.

The velocity of the vortex at the point P is given by the following equation.

$$\bar{v}_{z=ih}=2ku_0\Big\{\wp(im)-\wp(im+2a)\Big\}\Big(\frac{ds}{dz}\Big)_{s=im}$$
$$+\frac{i\Gamma}{2\pi}\Big\{\zeta(2a+2im)-\zeta(2a)-\zeta(2im)\Big\}\Big(\frac{ds}{dz}\Big)_{s=im}$$
$$+\frac{i\Gamma}{2\pi}\Big\{\zeta(s-im)\frac{ds}{dz}-\frac{1}{z-ih}\Big\}_{s=im}$$

* N.A.C.A., Report No. 648.

** M. Lagally, Z.a.M.M., Heft 4, 1929.

or

$$\bar{v}_{z=ih}=2ku_0\left\{2\wp(im)+e_1-\frac{1}{4}\frac{\wp'(im)^2}{\{\wp(im)-e_1\}^2}\right\}\left(\frac{ds}{dz}\right)_{s=im}$$
$$+\frac{i\Gamma}{2\pi}\frac{\wp'(2im)}{2\{\wp(2im)-e_1\}}\left(\frac{ds}{dz}\right)_{s=im}$$
$$+\frac{i\Gamma}{2\pi}\left\{\frac{\frac{d^2s}{dz^2}}{2\frac{ds}{dz}}\right\}_{s=im}, \tag{4}$$

where $\wp(2a)=e_1$ and

$$\frac{ds}{dz}=\frac{2k}{z^2-k^2},\quad \frac{d^2s}{dz^2}=-\frac{4kz}{(z^2-k^2)^2},$$
$$\frac{d^3s}{dz^3}=\frac{4k(3z^2+k^2)}{(z^2-k^2)^3}.$$

Suffices $s=im$ etc. show to take the value in the bracket at $s=im$ etc.

If the vortex at $z=ih$ is in equilibrium, then $\bar{v}_{z=ih}=0$ hence

$$\frac{\Gamma}{2\pi}\left\{\frac{h}{h^2+k^2}-\frac{i}{2}\frac{\wp'(2im)}{\wp(2im)-e_1}\left(\frac{ds}{dz}\right)_{s=im}\right\}$$
$$=2ku_0\left\{2\wp(im)+e_1-\frac{1}{4}\frac{\wp'(im)^2}{\{\wp(im)-e_1\}^2}\right\}\left(\frac{ds}{dz}\right)_{s=im}. \tag{5}$$

From this equation we can determine the position of equilibrium i.e. the value of h or conversely we can determine the magnitude of Γ for given position.

The same equation holds for the equilibrium of the vortex at the point Q or $z=-ih$.

3. Velocities resulting from small disturbances.

Now we examine the stability of these vortices which satisfy Eq. (5). For this purpose we give small disturbances to these vortices and see whether they deviate from their initial positions of equilibrium or not. Let the disturbances in z-plane be $\Delta_1 z$ and $\Delta_2 z$ at points P and Q respectively and corresponding disturbances in s-plane by $\Delta_1 s$ and $\Delta_2 s$. If we assume $\Delta_1 z$, $\Delta_2 z$, $\Delta_1 s$ and $\Delta_2 s$ are very small, then

$$\Delta_1 s=\Delta_1 z\left(\frac{ds}{dz}\right)_{s=im}\quad\text{and}\quad \Delta_2 s=\Delta_2 z\left(\frac{ds}{dz}\right)_{s=-im},$$

and moreover in the present case $\left(\frac{ds}{dz}\right)_{s=im}$ and $\left(\frac{ds}{dz}\right)_{s=-im}$ are real quantities, so if we express the conjugate complexes of $\Delta_1 z$, $\Delta_2 z$, $\Delta_1 s$ and $\Delta_2 s$ by $\overline{\Delta_1 z}$, $\overline{\Delta_2 z}$, $\overline{\Delta_1 s}$ and $\overline{\Delta_2 s}$, then

$$\overline{\Delta_1 s}=\overline{\Delta_1 z}\left(\frac{ds}{dz}\right)_{s=im}\quad\text{and}\quad \overline{\Delta_2 s}=\overline{\Delta_2 z}\left(\frac{ds}{dz}\right)_{s=-im}.$$

Neglecting terms of higher orders, we get after some calculations the following equations for the velocities of two vortices at new positions.

For the vortex at P or at $z=ih+\Delta_1 z$

$$\bar{v}_{z=ih+\Delta_1 z}=ia\Delta_1 z+ib\overline{\Delta_1 z}+ic\overline{\Delta_2 z}+id\Delta_2 z. \tag{6}$$

For the vortex at Q or at $z=-ih+\Delta_2 z$

$$\bar{v}_{z=-ih+\Delta_2 z}=-ia\Delta_2 z-ib\overline{\Delta_2 z}-ic\overline{\Delta_1 z}-id\Delta_1 z, \tag{7}$$

where

$$\left.\begin{aligned}
a=&-\Big\{2ku_0 i[\wp'(im)-\wp'(im+2a)]\\
&+\frac{\Gamma}{2\pi}[\wp(2im+2a)-\wp(2a)-\wp(2im)]\Big\}\left(\frac{ds}{dz}\right)^2_{s=im}\\
&+\frac{\Gamma}{2\pi}\left(\frac{\frac{d^3s}{dz^3}}{2\frac{ds}{dz}}\right)_{s=im}-\frac{\Gamma}{2\pi}\left(\frac{\frac{d^2s}{dz^2}}{\frac{ds}{dz}}\right)^2_{s=im},\\
b=&\frac{\Gamma}{2\pi}\wp(2a)\left(\frac{ds}{dz}\right)^2_{s=im},\\
c=&-\frac{\Gamma}{2\pi}\wp(2im+2a)\left(\frac{ds}{dz}\right)^2_{s=im},\\
d=&-\frac{\Gamma}{2\pi}\wp(2im)\left(\frac{ds}{dz}\right)^2_{s=im},
\end{aligned}\right\} \tag{8}$$

a, b, c and d are all real quantities and $\wp'$ is the derivative of $\wp$.

4. Criterion for stability.

If we express time by t and divide $\Delta_1 z$ and $\Delta_2 z$ into real and imaginary parts viz. $\Delta_1 z=\Delta_1 x+i\Delta_1 y$ and $\Delta_2 z=\Delta_2 x+i\Delta_2 y$, then from Eqs. (6) and (7) we get

$$\left.\begin{aligned}
&\frac{d\Delta_1 x}{dt}=-(a-b)\Delta_1 y+(c-d)\Delta_2 y,\\
&\frac{d\Delta_1 y}{dt}=-(a+b)\Delta_1 x-(c+d)\Delta_2 x,\\
&\frac{d\Delta_2 x}{dt}=-(c-d)\Delta_1 y+(a-b)\Delta_2 y,\\
\text{and}\quad &\frac{d\Delta_2 y}{dt}=(c+d)\Delta_1 x+(a+b)\Delta_2 x.
\end{aligned}\right\} \tag{9}$$

In the case of symmetrical disturbances i.e. $\Delta_1 x=\Delta_2 x$, $\Delta_1 y=-\Delta_2 y$ or $\Delta_1 z=\overline{\Delta_2 z}$, if $(a+c)^2>(b+d)^2$ the disturbances increase with time. Hence the condition of stability in this case is

$$(a+c)^2<(b+d)^2. \tag{10}$$

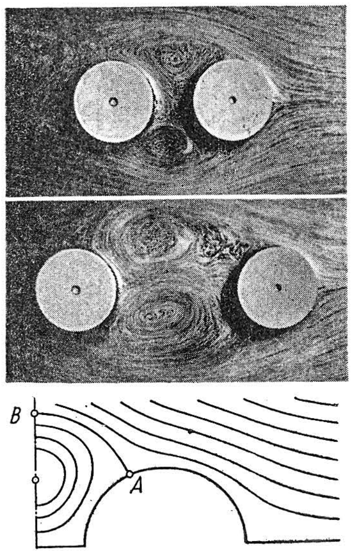
For other types of disturbances i.e. $\Delta_1 x=-\Delta_2 x$, $\Delta_1 y=\Delta_2 y$ or $\Delta_1 z=-\overline{\Delta_2 z}$, if $(a-c)^2>(b-d)^2$ the disturbances increase with time. Hence the condition of stability in this case is

$$(a-c)^2<(b-d)^2. \tag{11}$$

5. Numerical example.

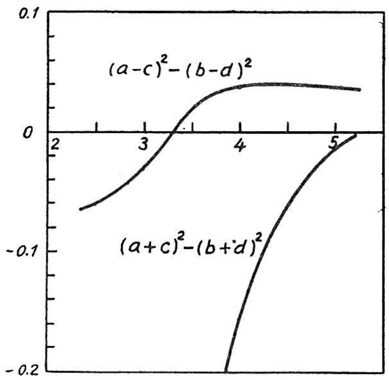
The positions of vortices or in other words the value of h is theoretically indeterminate so far we assume the flow of perfect fluid. To avoid much labor of numerical calculation and also from the point of view of practical application, the positions were determined from photographs of flow taken from small water tank. Figs. 2 *a*, *b* show

Fig. 2 *a*, *b*, *c*.*



two of them. From them we see $h \simeq 0.9r$ where r is the radius of the cylinders. Fig. 3 shows the results of numerical calculations, one curve shows the value of $(a-c)^2-(b-d)^2$ and the other shows

Fig. 3.



the value of $(a+c)^2-(b+d)^2$ both divided by $\frac{\Gamma^2}{4\pi^2}$, the abscissa being the distance between two centers of the cylinders in term of the radius r. From this diagram we see the vortices are stable if the distance between two cylinders is less than $3.3r$. The author had an opportunity to see a film of the flow as discussed here which was taken in a small water tank by Prof. Uematu of the Osaka Imperial University. From this film, it was noticed that the crosswise flow through the gap between two cylinders becomes remarkably strong if the distance is larger than about $4r$, and this agrees quite satisfactorily with our present result.

* Fig. 2 *c* shows the probable stream lines of the flow at the limit of the stability of the vortices. The point of stagnation A and point B are calculated approximately using the following simple transfermation, $\zeta = z + \frac{2}{z^2 - n^2}$.