A New Criterion for the Strength of Metals under Combined Alternating Stresses.

By

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Abstract.

Matsumura's theory of malleable materials, stating that "the elastic failure occurs in malleable materials when the maximum shear stress reaches a definite value depending on shear strain energy" is fairly well applicable to results of experiments, so far as the material is ductile; but the theory is not applicable to brittle materials. Extending Matsumura's theory to all materials, ductile and brittle, the authors propose now a new criterion and applying it to the case of fatigue, they think that "the fatigue failure occurs in ductile materials when the greatest maximum shear stress induced by combined repeated stresses reaches a definite value depending on shear strain energy at the instant, and that the fatigue occurs in brittle materials when the greatest maximum principal stress induced by the combined repeated stresses reaches a definite value depending also on shear strain energy at the instant." From this new criterion on fatigue, they derive the conditions of fatigue failure under the combination of various stresses. Comparing the results of calculation with those of experiments, it is established that the new criterion is fairly well applicable to results of experiments in every case.

I. Introduction.

There are two kinds of failure in metals, i. e., static failure and fatigue failure, the mechanisms of which seem at the first glance to be essentially different from each other. When materials are broken under static stress, considerable deformation generally takes place before destruction. So it is reasonable to regard the static strength of materials as the resistance to elastic break-down, that is, the resistance to generation of slip bands in crystals. But the resistance cannot be ascertained practically without some troublesome measurement. Then we can regard the yielding point of the materials approximately as the static strength. On the other hand, when materials are broken by fatigue, the relation is quite similar: namely when they are subjected to repeated stresses, they receive the effect which is similar to the so-called strain-hardening and become brittle before destruction. At the time there are many slip bands in crystals, which grow gradually to cracks and lead to destruction. So it is also reasonable to regard the magnitude of the cyclical stress, which is just enough to generate slip bands in crystals, as the fatigue resistance in the true sense. But it is not easy to obtain the magnitude from experiments. Therefore usually we regard the magnitude of the cyclical stress which is just enough to break down the material as approximately the fatigue resistance. Generally the stress which is just enough to generate slip bands in crystals is somewhat lower than the stress which is just enough to break down the material.(1) As above mentioned, when we attribute the strength of materials to the resistance to slip in crystals, we are convinced that both the static and fatigue failures are essentially analogous.

The authors have previously carried out the experiments on the strength of metals under combined alternating bending and torsion, and compared the results with some of the theories concerning the elastic failure of metals under static stresses.(2) Let σ be direct stress and τ be shear stress in static cases, then many theories of elastic failure under combined direct and shear stresses are generally represented by the following relation:

$$f(\sigma,\ \tau)=c \quad \text{(1)}$$

where f is a certain function of σ and τ, and c is a constant. The form of the function f is determined according to each theory. For example, in the theory of constant maximum principal stress,

$$f=\frac{1}{2}\sigma+\frac{1}{2}\sqrt{\sigma^2+4\tau^2}.$$

In the theory of constant maximum principal strain,

$$f=\frac{1}{2}\left(1-\frac{1}{m}\right)\sigma+\frac{1}{2}\left(1+\frac{1}{m}\right)\sqrt{\sigma^2+4\tau^2}$$

where m is Poisson's constant. In the theory of constant total strain energy,

$$f=\sigma^2+2\left(1+\frac{1}{m}\right)\tau^2\ ;$$

(1) T. Nishihara and M. Kawamoto, "Studies on Fatigue of Mild Steel by a Corrosion Method", Memoirs of the College of Engineering, Kyoto Imperial University, Vol. XI, No. 3. (1943), p. 31.

(2) T. Nishihara and M. Kawamoto, "The Strength of Metals under Combined Alternating Bending and Torsion", Memoirs of the College of Engineering, Kyoto Imperial University, Vol. X, No. 6, (1941), p. 117.

in the theory of constant shear strain energy,

$$f=\sigma^2+3\tau^2\,;\quad\text{and}$$

in the theory of constant maximum shear stress,

$$f=\frac{1}{2}\sqrt{\sigma^2+4\tau^2}\,.$$

Then we apply these theories to the case of fatigue, in which both direct and shear stresses alternate between two equal magnitudes of mutually opposite signs. Now we consider σ as the maximum direct stress and τ as the maximum shear stress in fatigue cases. Then similarly to equation (1), we can represent the condition of fatigue failure under combined alternating direct and shear stresses as follows:

$$f(\sigma,\ \tau)=c' \qquad\qquad (2)$$

where c' is a constant, which is different from c and should be determined from fatigue tests. In the case of static failure, it is a well-known fact that equation (1) is fairly well applicable to ductile materials, if we take the maximum shear stress as the function f, while it is applicable to brittle materials, if we take the maximum principal stress as the function f. Similarly, in the case of fatigue failure, equation (2) is applicable to ductile or brittle materials, if we take the maximum shear stress or the maximum principal stress as the function f, respectively. Therefore we can consider both static and fatigue failures to be essencially the same.

In the above-mentioned theories, every criterion contains no quantity varying according to the nature of materials. That is to say, these theories propose the criterion of elastic failure of materials with a definite expression, no matter whether the materials are ductile or brittle. In this point there is unreasonableness of these theories. So that, these theories are not generally applicable to every material, though each theory is fairly well applicable to a particular material. So it is proper to attempt to make the expressions of these theories to contain a variable quantity, which may be chosen appropriately according to the characteristic of materials. Thus we can make the theory applicable to every material, ductile or brittle. Let us consider some of the theories in such a way.

1. Matsumura's theory.[3]

According to this theory, the condition of elastic failure is given by the following equation:

$$k\sigma+\sqrt{\sigma^2+4\tau^2}=c \qquad\qquad (3)$$

where k is a constant which varies with the kind of materials and must be determined by experiments. If we put $k=0$ in equation (3), this theory becomes the constant maximum shear stress theory, and if we put $k=1$, it becomes the constant maximum principal stress theory. So it can be said that this theory is the combined criterion of both the extreme theories, i.e., the maximum shear stress and the maximum principal stress theories.

Let

σ_e = elastic limit under pure direct stress due to tension.

τ_e = elastic limit under pure shear stress.

then, determining the constants k and c in equation (3), we obtain the following equation:

$$\left(1-\frac{\sigma_e}{2\tau_e}\right)\sigma+\frac{\sigma_e}{2\tau_e}\sqrt{\sigma^2+4\tau^2}=\sigma_e \qquad\qquad (4)$$

2. Mohr's theory.[4]

In this theory the condition of elastic failure is given by the envelope of many stress circles in Mohr's stress diagram, which is obtained by experiments for each material. But if the envelope cannot be obtained without making many experiments, the value of this theory becomes insignificant in practice. Mohr then proposed to use the following straight lines approximately as the envelope: that is, common tangent lines to two stress circles which correspond to cases of pure tension and pure compression, as shown in Fig. 1. Now let us apply this theory to the case in which direct and shear stresses are combined.

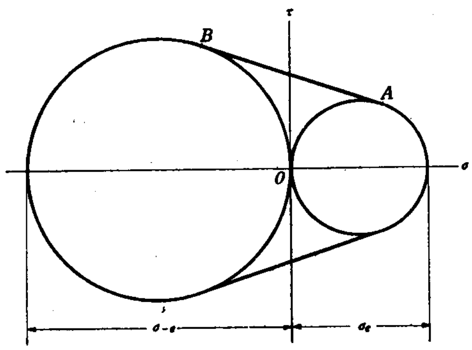

Fig. 1.

Mohr's Stress Diagram.

Let

σ_{-e} = elastic limit under pure direct stress due to compression.

then, the condition of elastic failure, which is represented by the approximate envelope, becomes as follows:

$$\frac{\sigma_{-e}-\sigma_e}{\sigma_{-e}+\sigma_e}\sigma+\sqrt{\sigma^2+4\tau^2}=\frac{2\sigma_{-e}\sigma_e}{\sigma_{-e}+\sigma^e} \qquad\qquad (5)$$

When σ_e is equal to σ_{-e}, equation (5) agrees with the constant maximum shear stress theory. When

(3) T. Matsumura, J. of the Society of Mech. Eng., Japan, Vol. XIII, No. 23, (1910), p. 1.

(4) O. Mohr, V. D. I., Bd. XXXXIV, Nr. 45, (1900), p. 1524 and 1572.

σ_{-e} becomes infinity, equation (5) agrees with the constant maximum principal stress theory. And when σ_{-e}/σ_e is equal to Poisson's constant m, equation (5) agrees with the constant maximum principal strain theory. Thus Mohr's theory contains many other theories as special cases.

3. *Ono's theory.*[5]

In this theory, the common tangent lines to the three stress circles corresponding to cases of pure tension, pure compression and pure torsion, are adopted approximately as the envelope in Mohr's theory; that is, common tangent lines AB and CD in Fig. 2. According to this theory, conditions of elastic failure become

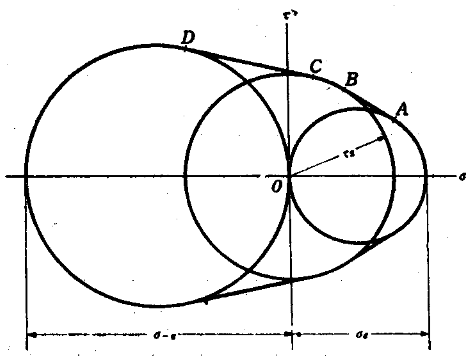

Fig. 2.

Mohr's Stress Diagram.

$$\text{for AB}\quad \left(1-\frac{\sigma_e}{2\tau_e}\right)\sigma+\frac{\sigma_e}{2\tau_e}\sqrt{\sigma^2+4\tau^2}=\sigma_e \quad \cdots(6)$$

$$\text{for CD}\quad \left(\frac{\sigma_{-e}}{2\tau_e}-1\right)\sigma+\frac{\sigma_{-e}}{2\tau_e}\sqrt{\sigma^2+4\tau^2}=\sigma_{-e} \cdots(7)$$

Equation (6) is the same as equation (4). So Mohr's theory with Ono's envelope is identical with Matsumura's theory.

Starting from Mohr's theory, a criterion on fatigue resistance under combined stresses has been derived by Dr. Ono.[6]

4. *Bailey's theory on ductile materials.*[7]

According to this theory, the condition of yielding of an isotropic material is given by the following equation:

$$(\sigma_1-\sigma_3)^2+\lambda(\sigma_1-\sigma_2)^2+\lambda(\sigma_2-\sigma_3)^2=\mu \cdots\cdots\cdots(8)$$

where σ_1, $\sigma_2(=0)$, and σ_3 are the three principal stresses at or near the surface of a test piece, and λ and μ are constants. Applying this theory to the case in which direct and shear stresses are combined, we can obtain

$$\sigma^2+2\frac{2+\lambda}{1+\lambda}\tau^2=\mu \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

If we put $\lambda=0$, $\lambda=1$ and $\lambda=m-1$ respectively in equation (9), this theory agrees with the constant maximum shear stress theory, the constant shear strain energy theory and the constant total strain energy theory, respectively. From equation (9), a further discussion has been made on fatigue resistance of ductile materials under combined stresses.

5. *Matsumura's theory on malleable materials.*[8]

This theory is represented by saying that "Elastic failure occurs in malleable materials, when the maximum shear stress reaches a definite value depending on shear strain energy. The relation can be written as follows:

$$(\sigma_1-\sigma_3)^2=a-b[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]\cdots\cdots(10)$$

where σ_1, σ_2, $\sigma_3(\sigma_1>\sigma_2>\sigma_3)$ are the three principal stresses, and a and b are constants which depend on material and stress distribution. Applying this theory to the case in which direct and shear stresses are combined, we obtain

$$\sigma^2+2\frac{3b+2}{2b+1}\tau^2=\frac{a}{2b+1} \quad \cdots\cdots\cdots\cdots\cdots\cdots(11)$$

If we put $b=0$, $b=\infty$ and $b=-\frac{m-1}{m-2}$ respectively in equation (11), this theory agrees with the constant maximum shear stress theory, the constant shear strain energy theory and the constant total strain energy theory, respectively.

II. A New Criterion for Static Failure.

Bailey's theory and Matsumura's theory for malleable materials are an elliptic law about direct stress σ and shear stress τ, as can be seen in equations (9) and (11). The elliptic formula is generally applicable with good accordance to every ductile material, but it is not applicable to brittle materials. According to the authors' experiments, the elliptic law is also not applicable to ductile materials with comparatively small ductility.

Then extending Matsumura's theory to all materials, ductile and brittle, the authors propose the following criterion: "Elastic failure occurs in ductile materials when the maximum shear stress induced by combined stresses reaches a definite value depending on shear strain energy, and it occurs in brittle materials when the maximum principal stress induced by combined stresses reaches

(5) A. Ono, J. of the Society of Mech. Eng., Japan, Vol. XVI, No. 29, (1912), p. 37.

(6) A. Ono, Trans. of the Society of Mech. Eng., Japan, Vol. 6, No. 25, (1940), Part 1, p. 30, and Vol. 7, No. 29, (1941), Part 1, p. 7.

(7) R. Bailey, Inst. of Mech. Eng. Proc., Vol. 143, No. 2, (1940), p. 101.

(8) T. Matsumura, J. of the Society of Mech. Eng., Japan, Vol. 33, No. 156, (1930), p. 181.

a definite value depending also on shear strain energy". According to this new criterion, the conditions of elastic failure can be expressed as follows:

for ductile materials

$$\xi(\sigma_1-\sigma_3)^2+[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]=p \quad \ldots\ldots(12)$$

for brittle materials

$$\eta\sigma_1^2+[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]=q \quad \ldots\ldots(13)$$

where σ_1, σ_2, and $\sigma_3(\sigma_1>\sigma_2>\sigma_3)$ are the three principal stresses, and ξ, η, p and q are constants depending on material and stress distribution and may be determined by experiments.

Now we consider the case in which direct stress σ and shear stress τ are combined,

$$\left.\begin{aligned}\sigma_1&=\frac{1}{2}\sigma+\frac{1}{2}\sqrt{\sigma^2+4\tau^2}\\ \sigma_2&=0\\ \sigma_3&=\frac{1}{2}\sigma-\frac{1}{2}\sqrt{\sigma^2+4\tau^2}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(14)$$

Substituting these in equations (12) and (13), the conditions of elastic failure become:

for ductile materials

$$(\xi+2)\sigma^2+2(2\xi+3)\tau^2=p \quad \ldots\ldots\ldots\ldots(15)$$

for brittle materials

$$\eta\left[\sigma\left\{\frac{1}{2}\sigma+\frac{1}{2}\sqrt{\sigma^2+4\tau^2}\right\}+\tau^2\right]+2(\sigma^2+3\tau^2)=q \quad \ldots\ldots\ldots\ldots(16)$$

Let us then determine the constants ξ, η, p and q. Let σ_e be the elastic limit under pure direct stress. From equations (15) and (16)

$$p=(\xi+2)\sigma_e^2 \quad \ldots\ldots\ldots\ldots(17)$$

$$q=(\eta+2)\sigma_e^2 \quad \ldots\ldots\ldots\ldots(18)$$

Let τ_e be the elastic limit under pure shear stress. From equations (15) and (16)

$$p=2(2\xi+3)\tau_e^2 \quad \ldots\ldots\ldots\ldots(19)$$

$$q=(\eta+6)\tau_e^2 \quad \ldots\ldots\ldots\ldots(20)$$

Eliminating p or q from equations (17) and (19) or equations (18) and (20) respectively, we can determine the constant ξ or η. And putting

$$\psi=\frac{\tau_e}{\sigma_e} \quad \ldots\ldots\ldots\ldots(21)$$

in the results, we obtain

for ductile materials

$$\psi^2=\frac{\xi+2}{2(2\xi+3)} \quad \text{or} \quad \xi=2\frac{1-3\psi^2}{4\psi^2-1} \quad \ldots\ldots(22)$$

for brittle materials

$$\psi^2=\frac{\eta+2}{\eta+6} \quad \text{or} \quad \eta=2\frac{3\psi^2-1}{1-\psi^2} \quad \ldots\ldots(23)$$

Now let us consider in what cases we should use the equations for ductile materials, and in what cases we should use the equations for brittle materials. Constants ξ and η are always greater than or equal to zero, so that from equations (22) and (23) we can derive the following relations:

for ductile materials $\quad \psi\leqq\frac{1}{\sqrt{3}}$

for brittle materials $\quad \psi\geqq\frac{1}{\sqrt{3}}$

So it is known that we should use the equations for ductile materials when $\psi\leqq\frac{1}{\sqrt{3}}$, and that we should use the equations for brittle materials when $\psi\geqq\frac{1}{\sqrt{3}}$. When $\psi=\frac{1}{\sqrt{3}}$, both equations for ductile and brittle materials become the same. But it is of course the better to use the equations for ductile materials when $\psi=\frac{1}{\sqrt{3}}$, because of simplicity.

In short, the author's criterion can be summarized as follows:

Let

$$U=\xi(\sigma_1-\sigma_3)^2+[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] \quad \ldots\ldots(24)$$

$$V=\eta\sigma_1^2+[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] \quad \ldots(25)$$

Then the values of function U or V determine the elastic failure of materials. And the conditions of elastic failure can be represented as

$$U=p \quad \text{when} \quad \psi\leqq\frac{1}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots(26)$$

$$V=q \quad \text{,,} \quad \psi>\frac{1}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots(27)$$

It must be mentioned that the static case is analogous to the fatigue case in which combined stresses are alternating between two equal magnitudes of mutually opposite signs. (See chapter V for details.)

III. A New Criterion for Fatigue Failure.

We apply the above-mentioned new criterion of static failure to the case of fatigue, and we consider that " fatigue failure occurs in ductile materials when the greatest maximum shear stress (the greatest value of the maximum shear stress) induced by combined repeated stresses reaches a definite value depending on shear strain energy at the instant, and it occurs in brittle materials when the greatest maximum principal stress, induced by the combined repeated stresses reaches a definite value depending also on shear strain energy at the instant." Let σ_1 be the greatest maximum principal stress and σ_2 $\sigma_3(\sigma_2>\sigma_3)$ be the other principal stresses at the same instant, and also let

$$U_{max}=\xi(\sigma_1-\sigma_3)^2+[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] \quad\ldots\ldots\ldots\ldots\ldots(28)$$

$$V_{max}=\eta\sigma_1^2+[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] \quad\ldots\ldots\ldots\ldots\ldots(29)$$

then the value U_{max} or V_{max} is considered to exert an influence upon the occurrence of fatigue failure. In equations (28) and (29), we assumed constants ξ and η to be the same as those in static cases. About the propriety of this assumption we shall discuss later.

But we connot, of course, consider the value U_{max} or V_{max} as the only factor that causes fatigue failure. Because fatigue failure should be caused not only by magnitude of the three principal stresses σ_1, σ_2, and σ_3, but also by the stress ranges of those alternating stresses. Then we divide those alternating stresses into the tow components: namely, the one is the mean stress acting statically and the other is the stress amplitude acting alternately between the two equal magnitudes of mutually opposite signs. Let σ_{1m}, σ_{2m}, and σ_{3m} be the static components of σ_1, σ_2 and σ_3, respectively, and also let σ_{1a}, σ_{2a} and σ_{3a} be the stress amplitudes of σ_1, σ_2 and σ_3; then

$$\sigma_1=\sigma_{1m}+\sigma_{1a},\quad \sigma_2=\sigma_{2m}+\sigma_{2a},\quad \sigma_3=\sigma_{3m}+\sigma_{3a}$$

Now similarly as in equations (28) and (29), we put

$$U_m=\xi(\sigma_{1m}-\sigma_{3m})^2+[(\sigma_{1m}-\sigma_{2m})^2+(\sigma_{2m}-\sigma_{3m})^2+(\sigma_{3m}-\sigma_{1m})^2] \ldots\ldots(30)$$

$$U_a=\xi(\sigma_{1a}-\sigma_{3a})^2+[(\sigma_{1a}-\sigma_{2a})^2+(\sigma_{2a}-\sigma_{3a})^2+(\sigma_{3a}-\sigma_{1a})^2] \quad\ldots\ldots(31)$$

$$V_m=\eta\sigma_{1m}^2+[(\sigma_{1m}-\sigma_{2m})^2+(\sigma_{2m}-\sigma_{3m})^2+(\sigma_{3m}-\sigma_{1m})^2] \quad\ldots(32)$$

$$V_a=\eta\sigma_{1a}^2+[(\sigma_{1a}-\sigma_{2a})^2+(\sigma_{2a}-\sigma_{3a})^2+(\sigma_{3a}-\sigma_{1a})^2] \quad\ldots\ldots(33)$$

And we regard these values U_m and U_a, or V_m and V_a, also as factors exerting an influence upon the occurence of fatigue failure.

Now we apply equations (30), (31), (32) and (33) to the case in which direct and shearing stresses are combined. Let σ_{max} be the maximum direct stress, and σ_m and σ_a be the mean stress and the stress amplitude. Similarly let τ_{max} be the maximum shear stress, and τ_m and τ_a be the mean stress and the stress amplitude, then

$$\sigma_{max}=\sigma_m+\sigma_a,\quad \tau_{max}=\tau_m+\tau_a$$

Therefore, if direct and shear stresses are in phase, we obtain the following relations similarly as in equation (14):

$$\left.\begin{aligned}\sigma_1&=\frac{1}{2}(\sigma_m+\sigma_a)+\frac{1}{2}\sqrt{(\sigma_m+\sigma_a)^2+4(\tau_m+\tau_a)^2}\\ \sigma_2&=0\\ \sigma_3&=\frac{1}{2}(\sigma_m+\sigma_a)-\frac{1}{2}\sqrt{(\sigma_m+\sigma_a)^2+4(\tau_m+\tau_a)^2}\end{aligned}\right\} \quad (34)$$

$$\left.\begin{aligned}\sigma_{1m}&=\frac{1}{2}\sigma_m+\frac{1}{2}\sqrt{\sigma_m^2+4\tau_m^2}\\ \sigma_{2m}&=0\\ \sigma_{3m}&=\frac{1}{2}\sigma_m-\frac{1}{2}\sqrt{\sigma_m^2+4\tau_m^2}\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

$$\left.\begin{aligned}\sigma_{1a}&=\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}\\ \sigma_{2a}&=0\\ \sigma_{3a}&=\frac{1}{2}\sigma_a-\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(36)$$

Substituting these relations in equations (28), (29), (30), (31), (32) and (33), we obtain

$$\left.\begin{aligned}U_{max}&=(\xi+2)(\sigma_m+\sigma_a)^2+2(2\xi+3)(\tau_m+\tau_a)^2\\ U_m&=(\xi+2)\sigma_m^2+2(2\xi+3)\tau_m^2\\ U_a&=(\xi+2)\sigma_a^2+2(2\xi+3)\tau_a^2\end{aligned}\right\} \quad (37)$$

$$\left.\begin{aligned}V_{max}&=\eta[(\sigma_m+\sigma_a)\Big\{\frac{1}{2}(\sigma_m+\sigma_a)+\frac{1}{2}\sqrt{(\sigma_m+\sigma_a)^2+4(\tau_m+\tau_a)^2}\Big\}\\ &\quad+(\tau_m+\tau_a)^2]+2\{(\sigma_m+\sigma_a)^2+3(\tau_m+\tau_a)^2\}\\ V_m&=\eta[\sigma_m\Big\{\frac{1}{2}\sigma_m+\frac{1}{2}\sqrt{\sigma_m^2+4\tau_m^2}\Big\}+\tau_m^2]+2(\sigma_m^2+3\tau_m^2)\\ V_a&=\eta[\sigma_a\Big\{\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}\Big\}+\tau_a^2]+2(\sigma_a^2+3\tau_a^2)\end{aligned}\right\} \quad (38)$$

As above mentioned, we consider U_m and U_a or V_m and V_a, to have also an influence upon the occurence of fatigue failure.

Consequently we can represent the conditions of fatigue failure by the following equations:

for ductile materials

$$U_{max}+f(U_m,\ U_a)=p \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39)$$

for brittle materials

$$V_{max}+f(V_m,\ V_a)=q \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40)$$

The second terms of the left sides of these equations represent the influence of alternation of applied stresses, and are a certain function of U_m and U_a, or V_m and V_a. If we omit these terms, equations (39) and (40) become the same as equations (26) and (27), namely of the case of static failure. How should the form of the function f be determined? Of course it should be determined applicable to experimental results. Now we adopt the following form as the function f:

$$f(U_m,U_a)=-\alpha U_m+\beta U_a-\gamma\sqrt{U_m\cdot U_a} \ldots\ldots(41)$$

$$f(V_m,\ V_a)=-\alpha V_m+\beta V_a-\gamma\sqrt{V_m\cdot V_a} \ldots\ldots(42)$$

where α, β and γ are constants which should be determined by experiments, and the sign of each term in these expressions is chosen as above for convenience. Substituting equations (41) and (42) in equations (39) and (40), the conditions of fatigue failure become as follows:

for ductile materials

$$U_{max}-\alpha U_m+\beta U_a-\gamma\sqrt{U_m\cdot U_a}=p \quad\cdots\cdots(43)$$

for brittle materials

$$V_{max}-\alpha V_m+\beta V_a-\gamma\sqrt{V_m\cdot V_a}=q\cdots\cdots\cdots(44)$$

Thus we obtain the conditions of fatigue failure under combined bending and torsional stresses.

But in the above representations, the values of constants ξ, η, α, β and γ are yet unknown. Let us now consider the values of these constants.

1. Determination of constants ξ and η.

First we consider the case in which only direct stress alternates between two equal magnitudes of mutually opposite signs, e.g., the case of rotating bending tests or the case of alternating tension compression tests with zero mean stress. Let σ_w be the endurance limit in this case, then

$$\sigma_a=\sigma_w,\quad \sigma_m=\tau_m=\tau_a=0$$

Applying these relations to equations (37) and (38), we obtain

$$U_{max}=U_a=(\xi+2)\sigma_w^2,\quad U_m=0$$
$$V_{max}=V_a=(\eta+2)\sigma_w^2,\quad V_m=0$$

Then applying these relations to equations (43) and (44), we obtain

$$(1+\beta)(\xi+2)\sigma_w^2=p\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(45)$$
$$(1+\beta)(\eta+2)\sigma_w^2=q\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(46)$$

In a like manner, we consider the case in which only shear stress alternates between two equal magnitudes of mutually opposite signs, e.g., the case of reversed torsional tests with zero mean stress. Let τ_w be the endurance limit in this case, then

$$\tau_a=\tau_w,\quad \tau_m=\sigma_m=\sigma_a=0$$

Applying these relations to equations (37) and (38), we obtain

$$U_{max}=U_a=2(2\xi+3)\tau_w^2,\quad U_m=0$$
$$V_{max}=V_a=(\eta+6)\tau_w^2,\quad V_m=0$$

Then applying these relations to equations (43) and (44), we obtain

$$2(1+\beta)(2\xi+3)\tau_w^2=p\cdots\cdots\cdots\cdots\cdots\cdots\cdots(47)$$
$$(1+\beta)(\eta+6)\tau_w^2=q\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(48)$$

Eliminating p, q and β from equations (45) and (47) or equations (46) and (48) respectively, we can determine the constant ξ or η. And putting

$$\varphi=\frac{\tau_w}{\sigma_w}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(49)$$

in the results, we obtain

for ductile materials

$$\varphi^2=\frac{\xi+2}{2(2\xi+2)}\quad\text{or}\quad \xi=2\frac{1-3\varphi^2}{4\varphi^2-1}\cdots\cdots\cdots(50)$$

for brittle materials

$$\varphi^2=\frac{\eta+2}{\eta+6}\quad\text{or}\quad \eta=2\frac{3\varphi^2-1}{1-\varphi^2}\cdots\cdots\cdots(51)$$

Now let us consider in what case we should use the equations for ductile materials, and in what case the equations for brittle materials. Constants ξ and η are always greater than or equal to zero, so that from equations (50) and (51) we can derive the following relations:

for ductile materials $\quad \varphi\leqq\frac{1}{\sqrt{3}}$

for brittle materials $\quad \varphi\geqq\frac{1}{\sqrt{3}}$

So it is known that we should use the equations for ductile materials when $\varphi\leqq\frac{1}{\sqrt{3}}$, and that we should use the equations for brittle materials when $\varphi\geqq\frac{1}{\sqrt{3}}$. When $\varphi=\frac{1}{\sqrt{3}}$, both equations for ductile and brittle materials become the same, but it is of course the better to use the equations for ductile materials when $\varphi=\frac{1}{\sqrt{3}}$, because of simplicity.

As we have noticed at the beginning of this chapter, we assumed that constants ξ and η in fatigue cases are equal to those in static cases. According to this assumption, we can derive the following relation from equations (22) and (50) or (23) and (51):

$$\varphi=\psi\quad\text{or}\quad \frac{\tau_w}{\sigma_w}=\frac{\tau_e}{\sigma_e}\cdots\cdots\cdots\cdots\cdots\cdots\cdots(52)$$

Therefore it can be seen that the above-mentioned assumption is appropriate, if we can prove the relation of equation (52) to be true, and we have already shown that the relation of equation (52) is very likely to be true, because both static and fatigue failures are essentially the same, as mentioned in introduction. Therefore it is quite proper to regard that constants ξ and η in fatigue cases are equal to those in static cases.

2. Determination of constant α.

Let us consider the case in which only the direct stress is working and its amplitude σ_a becomes infinitesimally small. To bring about fatigue failure in this case, the mean stress σ_m must be equal to breaking tensile strength σ_T of the material. Hence in this case

$$\sigma_m=\sigma_T,\quad \sigma_a=\tau_m=\tau_a=0$$

Applying these relations to equations (37) and (38), we obtain

$$U_{max}=U_m=(\xi+2)\sigma_T^2,\quad U_a=0$$
$$V_{max}=V_m=(\eta+2)\sigma_T^2,\quad V_a=0$$

Applying these relations to equations (43) and (44), we obtain

$$(1-a)(\xi+2)\sigma_T^2=p \quad \text{.........................(53)}$$
$$(1-a)(\eta+2)\sigma_T^2=q \quad \text{.........................(54)}$$

In a like manner, let us consider the case in which only the shear stress is working and its amplitude τ_a becomes infinitesimally small. To bring about fatigue failure in this case, the mean stress τ_m must be equal to breaking shear stress τ_T of the material. Hence

$$\tau_m=\tau_T, \quad \tau_a=\sigma_m=\sigma_a=0$$

Applying these relations to equations (37) and (38), we obtain

$$U_{max}=U_m=2(2\xi+3)\tau_T^2, \quad U_a=0$$
$$V_{max}=V_m=(\eta+6)\tau_T^2, \quad V_a=0$$

Then applying these relations to equations (43) and (44), we obtain

$$2(1-a)(2\xi+3)\tau_T^2=p \quad \text{.........................(55)}$$
$$(1-a)(\eta+6)\tau_T^2=q \quad \text{.........................(56)}$$

Substituting for p or q in equations (53) or (54) values from equations (17) or (18), or substituting for p or q in equations (55) or (56) values from equations (19) and (20), we can determine the constant a. And putting

$$v=\frac{\sigma_e}{\sigma_T} \quad \text{or} \quad \frac{\tau_e}{\tau_T} \quad \text{.........................(57)}$$

in the result, we obtain the following relation for both ductile and brittle materials:

$$1-a=v^2 \quad \text{or} \quad a=1-v^2 \quad \text{.................(58)}$$

Here we should use the former value in equation (57) as the value of v when direct stress acts statically, and we should use the latter value in equation (57) when shear stress acts statically.

The constant a is thus determined.

3. *Determination of constant β.*

The value of canstant β is considered to be different according to whether the combined direct and shear stresses are in the same phase or not.

(i) When combined direct and shear stresses are in the same phase.

Substituting for p or q in equations (45) or (46) values from equations (17) and (18), or substituting for p or q in equations (47) or (48) values from equations (19) and (20), we can determine the constant β. And putting

$$w=\frac{\sigma_e}{\sigma_w}=\frac{\tau_e}{\tau_w} \quad \text{.............................(59)}$$

in the result, we obtain the following relation for both ductile and brittle materials:

$$1+\beta=w^2 \quad \text{or} \quad \beta=w^2-1 \quad \text{.................(60)}$$

The constant β is thus determined.

(ii) When combined direct and shear stresses are not in the same phase.

In this case, constant β cannot be given with equation (60), because β is considered to be influenced not only by mechanical properties of materials, but also by the phase difference of applied stresses. So we should determine the value of β by the combined fatigue tests with phase differences.

It must be noticed that U_{max} and V_{max} cannot be given by equations (37) and (38) in this case, because those equations correspond to the case in which direct and shear stresses are in phase. When direct and shear stresses are not in phase, let

δ = angle of phase lag of shear stress to direct stress
ω = angular velocity
t = time

then U_{max} and V_{max} become as follows:

$$U_{max}=[(\xi+2)\{\sigma_m+\sigma_a\cos\omega t\}^2+2(2\xi+3)\{\tau_m+\tau_a\cos(\omega t-\delta)\}^2]_{max} \quad \text{...(61)}$$

$$V_{max}=\Big[\eta[(\sigma_m+\sigma_a\cos\omega t)\Big\{\frac{1}{2}(\sigma_m+\sigma_a\cos\omega t)+\frac{1}{2}\sqrt{(\sigma_m+\sigma_a\cos\omega t)^2+4\{\tau_m+\tau_a\cos(\omega t-\delta)\}^2}\Big\}+\{\tau_m+\tau_a\cos(\omega t-\delta)\}^2]+2[(\sigma_m+\sigma_a\cos\omega t)^2+3\{\tau_m+\tau_a\cos(\omega t-\delta)\}^2]\Big]_{max} \quad \text{.........(62)}$$

We regard that U_m, U_a, V_m and V_a in this case are also the same as given in equations (37) and (38).

4. *Determination of constant γ.*

We regard γ to be a constant which depends not only on material, but also on the kind of combination of stresses, stress distributions, methods of experiments, and others. Therefore the value of γ should be determined by experiments in each case.

In the above, we have derived from the author's theory of failure the method of calculation to obtain the fatigue limit under combined alternating stresses. Let us now apply this method of calculation to various special cases and compare the results of calculation with experimental results.

IV. When direct or shear stress alternates between certain maximum and minimum values.

1. *When only direct stress alternates between certain maximum and minimum values.*

In this case

$$\tau_m=\tau_a=0$$

Applying this relation to equations (37) and (38), we obtain

$$U_{max}=(\xi+2)(\sigma_m+\sigma_a)^2, \quad V_{max}=(\eta+2)(\sigma_m+\sigma_a)^2$$
$$U_m=(\xi+2)\sigma_m^2, \quad V_m=(\eta+2)\sigma_m^2$$
$$U_a=(\xi+2)\sigma_a^2, \quad V_a=(\eta+2)\sigma_a^2$$

Applying the relations to equations (43) and (44), we obtain

for ductile materials

$$(\xi+2)[(1-\alpha)\sigma_m^2+(2-\gamma)\sigma_m\sigma_a+(1+\beta)\sigma_a^2]=p$$

for brittle materials

$$(\eta+2)[(1-\alpha)\sigma_m^2+(2-\gamma)\sigma_m\sigma_a+(1+\beta)\sigma_a^2]=q$$

Substituting for p and q values from equations (17) and (18) respectively, we obtain the following relation for both ductile and brittle materials:

$$(1-\alpha)\sigma_m^2+(2-\gamma)\sigma_m\sigma_a+(1+\beta)\sigma_a^2=\sigma_e^2$$

Substituting for α and β values from equations (58) and (60) respectively, we obtain

$$v^2\sigma_m^2+(2-\gamma)\sigma_m\sigma_a+w^2\sigma_a^2=\sigma_e^2 \quad \text{..............(63)}$$

The constant γ should be determined from results of experiments. Now let us determine the constant γ from the following experimental results of the authors: that is, "In the endurance limit diagram taking mean stress as abscissa and stress amplitude as ordinate, the three points, i. e. the point of endurance limit with zero mean stress, the point of endurance limit with zero minimum stress, and the point of breaking static strength, are always on a straight line." Let σ_u be the endurance limit, when the applied minimum stress is zero; by putting

$$\frac{\sigma_e}{\sigma_u}=u \quad \text{..................................(64)}$$

we can represent the above-mentioned condition by the following equation:

$$2u=v+w \quad \text{..................................(65)}$$

In equation (63), when σ_m is equal to $\frac{1}{2}\sigma_u$, σ_a should also be equal to $\frac{1}{2}\sigma_u$; hence we can derive

$$v^2+2-\gamma+w^2=4u^2$$

Substituting for u the value from equation (65), we obtain

$$\gamma=2(1-v\cdot w) \quad \text{............................(66)}$$

Thus γ is determined. Applying this value of γ to equation (63), we can obtain the following relation:

$$v\sigma_m+w\sigma_a=\sigma_e \quad \text{..............................(67)}$$

Therefore the relation between σ_m and σ_a becomes linear. For example, when we apply $v=0.4$ and $w=1.1$ in equation (67), the relation between σ_m and σ_a becomes a straight line as shown in Fig. 3. Thus the result of calculation agrees with the following experimental results, previously obtained by the authors: that is, "If we take mean stress σ_m and stress amplitude σ_a as both coordinate axes, the endurance limit can be represented by the straight line through the two points, i. e., the point of endurance limit with zero mean stress and the point of breaking static strength."

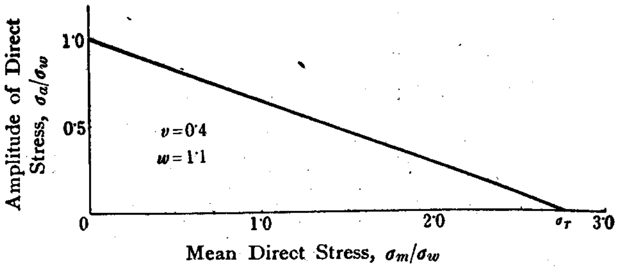

Fig. 3.
When Only Direct Stress Alternates Certain Maximum and Minimum Values.

2. *When only shear stress alternates between certain maximum and minimum values.*

Iu this case

$$\sigma_m=\sigma_a=0$$

Applying this relation to equations (37) and (38), we obtain

$$U_{max}=2(2\xi+3)(\tau_m+\tau_a)^2,\quad V_{max}=(\eta+6)(\tau_m+\tau_a)^2$$
$$U_m\ =2(2\xi+3)\tau_m^2,\quad V_m\ =(\eta+6)\tau_m^2$$
$$U_a\ =2(2\xi+3)\tau_a^2,\quad V_a\ =(\eta+6)\tau_a^2$$

Applying these relations to equations (43) and (44), we obtain

for ductile materials

$$2(2\xi+3)[(1-\alpha)\tau_m^2+(2-\gamma)\tau_m\tau_a+(1+\beta)\tau_a^2]=p$$

for brittle materials

$$(\eta+2)[(1-\alpha)\tau_m^2+(2-\gamma)\tau_m\tau_a+(1+\beta)\tau_a^2]=q$$

Substituting for p and q values from equations (19) and (20) respectively, we obtain the following relation for both ductile and brittle materials:

$$(1-\alpha)\tau_m^2+(2-\gamma)\tau_m\tau_a+(1+\beta)\tau_a^2=\tau_e^2$$

Substituting for α and β values from equations (58) and (60) respectively, we obtain

$$v^2\tau_m^2+(2-\gamma)\tau_m\tau_a+w^2\tau_a^2=\tau_e^2 \quad \text{.................(68)}$$

The constant γ should be determined by results of experiments. Now let us determine the constant γ, using the fatigue limit of the case in which minimum stress is zero. Let τ_u be the endurance limit, when applied minimum stress is zero; by putting

$$\frac{\tau_e}{\tau_u}=u' \quad \text{......................................(69)}$$

and applying the condition, that $\tau_a=\frac{1}{2}\tau_u$ for $\tau_m=\frac{1}{2}\tau_u$, to equation (68), we can derive the value of γ as follows:

$$\gamma=v^2+w^2-4u'+2$$

Thus γ is determined. Applying this value of γ in equation (68), we obtain

$$v^2\tau_m^2+(4u'^2-v^2-w^2)\tau_m\tau_a+w^2\tau_a^2=\tau_e^2 \quad \text{......(70)}$$

This is the relation between τ_m and τ_a. When we represent equation (70) in a diagram, taking τ_m and

τ_a in the two coordinate axes, it generally becomes an ellipse having its center at the origin and principal axes generally inclined to the two coordinate axes. For special cases, when $u'=\frac{1}{2}(v+w)$, equation (70) becomes a straight line, and when $u'=\frac{1}{2}\sqrt{v^2+w^2}$, it becomes an ellipse with its principal axes parallel to the coordinate axes. For example, when we apply $v=0.4$ and $w=1.1$ in equation (70), the relation between τ_m and τ_a becomes as in Fig. 4. That is when $u'=0.75$, it becomes a straight line, and when $u'=0.585$, it becomes an ellipse with its principal axes parallel to the coordinate axes.

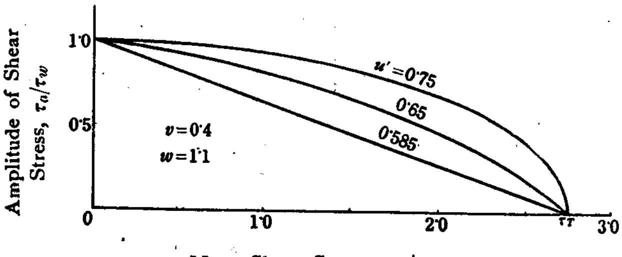

Fig. 4.
When Only Shear Stress Alternates Certain Maximum and Minimum Values.

V. When both the direct and shear stresses alternate between two equal magnitudes of mutually opposite signs and are in phase.

In this case

$$\sigma_m=\tau_m=0 \quad \ldots\ldots(71)$$

1. When $\varphi \leqq \frac{1}{\sqrt{3}}$

Applying the relations of equation (71) to equation (37), we obtain

$$U_{max}=U_a=(\xi+2)\sigma_a^2+2(2\xi+3)\tau_a^2$$
$$U_m=0$$

Applying these relation to equation (43) we obtain

$$(1+\beta)[(\xi+2)\sigma_a^2+2(2\xi+3)\tau_a^2]=p \quad \ldots\ldots(72)$$

Substituting for p, ξ and β values from equations (17), (50) and (60) respectively, we can derive the following relation:

$$\sigma_a^2+\frac{1}{\varphi^2}\tau_a^2=\sigma_w^2 \quad \ldots\ldots(73)$$

Equation (73) is the required relation between the direct and shear stresses at fatigue limits of the combination.

2. When $\varphi > \frac{1}{\sqrt{3}}$

Applying the relations of equation (71) to equation (38), we obtain

$$V_{max}=V_a=\eta\left[\sigma_a\left(\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}\right)+\tau_a^2\right]+2(\sigma_a^2+3\tau_a^2)$$
$$V_m=0$$

Applying these relations to equation (44), we obtain

$$(1+\beta)\left[\eta\left\{\sigma_a\left(\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}\right)+\tau_a^2\right\}+2(\sigma_a^2+3\tau_a^2)\right]=q \quad \ldots(74)$$

Substituting for q and β values from equations (18) and (60) respectively, equation (74) becomes

$$\eta\left[\sigma_a\left(\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}\right)+\tau_a^2-\sigma_w^2\right]+2(\sigma_a^2+3\tau_a^2-\sigma_w^2)=0 \quad \ldots(75)$$

Substituting for η value from equation (51), we obtain the following equation:

$$(1+\varphi^2)\sigma_a^2+(3\varphi^2-1)\sigma_a\sqrt{\sigma_a^2+4\tau_a^2}+4\tau_a^2=4\varphi^2\sigma_w^2 \quad \ldots(76)$$

Equation (76) is the required relation between direct and shear stresses at fatigue limits of the combination.

But it is somewhat tedious to calculate the values of σ_a and τ_a from equation (76). So let us derive an approximate equation for equation (76). Considering equation (75), it is seen that an equation obtained by putting the first term of equation (75) equal to zero corresponds to the theory of constant maximum principal stress, and that an equation obtained by putting the second term of equation (75) equal to zero corresponds to the theory of constant shear strain energy. Because the former equation can be obtained by squaring both sides of the equation

$$\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}=\sigma_w \quad \ldots\ldots(77)$$

which shows the theory of constant maximum principal stress, and the latter equation becomes

$$\sigma_a^2+3\tau_a^2=\sigma_w^2$$

which shows the theory of constant shear energy. Then substituting equation (77) in the first term of equation (75), we obtain

$$\eta(\sigma_w\sigma_a+\tau_a^2-\sigma_w^2)+2(\sigma_a^2+3\tau_a^2-\sigma_w^2)=0 \quad \ldots(78)$$

Similarly as in equation (75), there exists the following relation in equation (78): that is, an equation obtained by putting the first term of equation (78) equal to zero corresponds to the theory of constant maximum principal stress, and an equation obtained by putting the second term of equation (78) equal to zero corresponds to the theory of constant shear strain energy. Therefore we can use equation (78) instead of equation (75). Then substituting for η in equation (78) value from equation (51), we obtain the following equation:

$$(1-\varphi^2)\sigma_a^2+(3\varphi^2-1)\sigma_w\sigma_a+2\tau_a^2=2\varphi^2\sigma_w^2 \quad \ldots(79)$$

We can use equation (79) approximately for equation (76).

For example, substituting $\varphi=0.5$ in equation (73), and $\varphi=0.6$, 0.7, 0.8, 0.9 and 1.0 in equations (76) and (79), we obtain curves which are shown with thick lines in Fig. 5. Curves which are shown with full line in Fig. 5 correspond to exact equations (73) and (76), and curves shown with broken lines correspond to approximate equation (79). As seen in Fig. 5, approximate equation (79) is always in

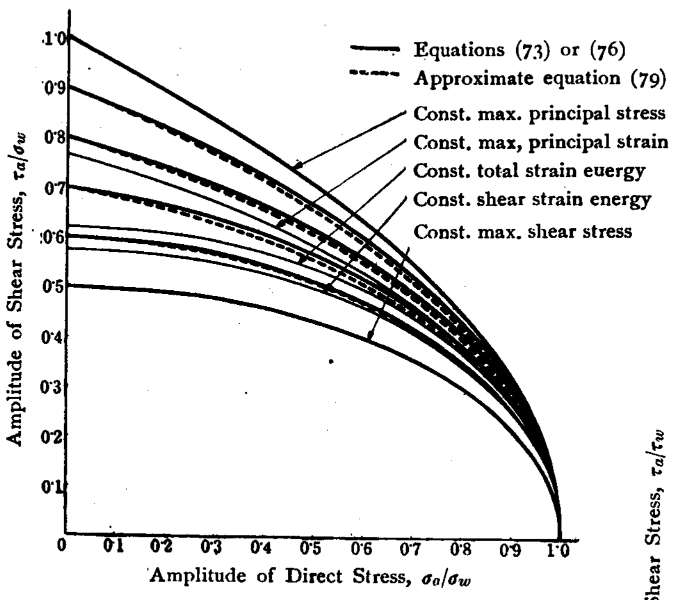

Fig. 5.

When Both Direct and Shear Stresses Alternate Between Two Equal Magnitudes of Mutualy Opposite Signs and Are In Phase. (The Authors' Theory.)

the safe side of exact equation (76). Now there is no need to compare these curves with results of experiments, because equations (73) and (79) are the same as the equations previously proposed by the authors and shown to be fairly applicable to experimental results.(9) For comparison, we showed also in Fig. 5 those curves, with thin lines, which correspond to theories of constant maximum principal stress, constant maximum principal strain, constant total strain energy, constant shear strain energy, and constant maximum shear stress (where we put Poisson's constant as 10/3.) When we put $\varphi=0.5$, $\frac{1}{\sqrt{3}}$ or 1.0, the authors' theory agrees with the theory of constant maximum shear stress, constant shear strain energy, or constant maximum principal stress, respectively.

The case of chapter V, i.e. the case when both direct and shear stresses alternate between two equal magnitudes of mutually opposite signs and are in phase, is quite analogous to the case of chapter II, i.e. the static case. If we put $w=1$, β becomes zero from equation (60). Then equations (72) and (74) become the same as equations (15) and (16) in the static case. Therefore if we regard σ_a and τ_a as static stresses and let $\sigma_w=\sigma_e$, then equations (73), (76) and (79) will represent the condition of elastic failure under static stresses.

Now let us for reference apply Matsumura's theory of static failure previously mentioned in introduction, to the case of fatigue. Regarding σ and τ in Matsumura's theory as the maximum values of the stresses, which alternate between two equal magnitudes of mutually opposite signs and are in phase, and letting $\sigma_e=\sigma_w$, the equation (4) becomes

$$(2\varphi-1)\sigma_a+\sqrt{\sigma_a^2+4\tau_a^2}=2\varphi\sigma_w \quad \ldots\ldots\ldots(80)$$

For example, substituting $\varphi=0.5$, 0.6, 0.7, 0.8, 0.9 and 1.0 in equation (80), we obtain curves shown with thick lines in Fig. 6. In Fig. 6 we also show for comparison the curves of the various other theories with thin lines.

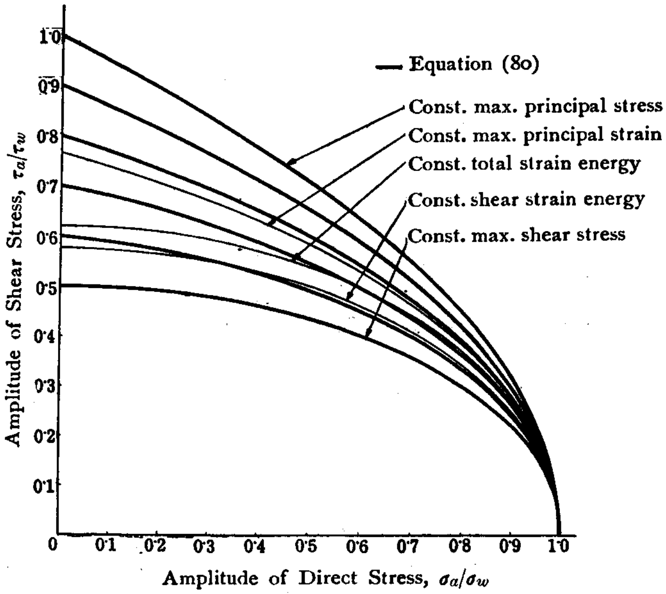

Fig. 6.

When Both Direct and Shear Stresses Alternate Between Two Equal Magnitudes of Mutually Opposite Signs and Are In Phase. (The Matsumura's Theory)

In equation (80) the relation between σ_a and τ_a for $\varphi=\frac{1}{\sqrt{3}}$ does not coincide with an ellipse having its center at origin. While in the authors' theory the relation between σ_a and τ_a for $\varphi=\frac{1}{\sqrt{3}}$ becomes simply an ellipse with its center at origin, it is in accordance with the theory of constant shear strain energy. As can be seen in Fig. 6, the curve of equation (80) for $\varphi=\frac{1}{\sqrt{3}}$ intersects with the curve of the theory of constant shear strain energy. But

(9) See foot-note (2).

according to the authors' experiments on various metals under combined alternating bending and torsion, the relation between σ_a and τ_a for $\varphi=\frac{1}{\sqrt{3}}$ must be properly represented with ellipse having its center at origin.

VI. When both direct and shear stresses alternate between two equal magnitudes of mutually opposite signs and are not in phase.

In the preceding chapter we considered the case in which both direct and shear stresses are in phase. In this chapter let us consider the case in which those stresses are not in phase, that is, they reach their maximum and minimum values at different instants.

1. When $\varphi<\frac{1}{\sqrt{3}}$.

In this case we should use equation (61), instead of equation (37), for the value of U_{max}. Therefore applying the relations of equation (71) to equations (61) and (37), we obtain

$$\begin{aligned} U_{max}&=[(\xi+2)\sigma_a^2\cos^2\omega t+2(2\xi+3)\tau_a^2 \\ &\qquad\cos^2(\omega t-\delta)]_{max} \\ U_m&=0 \\ U_a&=(\xi+2)\sigma_a^2+2(2\xi+3)\tau_a^2 \end{aligned}$$

Then applying these relations to equation (43), we obtain

$$[(\xi+2)\sigma_a^2\cos^2\omega t+2(2\xi+3)\tau_a^2\cos^2(\omega t-\delta)]_{max} \\ +\beta[(\xi+2)\sigma_a^2+2(2\xi+3)\tau_a^2]=p$$

Substituting for p and ξ values from equations (45) and (50) respectively, we obtain

$$[\sigma_a^2\cos^2\omega t+\frac{1}{\varphi^2}\tau_a^2\cos^2(\omega t-\delta)]_{max} \\ +\beta[\sigma_a^2+\frac{1}{\varphi^2}\tau_a^2]=(1+\beta)\sigma_w^2 \quad \text{......(81)}$$

where t should be determined as the first term of the left side of this equation to be maximum. Now let the time, when this first term becomes maximum, be as follows:

$$t=i\frac{\delta}{\omega} \quad \text{......(82)}$$

Putting the expression obtained from differentiating the first term of equation (81) equal to zero and substituting t by equation (82), we obtain the following relation:

$$\frac{\tau_a^2}{\sigma_a^2}=\varphi^2\frac{\sin i\delta\cdot\cos i\delta}{\sin(1-i)\delta\cos(1-i)\delta} \quad \text{......(83)}$$

Also substituting t by equation (82), equation (81) becomes

$$\sigma_a^2\cos^2 i\delta+\frac{1}{\varphi^2}\tau_a^2\cos^2(1-i)\delta \\ +\beta(\sigma_a^2+\frac{1}{\varphi^2}\tau_a^2)=(1+\beta)\sigma_w^2 \quad \text{......(84)}$$

From equations (83) and (84), σ_a and τ_a are obtained as follows:

$$\left.\begin{aligned} \sigma_a&=\sqrt{\frac{1+\beta}{h_1+(1+h_2)\beta}}\cdot\sigma_w \\ \tau_a&=\sqrt{h_2}\cdot\varphi\cdot\sigma_a \end{aligned}\right\} \quad \text{......(85)}$$

where

$$\left.\begin{aligned} h_1&=\cos^2 i\delta+\sin i\delta\cos i\delta\frac{\cos(1-i)\delta}{\sin(1-i)\delta} \\ h_2&=\frac{\sin i\delta\cdot\cos i\delta}{\sin(1-i)\delta\cdot\cos(1-i)\delta} \end{aligned}\right\} \quad \text{...(86)}$$

Considering i as a parameter in these equations, we can obtain the relation between σ_a and τ_a at the fatigue limit of the combination for an arbitrary value of δ, when σ_w and φ are given. But in above equations, β is a constant which depends not only on materials but also on the phase difference of applied stresses and must be determined from experiments in each case.

It must be noted that we cannot derive the relation between σ_a and τ_a from equations (85) and (86), when angle of phase difference δ is equal to 90 degree. In this case, the value of i becomes as follows:

$i=0$	when	$\sigma_a>\frac{1}{\varphi}\tau_a$
$i=1$	"	$\sigma_a<\frac{1}{\varphi}\tau_a$
$i=$ indeterminate	"	$\sigma_a=\frac{1}{\varphi}\tau_a$

Consequently the relation between σ_a and τ_a for $\delta=90$ degree becomes as follows:

$$\left.\begin{aligned} &\sigma_a^2+\frac{\beta}{1+\beta}\frac{1}{\varphi^2}\tau_a^2=\sigma_w^2 &&\text{when } \sigma_a>\frac{1}{\varphi}\tau_a \\ &\frac{\beta}{1+\beta}\sigma_a^2+\frac{1}{\varphi^2}\tau_a^2=\sigma_w^2 &&\text{"}\quad \sigma_a<\frac{1}{\varphi}\tau_a \\ &\sigma_a=\frac{1}{\varphi}\tau_a=\sigma_w &&\text{"}\quad \sigma_a=\frac{1}{\varphi}\tau_a \end{aligned}\right\} \text{...(87)}$$

It is needless to consider the cases in which phase differences are more than 90 degrees or less than 0 degree. Because those cases can be reduced always to the similar cases in which phase differences are between 0 and 90 degrees.

Thus the relations between σ_a and τ_a at the fatigue limit of the combination are made clear for every case. Then, as an example, we made calculation on the case in which φ is equal to 1/2, i.e., the case of maximum shear stress theory. Figs. 7 and 8 show the results of calculation for $\delta=90$ and 60 degrees, respectively. Of course, when β is infinity, the results of calculation agree with the case in which direct and shear stresses are in phase, i.e. $\delta=0$ degree. As seen in these figures, values of σ_a and τ_a at the fatigue limit become large, as β becomes small or δ becomes large. When $\delta=90$ degrees and $\beta=0$, values of σ_a or τ_a at the

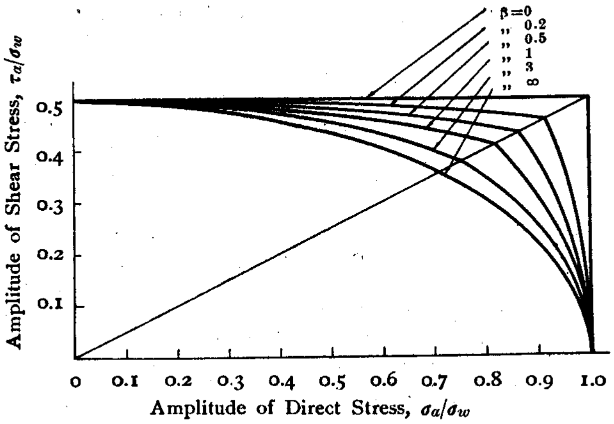

Fig. 7.
On Ductile Materials of $\varphi=0.5$, when Both Direct and Shear Stresses Alternate Between Two Equal Magnitudes of Mutually Opposite Signs and the Phase Difference $\delta=90°$.

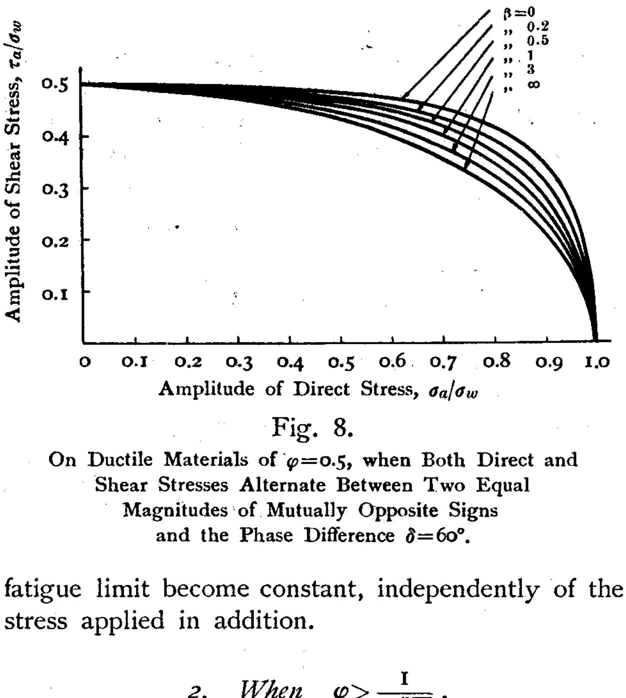

Fig. 8.
On Ductile Materials of $\varphi=0.5$, when Both Direct and Shear Stresses Alternate Between Two Equal Magnitudes of Mutually Opposite Signs and the Phase Difference $\delta=60°$.

fatigue limit become constant, independently of the stress applied in addition.

2. When $\varphi>\dfrac{1}{\sqrt{3}}$.

In this case we should use equation (62), instead of equation (37), for the value of V_{max}. Therefore applying the relations of equation (71) to equations (62) and (38), we obtain

$$V_{max}=\Big[\eta\,[\sigma_a\cos\omega t\Big\{\frac{1}{2}\sigma_a\cos\omega t+\frac{1}{2}\sqrt{\sigma_a^2\cos^2\omega t+4\tau_a^2\cos^2(\omega t-\delta)}\Big\}+\tau_a^2\cos^2(\omega t-\delta)]+2\{\sigma_a^2\cos^2\omega t+3\tau_a^2\cos^2(\omega t-\delta)\}\Big]_{max}$$

$$V_m=0$$

$$V_a=\eta[\sigma_a\Big\{\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_a^2}\Big\}+\tau_a^2]+2(\sigma_a^2+3\tau_a^2)$$

Applying these relations to equation (44), we obtain

$$[(\eta+4)\sigma_a^2\cos^2\omega t+\eta\sigma_a\cos\omega t\sqrt{\sigma_a^2\cos^2\omega t+4\tau_a^2\cos^2(\omega t-\delta)}+2(\eta+6)\tau_a^2\cos^2(\omega t-\delta)]_{max}+\beta[(\eta+4)\sigma_a^2+\eta\sigma_a\sqrt{\sigma_a^2+4\tau_a^2}+2(\eta+6)\tau_a^2]=2q$$

Substituting for q and η values from equations (46) and (51) respectively, we obtain

$$[(\varphi^2+1)\sigma_a^2\cos^2\omega t+(3\varphi^2-1)\sigma_a\cos\omega t\sqrt{\sigma_a^2\cos^2\omega t+4\tau_a^2\cos^2(\omega t-\delta)}+4\tau_a^2\cos^2(\omega t-\delta)]_{max}+\beta[(\varphi^2+1)\sigma_a^2+(3\varphi^2-1)\sigma_a\sqrt{\sigma_a^2+4\tau_a^2}+4\tau_a^2]=4(1+\beta)\varphi^2\sigma_w^2 \quad \ldots\ldots(88)$$

where t should be determined as the first term of the left side of this equation becomes maximum. Now let the time t when this first term becomes maximum be $i\dfrac{\delta}{\omega}$ similarly as before. Then putting the expression obtained by differentiation of the first term of equation (88) equal to zero, and substituting t by $i\dfrac{\delta}{\omega}$, we obtain the following relation:

$$[(\varphi^2+1)\sigma_a^2\sin i\delta\cos i\delta-4\tau_a^2\sin(1-i)\delta\cos(1-i)\delta]\sqrt{\sigma_a^2\cos^2 i\delta+4\tau_a^2\cos^2(1-i)\delta}+(3\varphi^2-1)\sigma_a[\sigma_a^2\sin i\delta\cos^2 i\delta+2\tau_a^2\{\sin i\delta\cos^2(1-i)\delta-\cos i\delta\sin(1-i)\delta\cos(1-i)\delta\}]=0 \quad \ldots\ldots(89)$$

Substituting t by $i\dfrac{\delta}{\omega}$, equation (88) becomes

$$(\varphi^2+1)\sigma_a^2\cos^2 i\delta+(3\varphi^2-1)\sigma_a\cos i\delta\sqrt{\sigma_a^2\cos^2 i\delta+4\tau_a^2\cos^2(1-i\delta)}+4\tau_a^2\cos^2(1-i)\delta+\beta[(\varphi^2+1)\sigma_a^2+(3\varphi^2-1)\sigma_a\sqrt{\sigma_a^2+4\tau_a^2}+4\tau_a^2]=4(1+\beta)\varphi^2\sigma_w^2 \quad \ldots\ldots(90)$$

Considering i as a parameter in equations (89) and (90), we can calculate values of σ_a and τ_a at the fatigue limit of the combination for an arbitrary value of δ, when σ_w and φ are given. β must be determined from experiments in each case.

It must be noted that from equation (89) we know

$$i=0 \quad \text{when} \quad \delta=90° \text{ and } \sigma_a>\frac{2}{\sqrt{\varphi^2+1}}\tau_a$$

Therefore the relation between σ_a and τ_a becomes as follows

$$4\varphi^2\sigma_a^2+\beta[(\varphi^2+1)\sigma_a^2+(3\varphi^2-1)\sigma_a\sqrt{\sigma_a^2+4\tau_a^2}+4\tau_a^2]=4(1+\beta)\varphi^2\sigma_w^2 \quad \ldots\ldots\ldots\ldots(91)$$

$$\text{when} \quad \delta=90° \quad \text{and} \quad \sigma_a>\frac{2}{\sqrt{\varphi^2+1}}\tau_a$$

Thus we can calculate values of σ_a and τ_a at the fatigue limit of the combination in every case. As an example, we made calculation for the case which φ is equal to 1, i.e., the case of maximum principal stress theory. Figs. 9 and 10 show the results of

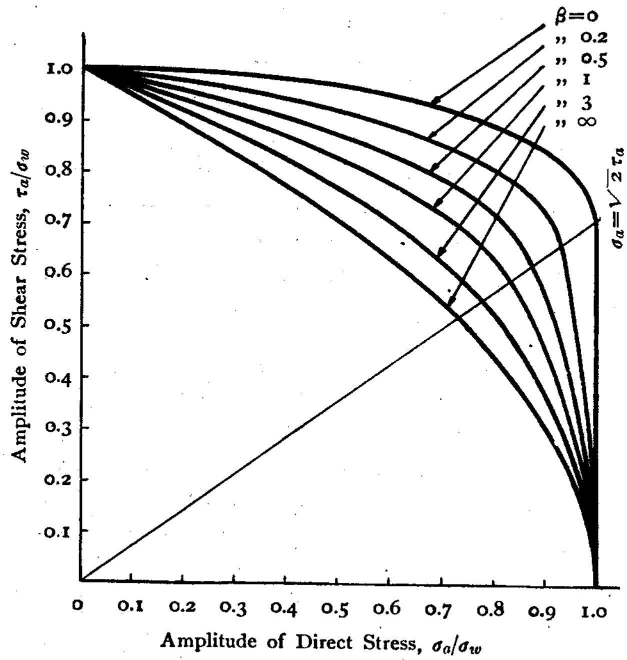

Fig. 9.
On Brittle Materials of $\varphi=1$, when Both Direct and Shear Stresses Alternate Between Two Equal Magnitudes of Mutually Opposite Signs and the Phase Difference $\delta=90°$.

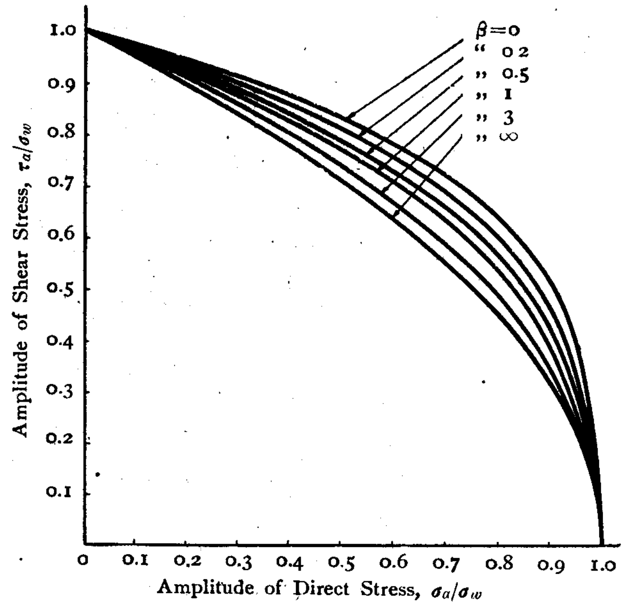

Fig. 10.
On Brittle Materials of $\varphi=1$, when Both Direct and Shear Stresses Alternate Between Two Equal Magnitudes of Mutually Opposite Signs and the Phase Difference $\delta=60°$.

calculation for $\delta=90$ and 60 degrees, respectively. Similarly as in the case of $\varphi<\frac{1}{\sqrt{3}}$, values of σ_a and τ_a at the fatigue limit become large, as β becomes small or δ becomes large. In the case of $\beta=0$ in Fig. 9, the value of σ_a at the fatigue limit is constant independently of the applied shear stress, when $\sigma_a>\sqrt{2}\,\tau_a$, but the value of τ_a at the fatigue limit is not constant, when $\sigma_a<\sqrt{2}\,\tau_a$.

The authors are now carrying out fatigue tests on several metals under combined alternating bending and torsional stresses with phase differences. Comparing with the test results obtained till now, the authors are convinced that the above criterion of the strength of metals under combined stresses with phase differences is surely applicable to the test results on every metal. As for the applicability of the above criterion to practical test results, we shall discuss fully in the forthcoming report.

VII. When direct stress alternates between two equal magnitudes of mutually opposite signs and shear stress acts statically.

Let the amplitude of direct stress at the fatigue limit be σ_a, when shear stress τ_{st} is working statically. In this case

$$\tau_m=\tau_{st},\quad \sigma_m=\tau_a=0 \quad\quad (92)$$

1. When $\varphi\leqq\frac{1}{\sqrt{3}}$.

Applying the relations of equation (92) to equation (37), we obtain

$$U_{max}=(\xi+2)\sigma_a^2+2(2\xi+3)\tau_{st}^2$$
$$U_m=2(2\xi+3)\tau_{st}^2$$
$$U_a=(\xi+2)\sigma_a^2$$

Applying these relations to equation (43), we obtain

$$(1+\beta)(\xi+2)\sigma_a^2-\gamma\sqrt{2(\xi+2)(2\xi+3)}\,\sigma_a\tau_{st}$$
$$+2(1-\alpha)(2\xi+3)\tau_{st}^2=p$$

Substituting for p, ξ, α and β values from equations (17), (50), (58) and (60) respectively, we can derive the following relation:

$$\varphi^2w^2\sigma_a^2-\gamma\varphi\sigma_a\tau_{st}+v^2\tau_{st}^2=\varphi^2\sigma_e^2 \quad\quad (93)$$

From this equation, when the applied static shear stress τ_{st} is given, we can calculate the amplitude of direct stress σ_a at the fatigue limit. Taking σ_a and τ_{st} in rectangular coordinate axes, equation (93) can be represented with an ellipse having its center at origin and the principal axes generally inclined to coordinate axes.

Then let us examine whether the results of calculation from equation (93) agree with experimental results. Fig. 11 gives the results fatigue test carried out by Ono[10] and Lea-Budgen[11] under the combined rotating bending and static torsional stresses. In these results, we see that the fatigue limit of rotating bending becomes rather higher, as

(10) A. Ono, J. of the Society of Mech. Eng., Japan, Vol. XXIII, No. 62, (1921), p. 201.
(11) F. Lea and H. Budgen, Engineering, Vol. CXXII, (1926), p. 242.

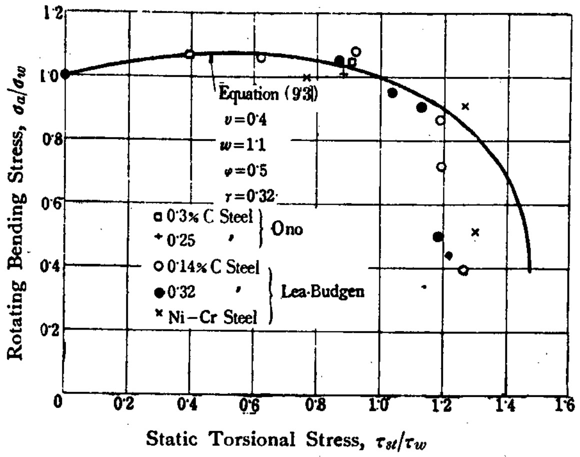

Fig. 11.

Test Results on Ductile Materials under Combined Rotating Bending and Static Torsion.

the applied static torsional stress becomes larger, so far as the torsional stress is under a certain limit. The curve shown in Fig. 11 is that which was drawn from equation (93), where we put $\varphi=0.5$, $\gamma=0.32$, $v=0.4$ and $w=1.1$, as in the previous example. As seen in the figure, the curve calculated from equation (93) is in good accordance with the experimental results.

2. *When* $\varphi>\frac{1}{\sqrt{3}}$.

Applying the relations of equation (92) to equation (38), we obtain

$$V_{max}=\eta\Big[\sigma_a\Big(\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_{st}^2}\Big)+\tau_{st}^2\Big]+2(\sigma_a^2+3\tau_{st}^2)$$

$$V_m=(\eta+6)\tau_{st}^2$$

$$V_a=(\eta+2)\sigma_a^2$$

Then applying these relations to equation (44), we obtain

$$\eta\Big[\sigma_a\Big(\frac{1}{2}\sigma_a+\frac{1}{2}\sqrt{\sigma_a^2+4\tau_{st}^2}\Big)+\tau_{st}^2\Big]+2(\sigma_a^2+3\tau_{st}^2)-\alpha(\eta+6)\tau_{st}^2+\beta(\eta+2)\sigma_a^2-\gamma\sqrt{(\eta+2)(\eta+6)}\,\sigma_a\cdot\tau_{st}^2=q$$

Substituting for q, η, α and β values from equations (18). (51), (58) and (60) respectively, the following relation can be derived:

$$(4w^2\varphi^2-3\varphi^2+1)\sigma_a^2+(3\varphi^2-1)\sigma_a\sqrt{\sigma_a^2+4\tau_{st}^2}-4\gamma\varphi\sigma_a\tau_{st}+4v^2\tau_{st}^2=4\varphi^2\sigma_e^2 \quad \text{.........(94)}$$

This is the required equation which represents the relation between static shear stress τ_{st} and the amplitude of direct stress σ_a at the fatigue limit.

To examine whether equation (94) is applicable to results of experiments, let us compare the results of calculation from equation (94) with the fatigue test results made by Dr. Ono for cast iron under combined rotating bending and static torsional stresses.[12] Fig. 12 shows the test results. As seen in this figure, the limit of cast iron at rotating bending decreases gradually from the start, as the applied static torsional stress increases. In this case, $\tau_T=20.7\,\text{kg/mm}^2$ and $\sigma_w=7.49\,\text{kg/mm}^2$. Now let $w=1$ and $\varphi=0.8$ (according to the authors' experiments under combined alternating bending and torsion, $\varphi=0.808$ for cast iron[13]), then v becomes 0.29. Value of γ should be determined from the results of experiments in this case. Adopting the condition that $\sigma_a=7.0\,\text{kg/mm}^2$ for $\tau_{st}=5.25\,\text{kg/mm}^2$, γ becomes 0.232. Using the above values of the constants, equation (94) can be represented as the curve shown in Fig. 12. We see that the curve of equation (94) is in good accordance with the results of experiments.

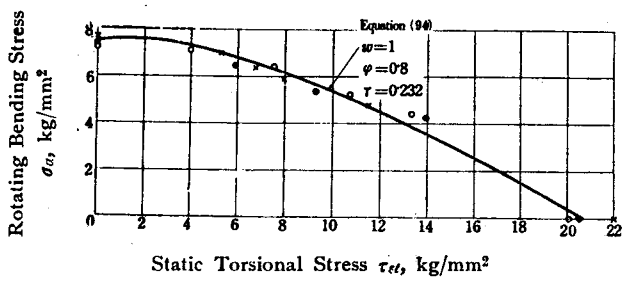

Fig. 12.

Test Results on Cast Iron under Combined Rotating Bending and Static Torsion.

VIII. When shear stress alternates between two equal magnitudes of mutually opposite signs and direct stress acts statically.

Let the amplitude of shear stress at the fatigue limit be τ_a, when direct stress σ_{st} is working statically. In this case

$$\sigma_m=\sigma_{st},\quad \sigma_a=\tau_m=0 \quad \text{.......................(95)}$$

1. *When* $\varphi\leqq\frac{1}{\sqrt{3}}$.

Applying the relations of equation (95) to equation (37), we obtain

$$U_{max}=(\xi+2)\sigma_{st}^2+2(2\xi+3)\tau_a^2$$

$$U_m=(\xi+2)\sigma_{st}^2$$

$$U_a=2(2\xi+3)\tau_a^2$$

Applying these relaiions to equation (43), we obtain

$$(1-\alpha)(\xi+2)\sigma_{st}^2-\gamma\sqrt{2(\xi+2)(2\xi+3)}\,\sigma_{st}\cdot\tau_a+2(1+\beta)(2\xi+3)\tau_a^2=p$$

Substituting for p, ξ, α and β values from equations

(12) A. Ono, Trans. of the Society of Mech. Eng., Japan, Vol. 6, No. 25, (1940), Part 1, p. 30.

(13) See foot-note (2).

(19), (40), (58) and (60) respectively, the following relation can be derived:

$$v^2\varphi^2\sigma_{st}^2-\gamma\varphi\sigma_{st}\tau_a+w^2\tau_a^2=\tau_e^2 \quad \text{.................}(96)$$

This is the required relation between static direct stress σ_{st} and the amplitude of shear stress τ_a at the fatigue limit. Taking τ_a and σ_{st} in the rectangular coordinate axes, equation (96) can be represented with an ellipse having its center at origin and the principal axes generally inclined to coordinate axes.

In order to examine whether the equation (96) is applicable to results of experiments, let us compare the results of calculation from equation (96) with the fatigue test results made by the authors with 0.1% and 0.34% carbon steels under combined alternating torsion and static tension.(14) Fig. 13

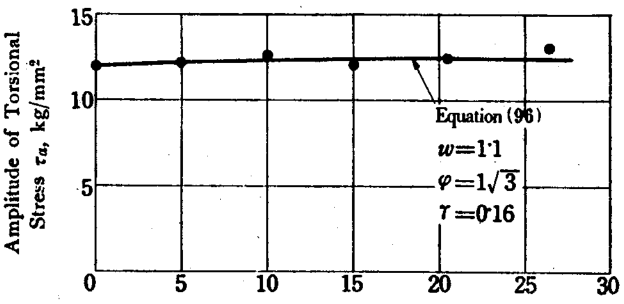

Fig. 13.
Test Results on 0.1% Carbon Steel under Combined Alternating Torsion and Static Tension.

shows the test results for 0.1% carbon steel. From the test results, it can be seen that the torsional fatigue limit undergoes a little influence of the applied static tensile stress, or it shows a tendency to become rather higher as the applied static tensile stress increases. In this case $\sigma_T=83.4\ \text{kg/mm}^2$ and $\tau_w=12\ \text{kg/mm}^2$. Now let $w=1.1$ and $\varphi=\frac{1}{\sqrt{3}}$, then v becomes 0.274. The value of γ should be determined by the results of experiments in this case. Adopting the condition that $\tau_a=12.44\ \text{kg/mm}^2$ for $\sigma_{st}=20.49\ \text{kg/mm}^2$, γ becomes 0.16. Using the

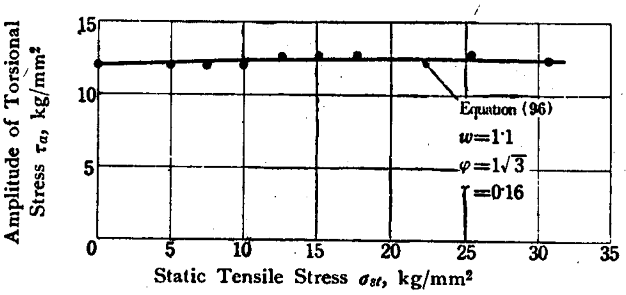

Fig. 14.
Test Results on 0.34% Carbon Steel and Combined Alternating Torsion and Static Tension.

above values of the constants, equation (96) can be represented as the curve shown in Fig. 13. As seen in this figure, the curve of equation (96) is in good accordance with the results of experiments.

Fig. 14 shows the test results for 0.34% carbon steel. In this case also, we see that the torsional fatigue limit has a tendency to become rather higher as the applied static tensile stress increases. In this case $\sigma_T=81.2\ \text{kg/mm}^2$ and $\tau_w=12\ \text{kg/mm}^2$. Now let $w=1.1$ aud $\varphi=\frac{1}{\sqrt{3}}$, then v becomes 0.28. By determining γ from the condition that $\tau_a=12.4\ \text{kg/mm}^2$ for $\sigma_{st}=30.7\ \text{kg/mm}^2$, it becomes 0.168. Using these values of constants, equation (96) can be represented as the curve shown in Fig. 14. As seen in this figure, the curve of equation (96) is in good accordance with the results of experiments also in this case.

2\. *When* $\varphi>\frac{1}{\sqrt{3}}$.

Applying the relations of equation (95) to equation (38), we obtain

$$V_{max}=\eta\left[\sigma_{st}\left(\frac{1}{2}\sigma_{st}+\frac{1}{2}\sqrt{\sigma_{st}^2+4\tau_a^2}\right)+\tau_a^2\right]+3(\sigma_{st}^2+3\tau_a^2)$$

$$V_m=(\eta+2)\sigma_{st}^2$$

$$V_a=(\eta+6)\tau_a^2$$

Then applying these relations to equation (44), we obtain

$$\eta\left[\sigma_{st}\left(\frac{1}{2}\sigma_{st}+\frac{1}{2}\sqrt{\sigma_{st}^2+4\tau_a^2}\right)+\tau_a^2\right]+2(\sigma_{st}^2+3\tau_a^2)-\alpha(\eta+2)\sigma_{st}^2+\beta(\eta+6)\tau_a^2-\gamma\sqrt{(\eta+2)(\eta+6)}\,\sigma_{st}\tau_a=q$$

Substituting for q, η, α and β values from equations (20), (51), (58) and (60) respectively, the following relation can be derived:

$$(4v^2\varphi^2-3\varphi^2+1)\sigma_{st}^2+(3\varphi^2-1)\sigma_{st}\sqrt{\sigma_{st}^2+4\tau_a^2}-4\gamma\varphi\sigma_{st}\tau_a+4w^2\tau_a^2=4\tau_e^2 \quad \text{.........}(97)$$

From this equation, when the applied static direct stress σ_{st} is given, we can calculate the amplitude of shear stress τ_a at the fatigue limit.

To examine whether equation (97) is applicable to results of experiments, let us compare the results of calculation from equation (97) with the fatigue test results made by the authors for 0.72% carbon steel and cast iron under combined alternating torsion and static tension.(15) Fig. 15 shows the test results for 0.72% carbon steel. From the test results, it is seen that the torsional fatigue limit decreases gradually as the applied static tensile stress increases. In this case $\sigma_T=102\ \text{kg/mm}^2$ and $\tau_w=20\ \text{kg/mm}^2$. Now let $w=1.1$ and $\varphi=0.68$, then

(14) T. Nishihara and M. Kawamoto, Nippon Kinzoku Gakkai-Si, Vol. 6, No. 6, (1942), p. 316.
(15) See foot-note (14).

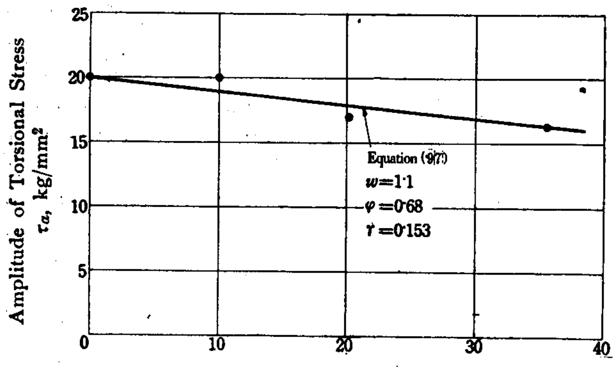

Fig. 15.
Test Results on 0.72% Carbon Steel under Combined Alternating Torsion and Static Tention.

v becomes 0.317, According to the author's experiments under combined alternating bending and torsion, $\varphi=0.667$ for 0.62% carbon steel. Value of φ has a tendency to increase with the carbon content in steel. And by determining γ from the condition that $\tau_a=16.3\,\text{kg/mm}^2$ for $\sigma_{st}=35.5\,\text{kg/mm}^2$, it becomes -0.153. Using these values of constants, equation (87) can be represented as the curve shown in Fig. 15, which is seen to be practically in accordance with the results of experiments.

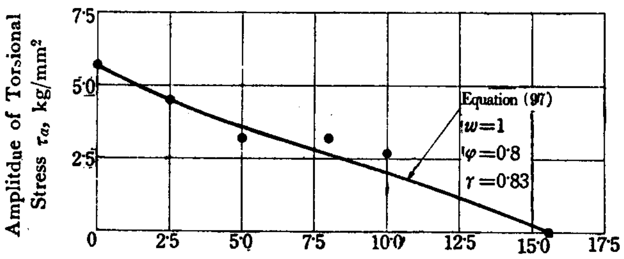

Fig. 16.
Test Results on Cast Iron under Combined Alternating Torsion and Static Tension.

Fig. 16 shows the test results for cast iron, in which we see that the torsional fatigue limit decreases rapidly, as the applied static tensile stress increases. In this case, $\sigma_T=15.55\,\text{kg/mm}^2$ and $\tau_w=5.7\,\text{kg/mm}^2$. Now let $w=1$ and $\varphi=0.8$ as in the previous case for cast iron, then v becomes 0.458. And by determining γ from the condition that $\tau_a=4.5\,\text{kg/mm}^2$ for $\sigma_{st}=2.5\,\text{kg/mm}^2$, it becomes -0.83. Using these values of constants, equation (97) can be represented as the curve shown in Fig. 16, which is seen also to be practically in accordance with the results of experiments.

IX. Summary.

(1) The authors have proposed a new criterion stating that "the elastic failure occurs in ductile materials when the maximum shear stress induced by combined stresses reaches a definite value depending on shear strain energy, and the elastic failure occurs in brittle materials when the maximum principal stress induced by combined stresses reaches a definite value depending also on shear strain energy." From this criterion, the conditions of elastic failure have been derived for ductile and brittle materials.

(2) Applying the above criterion to the case of fatigue, the authors have also proposed the criterion stating that "the fatigue failure occurs in ductile materials when the greatest maximum shear stress induced by combined repeated stresses reaches a definite value depending on shear strain energy at the instant, and that the fatigue failure occurs in brittle materials when the greatest maximum principal stress induced by combined repeated stresses reaches a definite value depending also on shear strain energy at the instant." From this criterion, the conditions of fatigue failure have been derived for ductile and brittle materials.

(3) When the new criterion is applied to the case in which only the direct stress alternates between certain maximum and minimum values, the results of calculation agree with the experimental results previously obtained by the authors: i. e. "If we take the mean stress σ_m and stress amplitude σ_a as the two coordinate axes, the endurance limit can be represented by the straight line through the two points, i. e. the point of endurance limit with zero mean stress and the point of breaking static strength."

(4) When the new criterion is applied to the case in which both direct and shear stresses alternate between two equal magnitudes of mutually opposite signs and are in phase, the results of calculation become the same as the equations which the authors have proposed in the previous report to be fairly applicable to experimental results.

(5) The conditions of fatigue failure have been calculated also for the case in which both direct and shear stresses alternate between two equal magnitudes of mutually opposite signs and are not in phase. But we cannot compare the results of calculation with results of experiments, because no experiments have ever been made for this case. The authors are now carrying out fatigue tests on several metals for this case. Comparing with the test results obtained up to now, the authors are convinced that the new criterion is surely applicable to results of experiments. As for this case, we shall discuss fully in the forthcoming report.

(6) Further when the new criterion is applied to the case in which either direct or shear stress alternate between two equal magnitudes of mutually opposite signs and the other stress acts statically,

the results of calculation are also applicable to the following experimental results; i. e. the fatigue limit of ductile material becomes rather higher, as the applied static stress increases, and the fatigue limit of brittle material becomes lower, as the applied static stress increases.

X. Appendix.

As previously mentioned in Chapters VII and VIII, according to the results of experiments made by Ono, Lea-Budgen and the authors on the case, in which either direct or shear stress alternates between two equal magnitudes of mutually opposite signs and the other stress acts statically, the effect of static stress upon the fatigue limit is very different according to whether the material is ductile or brittle. That is, in ductile materials the fatigue limit becomes higher as the applied static stress increases, and in brittle materials the fatigue limit becomes lower as the applied static stress increases. And the results of calculation from the new criterion are in good accordance with those results of experiments. That is, the curves of equations (93) and (96) for ductile materials take the form indicating that the fatigue limit becomes higher as static stress increases, and the curves of equations (94) and (97) for brittle materials take the form indicating that the fatigue limit becomes lower as static stress increases.

But attention must be paid to the fact that shapes of the curves represented by equations (93), (94), (96) and (97), can take either form according as the fatigue limit becomes higher or lower as static stress increases, according to an appropriate choice of the value of γ.[16] Therefore the new criterion is indeed a proper means to represent those experimental results in all cases. However, to our great regret, the reason for those results of experiments cannot be fully explained theoretically from the above criterion.

In explanation of those results of experiments, Ono[17] and Bailey[18] have reported their theories for the case in which direct stress alternates and shear stress acts statically. The authors have also reported a theory for the case in which shear stress alternates and direct stress acts statically.[19] Bailey has also recently reported a similar theory for ductile materials.[20]

Here the authors wish to state the fact that the above-mentioned experimental results can well be explained on the basis of the following two assumptions:

(i) Taking mean stress σ_m (or τ_m) as abscissa and stress amplitude σ_a (or τ_a) as ordinate, the fatigue limit can be represented as the straight line through the two points, i. e. the point of endurance limit with zero mean stress and the point of breaking tensile strength (or breaking shear strength.)

(ii) In ductile materials, fatigue failure is determined only by the shear stress acting in the plane of the greatest maximum shear stress, and in brittle materials, fatigue failure is determined only by the direct stress acting in the plane of the greatest maximum principal stress.

1. On ductile materials.

(1) When direct stress alternates between two equal magnitudes of mutually opposite signs and shear stress acts statically.

Let τ' be the shear stress which acts in the plane inclined at angle α with the cross section of a specimen, then

$$\tau' = \frac{1}{2}\sigma_a \sin 2\alpha + \tau_{st} \cos 2\alpha$$

Therefore, let the mean shearing stress in this plane be τ'_m, and the amplitude of the shearing stress be τ'_a, then

$$\left.\begin{aligned} \tau'_a &= \frac{1}{2}\sigma_a \sin 2\alpha \\ \tau'_m &= \tau_{st} \cos 2\alpha \end{aligned}\right\} \qquad (98)$$

From the above-mentioned assumption

$$\frac{\tau'_a}{\tau_w} + \frac{\tau'_m}{\tau_T} = 1 \quad \text{or} \quad \tau'_a + \frac{v}{w}\tau'_m = \tau_w$$

Substituting for τ'_a and τ'_m in this equation values from equation (98), we obtain

$$\frac{1}{2}\sigma_a \sin 2\alpha + \frac{v}{w}\tau_{st} \cos 2\alpha = \tau_w \qquad (99)$$

Whereas we know that the greatest maximum shear stress occurs in the plane of

$$\alpha = \frac{1}{2}\tan^{-1}\frac{\sigma_a}{2\tau_{st}}$$

Substituting this value of α in equation (99), we obtain

$$\sigma_a^2 + 4\frac{v}{w}\tau_{st}^2 = \sigma_w\sqrt{\sigma_a^2 + 4\tau_{st}^2} \qquad (100)$$

---

(16) But there are some reports which give the results of experiments, that the fatigue limit of ductile materials becomes lower from the first as static stress increases. For example, for the case of combined rotating bending and static torsion, Davies, Inst. of Mech. Eng. Proc., Vol. 131, (1335), p. 66, and for the case of combined alternating torsion and static tension, Hohenemser-Prager, Metallwirtschaft, Vol. 12, (1933), p. 342.

(17) See foot-note (10).

(18) R. Bailey, Engineering, Vol. CIV, (1916), p. 81.

(19) See foot-note (14).

(20) See foot-note (7).

This is the relation between static shear stress and the amplitude of direct stress at the fatigue limit. Fig. 17 shows the relation of equation (100) in a diagram. From this diagram, we understand that the amplitude of direct stress at the fatigue limit becomes higher as static shear stress increases, so far as the latter is under a certain limit.

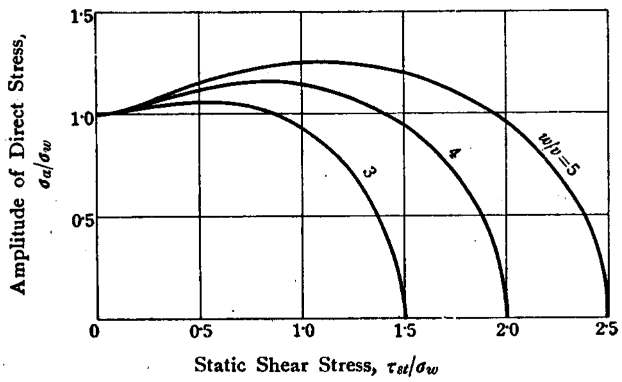

Fig. 17.

The Case of Combined Alternating Direct and Static Shear Stresses on Ductile Material.

(2) When shear stress alternates between two equal magnitudes of mutually opposite signs and direct stress acts statically.

In the similar manner as in the preceding case, we can derive the following relation in this case:

$$\tau_a^2 + \frac{1}{4}\,\frac{v}{w}\sigma_{st}^2 = \frac{1}{2}\tau_w\sqrt{\sigma_{st} + 4\tau_a^2} \quad \cdots\cdots\cdots 101)$$

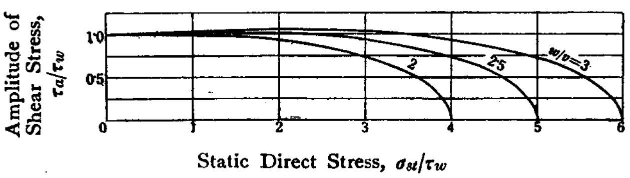

Fig. 18.

The Case of Combined Alternating Shear and Static Direct Stresses on Ductile Material.

Fig. 18 shows the relation of equation (101) in a diagram. From this diagram, we understand that the amplitude of shear stress at the fatigue limit becomes higher as static direct stress increases, so far as the latter is under a certain limit. But the degree of increase of the fatigue limit is smaller in comparison with the preceding case. Especially when w/v is small, the fatigue limit does not increase.

2. On brittle materials.

(1) When direct stress alternates between two equal magnitudes of mutually opposite signs and shear stress acts statically.

Let σ' be the direct stress which acts in the plane inclined at angle α with the cross section of a specimen, then

$$\sigma' = \frac{1}{2}\sigma_a(1 + \cos 2\alpha) + \tau_{st}\sin 2\alpha$$

Therefore, let the mean direct stress in this plane be σ'_m, and the amplitude of the direct stress be σ'_a, then

$$\left.\begin{aligned} \sigma'_a &= \frac{1}{2}\sigma_a(1 + \cos 2\alpha) \\ \sigma'_m &= \tau_{st}\sin 2\alpha \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (102)$$

From the above-mentioned assumption

$$\frac{\sigma'_a}{\sigma_w} + \frac{\sigma'_m}{\sigma_T} = 1 \quad \text{or} \quad \sigma'_a + \frac{v}{w}\sigma'_m = \sigma_w$$

Substituting for σ'_a and σ'_m in this equation values from equation (102), we obtain

$$\frac{1}{2}\sigma_a(1 + \cos 2\alpha) + \frac{v}{w}\tau_{st}\sin 2\alpha = \sigma_w \cdots\cdots (103)$$

Whereas we know that the greatest maximum principal stress occurs in the plane of

$$\alpha = \frac{1}{2}\tan^{-1}\frac{2\tau_{st}}{\sigma_a}$$

Substituting this value of α in equation (103), we obtain

$$\sigma_a^2 + 4\frac{v}{w}\tau_{st}^2 = (2\sigma_w - \sigma_a)\sqrt{\sigma_a^2 + 4\tau_{st}^2} \quad \cdots\cdots (104)$$

This is the relation between static shear stress and the amplitude of direct stress at the fatigue limit.

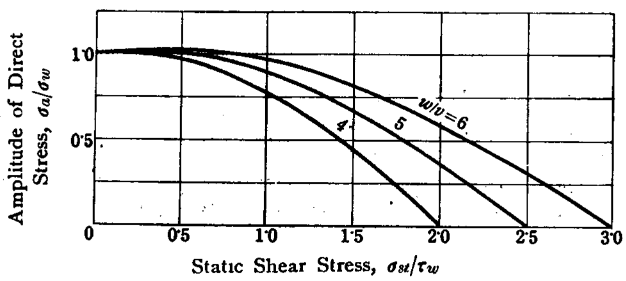

Fig. 19.

The Case of Combined Alternating Direct and Static Shear Stresses on Brittle Material.

Fig. 19 shows the relation of equation (104) in a diagram. From this diagram, we understand that the amplitude of direct stress at the fatigue limit becomes generally lower as the static shear stress increases. But it can be seen in Fig. 19 that the fatigue limit increases slightly at first, when w/v is large. The matter is quite similar as in equation (94) or Fig. 12.[21]

(2) When shear stress alternates between two equal magnitudes of mutually opposite signs and direct stress acts statically.

In the similar manner as in the preceding case, we can derive the following relation in this case:

$$\tau_a^2 + \frac{1}{4}\,\frac{v}{w}\sigma_{st}^2 = \frac{1}{4}\left(2\tau_w - \frac{v}{w}\sigma_{st}\right)\sqrt{\sigma_{st}^2 + 4\tau_a^2} \cdots\cdots\cdots (105)$$

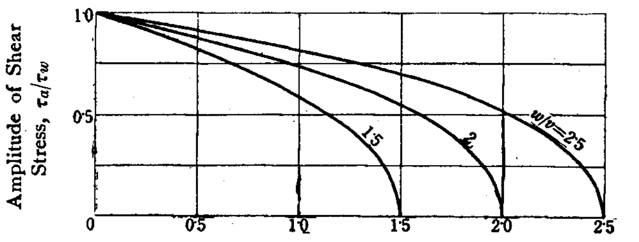

Fig. 20.

The Case of Combined Alternating Shear and Static Direct Stresses on Brittle Material.

Fig. 20 shows the relation of equation (105) in a diagram. From this diagram, we understand that the amplitude of shear stress at the fatigue limit becomes considerably lower from the first as the static direct stress increases.

(21) The matter is also analogous to the results of calculation made by Dr. Ono, that static shear stress exerts no influence upon the amplitude of direct stress at the fatigue limit, even when the fatigue failure is considered to be determined only by direct stress. See foot-note (10).