# New Theory of Fatigue of Metals and Various Applications.

By

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CONTENTS



## § 1 Rapid Determination of Endurance Limit.

According to the authors' view that fatigue of metals occurs owing to the accumulation of crystal distortion the fundamental formulæ for fatigue curve has been derived[1]. Namely, if we denote the given stress with $\sigma$, the endurance limit with $\sigma_w$, the number of stress repetitions with $N$, then the equation

$$H^{mpN} = \frac{\sigma^{1+m}}{\sigma^{1+m} - \sigma_w^{1+m}} \qquad (1)$$

holds. Here $H$ is the "hardening coefficient", $m$ is the power in the equation $\varepsilon_p = a\sigma^m$, and $p$ is the ratio of the number of stress repetitions till the first crack occurs to that till specimen fractures. Using the "relative stress" $S = \sigma/\sigma_w$ above equation becomes

$$H^{mpN} = \frac{S^{1+m}}{S^{1+m} - 1} \qquad (2)$$

where the right-hand side is a function of $S$ only and is called "fatigue function of $m$-th degree".

$$\varphi_m(S) = \frac{S^{1+m}}{S^{1+m} - 1} \qquad (3)$$

This is represented in fig. 1 for $m = 0, 1, 2, \ldots\ldots$ and it will be reduced to a hyperbola if we write

$$\varphi_m(S) = Y + 1, \quad S^{1+m} = X + 1.$$

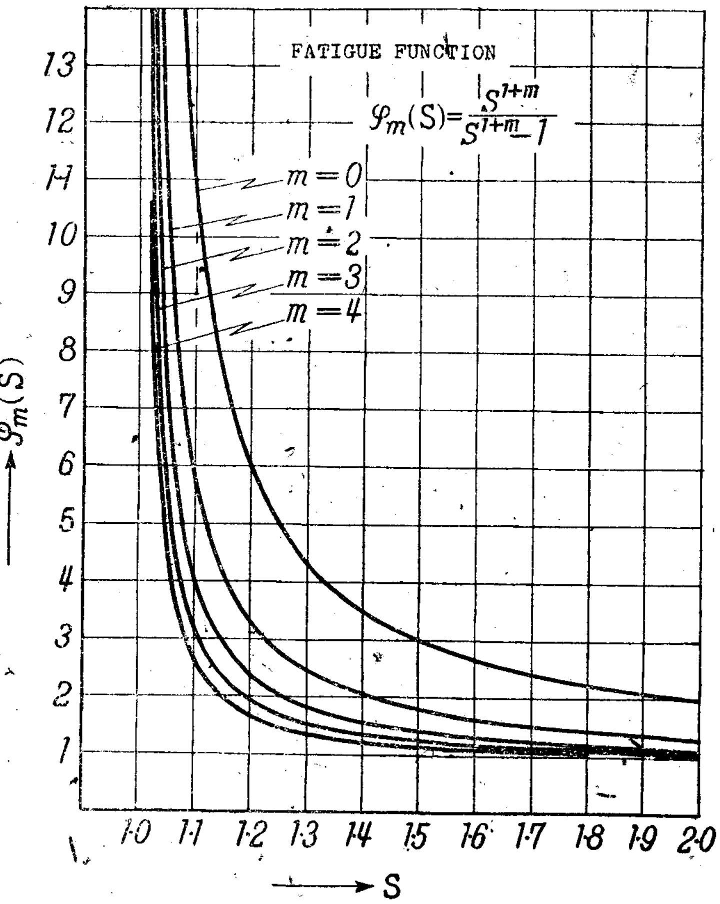


Fig. 1.

Now in the equation

$$H^{mpN} = \varphi_m(S)$$

take logarithms of both sides twice and we have

$$\log N = \log\log \varphi_m(S) - \log\log H - \log m - \log p. \qquad (4)$$

On the other hand

$$\log \sigma = \log S + \log \sigma_w \tag{5}$$

Hence we know from (4) and (5) that $\log N - \log \sigma$ relation is quite similar to $\log\log \varphi_m(S) - \log S$ relation. But the latter is calculable beforehand. For example, it is represented in fig. 2 for $m = 5 \sim 10$ and in fig. 3 for $m = 10 \sim 25$. Of course we can draw the curves as minutely as we want. Now we plot the experimental values of $\log N$ and $\log \sigma$ on a tracing paper using the same scale as in fig. 2 or in fig. 3. The relation between $\log N$ and $\log \sigma$ must coincide with one of the curves in fig. 2 or in fig. 3. Namely, we move the tracing paper up and down, right

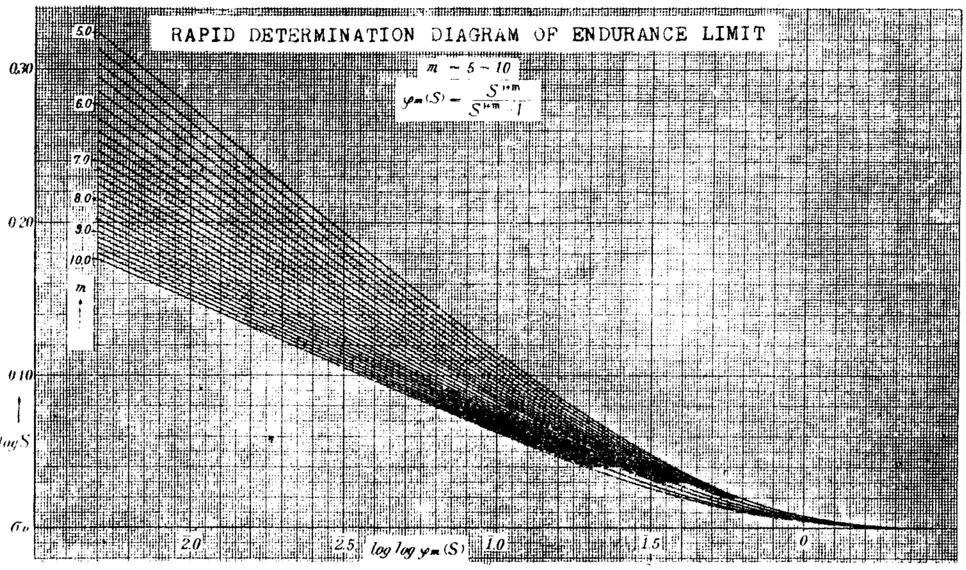


Fig. 2.

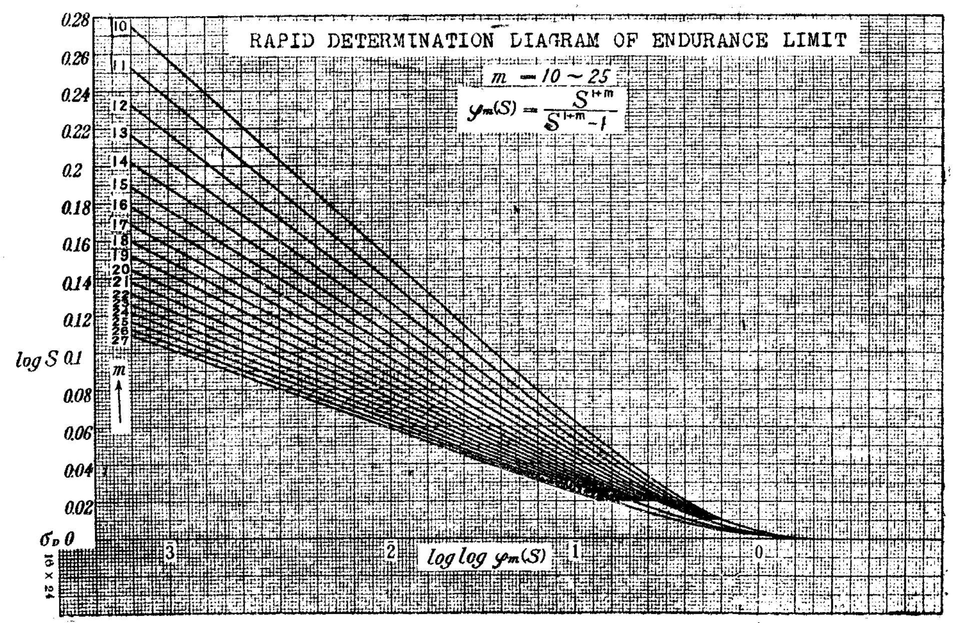


Fig. 3.

and left keeping both axes always parallel to those in fig. 2 (or in fig. 3) and find out a curve on which experimental data of $\log N$ and $\log \sigma$ lie as well as possible. When this is done, the point where the longitudinal axis on the tracing paper is cut by the transversal axis in fig. 2 or in fig. 3 is $\sigma_w$ itself corresponding to $\log S = 0$. Fig. 2 or fig. 3 may be called "rapid determination diagram of endurance limit". Also the value of $m$ can be read out instantly and $H$ is calculable from an arbitrary point on the determined curve reading the values of $\log N$, loglog $\varphi_m$(...) and $\log m$ (Cf. eq. 4).

**Example.** 0.22% carbon steel, repeated bending, mean stress $\sigma_m = 15$ kg/mm², true stress at fracture $\sigma_T = 91$ kg/mm² (experiment was carried out by Mr. Miyoshi in the Nishihara's laboratory).

**Table 1.**

| $\sigma$ kg/mm² | $N$, $\times 10^6$ | $\log \sigma$ | $\log N$ |
|---|---|---|---|
| 35.0 | 0.0464 | 1.5441 | 4.6665 |
| 24.0 | 3.62 | 1.3802 | 6.5578 |
| 22.6 | 12.2* | 1.3541 | 7.0864 |

* Not broken.

From the data in table 1, using the rapid determination diagram above stated we get $\sigma_w = 22.4$ kg/mm² and $m = 9.8$, also $\eta = 1.85 \times 10^{-8}$ from $\eta \cong \log_e H = 2.303 \log_{10} H$. The fatigue curve is represented in fig. 4. Only three sets of experimental values

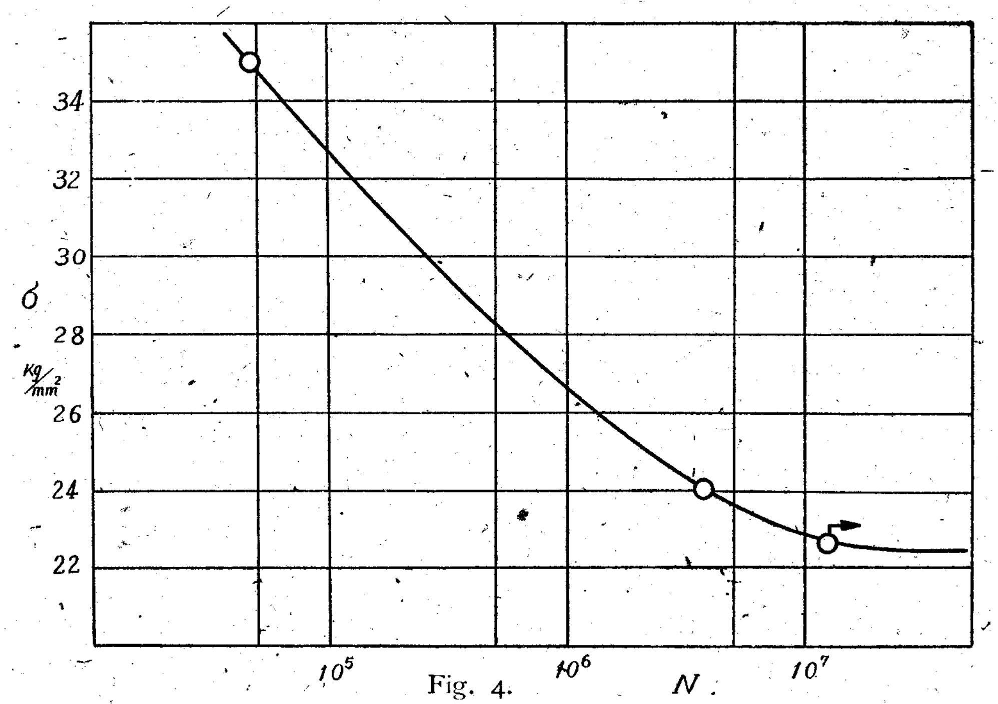


Fig. 4.

0.22%C steel, repeated bending. $m=9.8$, $\sigma_w=22.4$ kg/mm², $\eta=1.85\cdot 10^{-8}$, $\sigma_m=15$ kg/mm².

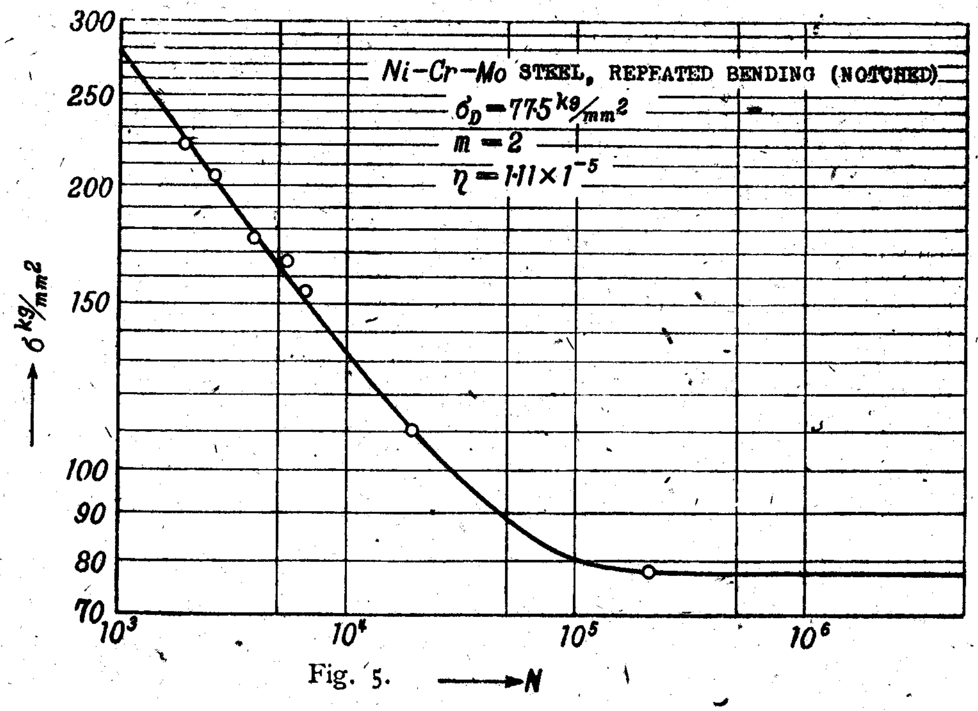


Fig. 5. ——→N

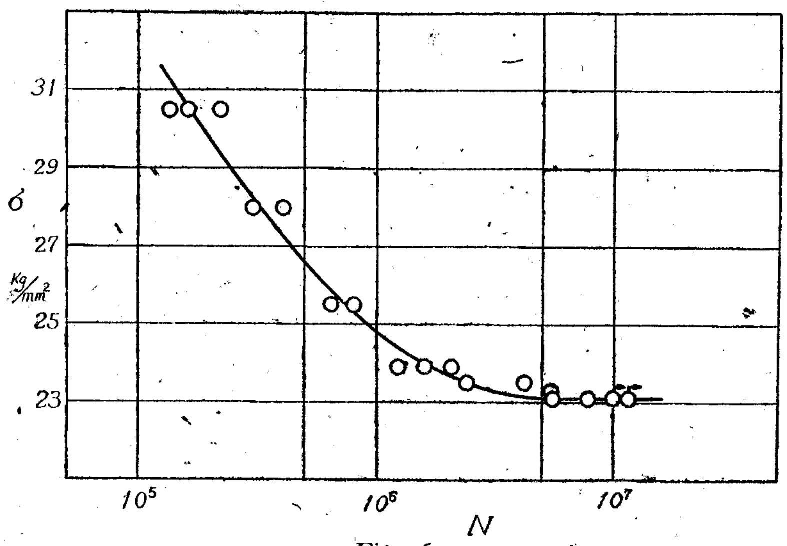


Fig. 6.
Carbon steel, rotating bending. $m=6.2$, $\sigma_w=23.1$ kg/mm², $\eta=1.45\cdot 10^{-7}$.

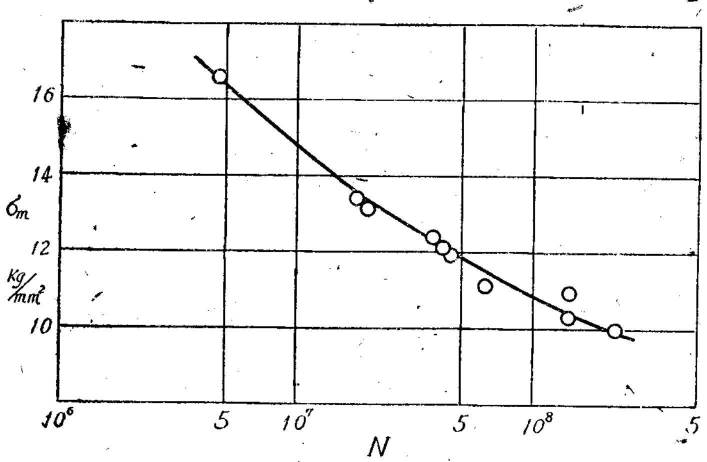


Fig. 7.
Duralumin, rotating bending. $m=5.8$, $\sigma_w=9.2$ kg/mm², $\eta=6\ 75\cdot 10^{-9}$.

sufficed to determine the reasonable value of endurance limit.

Also fig. 5 shows the fatigue curve for Ni-Cr steel (experiment was carried out by Mr. Miki in the Nishihara's laboratory). Similarly fig. 6 represents the case of 0.34% carbon steel[2] and fig. 7 that of duralumin[3].

## § 2 Life for the Case of Stepwise Change of Stress.

After a stress amplitude $\sigma_1$ was given to specimen to a certain number of repetitions stress was changed to another value, say $\sigma_2$. What is the life of the specimen? To answer this problem it is necessary to establish the degree of fatigue progress. As already stated in the previous report, the degree of fatigue at the weakest point in the specimen may be defined as the accmulated lattice distortion energy compared with the allowable limit $D$,

$$f=\sum \delta/D=S^{1+m}\left(1-\frac{1}{H^{mn}}\right) \tag{6}$$

where $n$ means the number of stress repetitions. $f$ must be smaller than or at most equal to unity which is ovbious from its definition. But there are cases where $f$ becomes greater than unity for given $n$, that is, for $n$ which lies between the number of stress repetitions $N$ till the specimen fractures and $pN$. There is no physical meaning for $f$ that is greater than unity but if we allow the value greater than unity for $f$, one to one correspondence between $f$ and $n$ will be preserved. On this convention we have from equation (6),

$$H^{mn}=\frac{S^{1+m}}{S^{1+m}-f} \tag{7}$$

and we can obtain $n$ till the degree of fatigue progress at the weakest point reaches $f$ when stressed at the relative stress S. Equation (2) on the other hand involves the number of stress repetitions $N$ till specimen fractures and is slightly different from equation (7). To obtain the same $N$ however, we can also use the equation (7) putting $f=f_{\max}$ which is now greater than unity. Of course $f_{\max}$ depends upon the applied stress $S$ and is easily proved to be

$$f_{\max}=S^{1+m}-S^{1+m}\left(1-\frac{1}{S^{1+m}}\right)^{\frac{1}{p}},\quad S>1 \tag{8}$$

from equations (2) and (6). When $S$ is smaller than unity we have from equation (6) putting $n\rightarrow\infty$

$$f_{\max}=S^{1+m},\quad S<1. \tag{8'}$$

In the equation (7) naturally $f$ must be smaller than $S$. And in general the number of repetitions $\varDelta n$ till the degree of fatigue progress reaches $f_2$ from $f_1$ can be calculated by the equation

$$H^{m\varDelta n}=\frac{S^{1+m}-f_1}{S^{1+m}-f_2}. \tag{9}$$

Now the relative stress $S_1$ was applied to a specimen $n_1$ times repeatedly then the degree of fatigue progress $f_1$ is from equation (6)

$$f_1=S_1^{1+m}\left(1-\frac{1}{H^{mn_1}}\right). \tag{6'}$$

During the repetitions of stress the material changes its mechanical properties and expresses different attitudes towards external load from before. Property of maiden material is expressed by the equation

$$\varepsilon_p=a\sigma^m \tag{10}$$

and work done in the first loading is

$$W_1=\frac{ma}{1+m}\sigma^{1+m}. \tag{11}$$

When stressed at $\sigma_1$, $n_1$ times repeatedly, the material hardens and its property towards $(n_1+1)$-th loading is expressed by the equation

$$\varepsilon_p'=\frac{a}{H^{mn_1}}\sigma^m \tag{12}$$

and work done in this $(n+1)$-th loading is

$$W_1'=\frac{ma}{1+m}\,\frac{\sigma^{1+m}}{H^{mn_1}} \tag{13}$$

Comparison of (11) with (13) shows clearly that the coefficient $a$ changes to $a/H^{mn_1}$ or the effect of stress changes from $\sigma$ to $\sigma/H^{\frac{mn_1}{1+m}}$. Consequently, after the stress was repeated $n_1$ times if it changed to $\sigma_2$ its effect will be equal to

$$\sigma_r=\sigma_2\cdot H^{-\frac{mn_1}{1+m}} \tag{14}$$

for maiden material. This may be called the "reduced stress", and

$$r=H^{-\frac{mn_1}{1+n}} \tag{15}$$

is called corresponding "coefficient of stress reduction". Accordingly the life $n_2$ after the stress was changed to $\sigma_2$ is obtained from the following equation (hence the residual degree of fatigue progress is $f_{\max}-f_1$ in the equation (7)),

$$H^{mn_2}=\frac{S_r^{1+m}}{S_r^{1+m}-(f_{\max}-f_1)} \tag{16}$$

Fig. 8 represents the change of coefficient of stress reduction in a certain case of 0.65% carbon steel. To obtain $n_2$ the value of $S_r^{1+m}$ or that of $r^{1+m}=H^{-mn_1}$ is necessary as shown in equation (16), therefore the value of $H^{-mn_1}$ is also represented in the same figure.

**Example 1.** Fatigue curve was determined from experimental data for 0.65% carbon steel

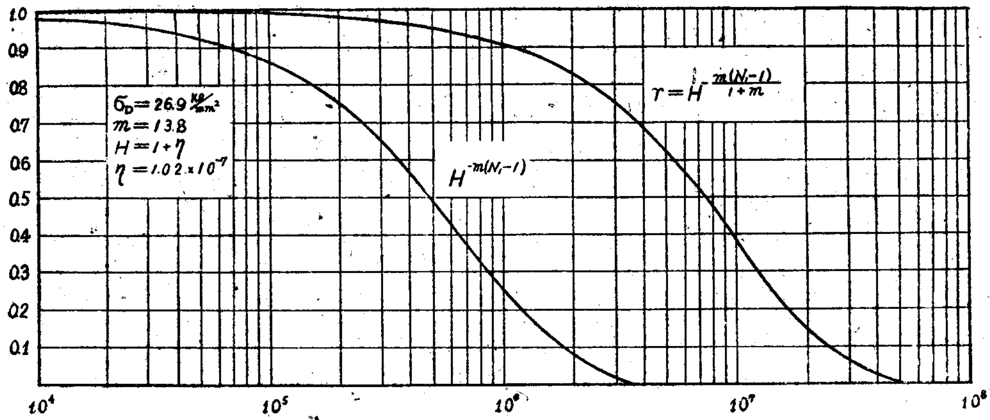


Fig. 8.
Coefficient of stress reduction (repeated tension and compression, 0.65% C steel).

in repeated tension and compression test. "Iso-fatigued line" for $F=0.9$, 0.8, 0.7,...... 0.1 were also drawn in fig. 9 ($p=1$ was assumed for simplicity). Now, $\sigma_1=26$ kg/mm$^2$ was repeated $N_1=1.32\times10^6$ times, degree of fatigue progress will be obtained to be $f=0.5$ by tracing the horizontal line corresponding $\sigma_1=26$ kg/mm$^2$. Let the stress be changed to $\sigma_2=34$ kg/mm$^2$. From

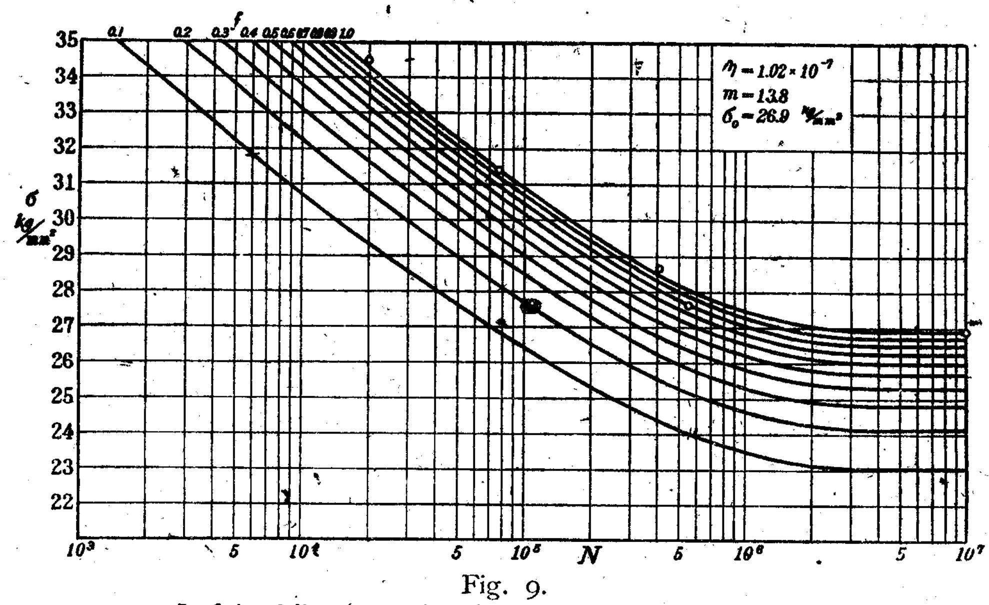


Fig. 9.
Iso-fatigued line. (repeated tension and compression, 0.65% C steel).

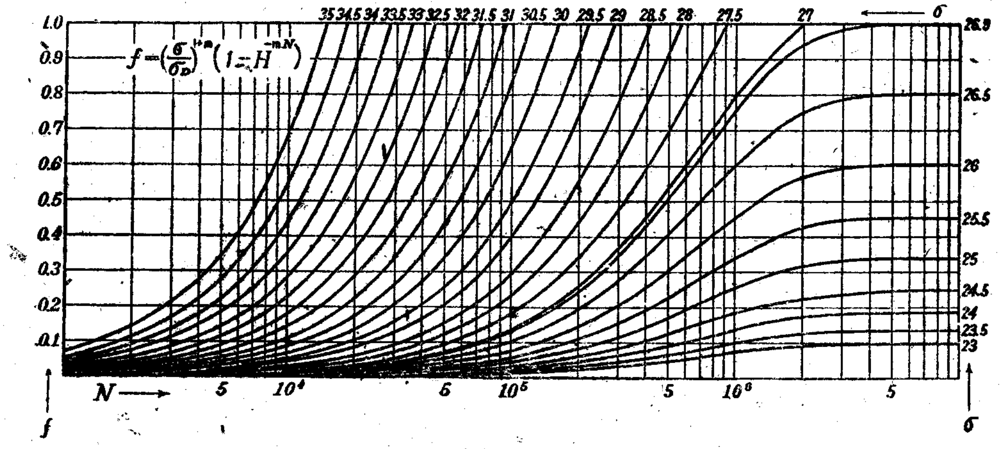


Fig. 10.
Degree of fatigue progress, 0.65% C steel.

fig. 8 we get for the coefficient of stress reduction $r=0.883$, and consequently, $\sigma_r=30.1$ kg/mm². In fig. 9 tracing the horizontal line of 30.1 kg/mm² the intercepting point with the curve $f=0.5$ yields $N_2=7.1\times10^4$. Life from the start is $N_1+N_2=139.1\times10^4$. If we neglect the change of properties of material, tracing the horizontal line of 34 kg/mm² will give $N=1.13\times10^4$ which is of course quite different from $N_2=7.1\times10^4$. Repetitions of stress $\sigma_1=34$ kg/mm² from start will cause fracture before $N_2=7.1\times10^4$. From this it is obvious that the order of application of stresses has extreme influence to the result.

In place of fig. 9 we can also use fig. 10 for the same purpose in which degree of fatigue progress is scaled along the longitudinal axis.

**Example 2.** Another interesting application of the theory is the problem of "damage line". According to Russel and Welker[4] material is said damaged if it fractures within the stress repetitians $10^7$ when stressed at the endurance limit $\sigma_w$ after it is exposed to the stress $\sigma_1$ and the stress repetitions $N_1$. The border line between the damaged and the undamaged region on the $(\sigma-N)$ plane is the "damage line".

Material in the service does not necessarily receive a stress wave of constant amplitude and the property of the material which endures short time overstressing is very desirable.

As already described, the authors' theory enables us to evaluate the degree of fatigue progress and to calculate the life of material from an arbitrary state of fatigue till another state of it and consequently the equation of the "damage line" can be derived in the following way.

When the relative stress $S_1$ was repeated $N_1$ times the degree of fatigue progress $f_1$ is given by the equation (6′). Stress is changed to $\sigma_w$, that is to $S=1$, and in order that the point $(S_1, N_1)$ lies on the damage line the specimen must fracture at exactly $N=10^7$ consuming the remaining degree of fatigue $(f_{max}-f_1)$. Hence from equation (16) we have

$$H^{m\cdot10^7}=\frac{S_r^{1+m}}{S_r^{1+m}-(f_{max}-f_1)} \qquad (16')$$

where $S_r=1\times r=H^{-\frac{mN_1}{1+m}}$. Substituting the value of $f_1$ from equation (6′) and that of $f_{max}$ from equation (8′) we get

$$S_1^{1+m}=\frac{1}{H^{m10^7}(H^{mN_1}-1)} \qquad (17)$$

This is the equation of damage line, and we can see that it is wholy determined by two constants $H$ and $m$, that is, by the knowledge of the fatigue curve.

In fig. 11 the experimental data according to Russel and Welker are compared with the

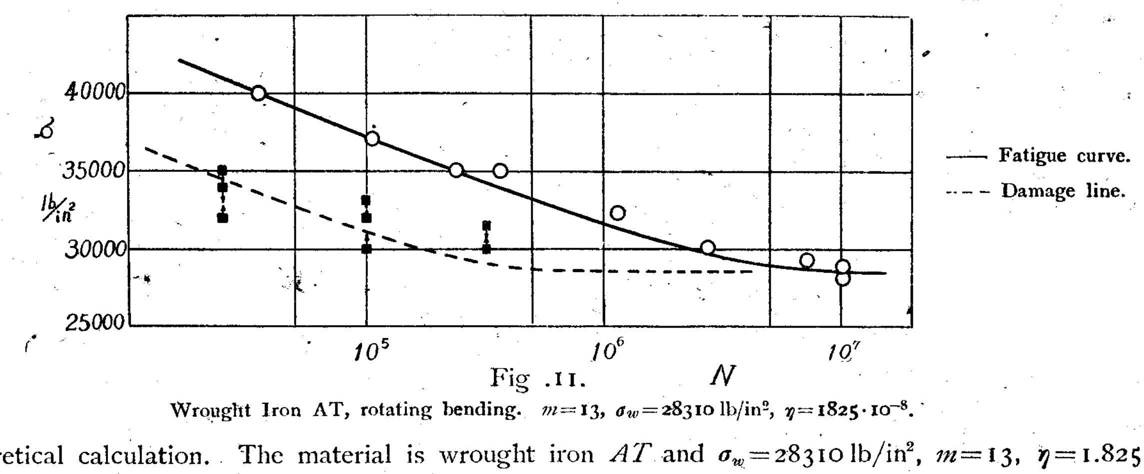


Fig. 11.

Wrought Iron AT, rotating bending. $m=13$, $\sigma_w=28310$ lb/in², $\eta=1825\cdot10^{-8}$.

theoretical calculation. The material is wrought iron $AT$ and $\sigma_w=28310$ lb/in², $m=13$, $\eta=1.825\times10^{-8}$. Agreement is seen to be tolerably good.

## § 3 Fatigue Strength for Simply Changing Stress Amplitude.

When a simple stress wave is repeated work done in the first loading is expressed by equation

(11). When each stress wave is simple but its maximum value changes, for instance, as in fig. 12a work done in the first complete cycle of loading is

$$W_1'=\frac{ma}{1+m}(\sigma_1^{1+m}+\sigma_2^{1+m}+\sigma_3^{1+m}+\ldots\ldots). \quad (18)$$

Lattice distortion has been assumed to be proportional to this work and hence if we calculate the reduced stress $\sigma_r$ by the equation

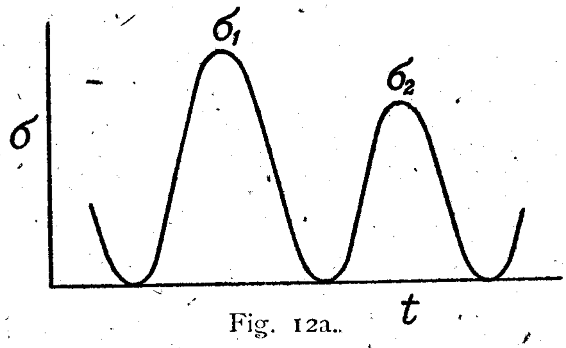

Fig. 12a.

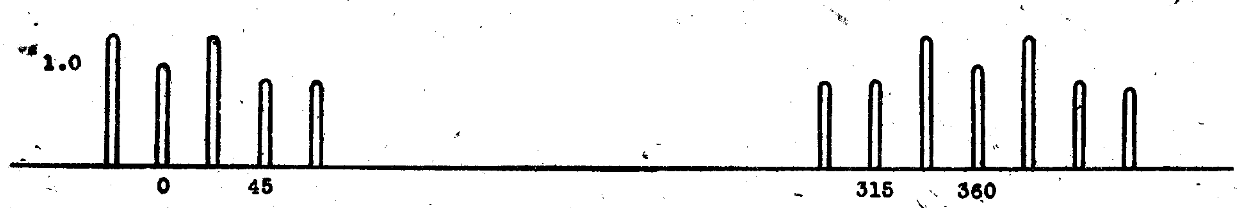

Fig. 12b.

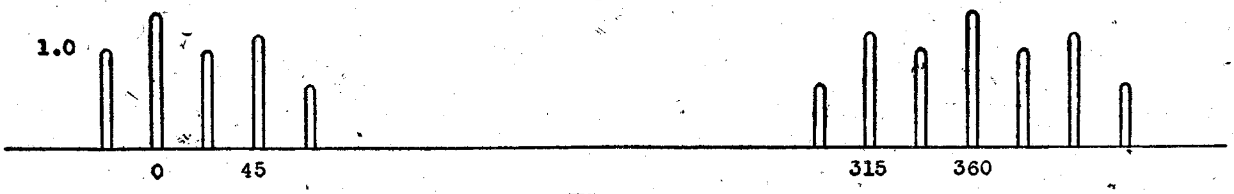

Fig. 12c.

$$\sigma_r^{1+m}=\sigma_1^{1+m}+\sigma_2^{1+m}+\sigma_3^{1+m}+\ldots\ldots\ldots\ldots \quad (19)$$

this $\sigma_r$ has the same effect for fatigue progress and accordingingly we can take it as the basis in calculating the life etc.

**Example.** A roller bearing manufactured by a certain company has dimensions as shown in fig. 13. Assuming that rollers and outer race continue pure rolling to each other contact of them are repeated at an interval of 44°31′ (approximately equal to 45°). One point of the roller (in the cross section), for instance, point $A$ which is in contact with the lowest point inside the outer race (fig. 13) repeats contacts three times with outer race and four times with inner race per one revolution of the bearing and is subjected to repeated compression. According to Hertz's formula this compressive stress $\sigma_m$ is given as

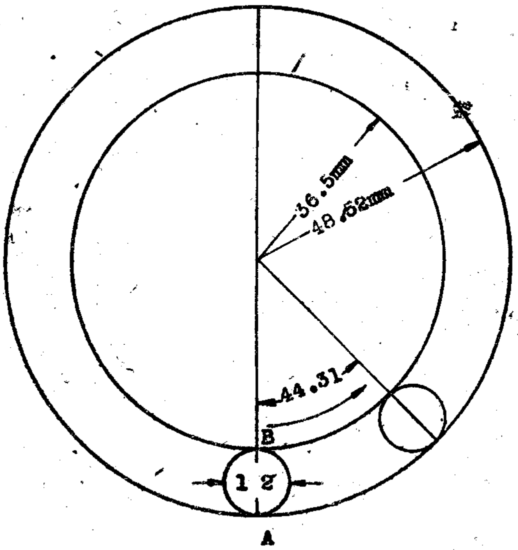

Fig. 13.

$$\sigma_m=0.581\sqrt{\frac{PE}{l}\left(\frac{1}{d_1}\pm\frac{1}{d_2}\right)} \quad (20)$$

where $\pm$ signs correspond to outer and inner contact respectively. Here $\left(\frac{1}{d_1}+\frac{1}{d_2}\right)\Big/\left(\frac{1}{d_1}-\frac{1}{d_2}\right)=1.765.$

Considering that the effective load $P$ varies proportionally to $\cos\theta$ corresponding to the position of roller the reduced compressive stress at a definite point $A$ of a roller is as follows

$${}_A\sigma_r^{1+m}=\left\{0.591\sqrt{\frac{PE}{l}\left(\frac{1}{d_1}-\frac{1}{d_2}\right)}\right\}^{1+m}\left\{2(1.765\cos 67.5°)^{\frac{1+m}{2}}+2(\cos 45°)^{\frac{1+m}{2}}+2(1.765\cos 22.5°)^{\frac{1+m}{2}}+(\cos 0°)^{\frac{1+m}{2}}\right\} \quad (21)$$

**Table 2.**

| | $m=9$ | $m=3$ |
|---|---|---|
| ${}_A\sigma_r$ | 1.38 | 1.69 |
| ${}_B\sigma_r$ | 1.45 | 1.84 |

Similarly the reduced stress ${}_B\sigma_r$ at a point $B$ (fig. 13) is

$$ {}_B\sigma_r^{1+m}=\left\{0.591\sqrt{\frac{PE}{l}\left(\frac{1}{d_1}-\frac{1}{d_2}\right)}\right\}^{1+m}\left\{2(\cos 67.5^\circ)^{\frac{1+m}{2}}+2(1.765\cos 45^\circ)^{\frac{1+m}{2}}+2(\cos 22.5^\circ)^{\frac{1+m}{2}}+(1.765\cos 0^\circ)^{\frac{1+m}{2}}\right\} \quad (21') $$

The value of $m$ for the material is yet unknown, but using $m=9$ and $m=3$ the values of ${}_A\sigma_r$ and ${}_B\sigma_r$ except the common term are summarized in table 2. Stresses at the point $A$ and $B$ are visualized in fig. 12b and 12c.

## § 4 Fatigue Strength for Complexly Changing Stress Amplitude.

**i) Correction for Unloading.** Hitherto we treated comparatively simple stress wave but in real cases obviously there are more complex stresses. It is impossible to experiment for almost infinitely large number of stress kinds and consequently we must carry out experiments for some fundamental types of stress wave and establish the method of calculation for real cases, that is why the fundamental theory of fatigue is necessary. For the purpose some points in authors' theory must be refined.

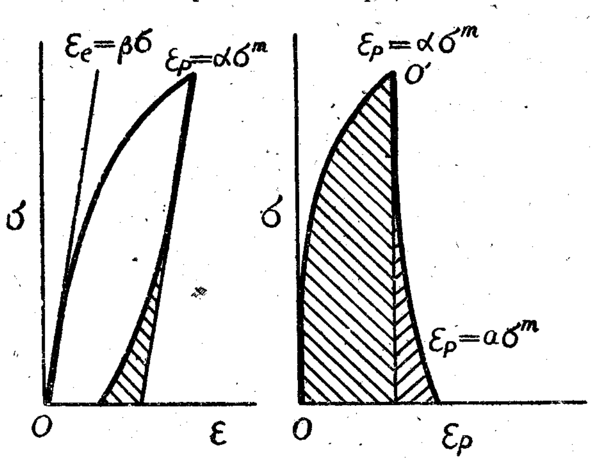


Fig. 14.

Up to this time we have considered that work must be done when loading but not when unloading, however, in real case as shown in fig. 14 the curve for unloading does not go downward parallel to the elastic line of loading and gets slightly inside, namely, some crystal grains are subjected to further strain when unloading (deformation after external load is removed is called elastic after-effect). Accompanying the work done during unloading lattice distortion occurs and naturally fatigue state of the material is influenced from the view point of the authors' theory. Work done during unloading is equal to the hatched part in the same figure. To visualize its area it was represented in fig. 14 right overturned to outside and elastic strain is omitted as it has no relation with the present problem. The curve for unloading may not coincide with that for loading, but it may be assumed to have the similar form such as $\varepsilon_p=a\sigma^m$. Of course the origin of coordinates is considered to transfer from O to O′ in the same figure. From this the excess work for unloading, that is, the area of outwards overturned part is calculated to be

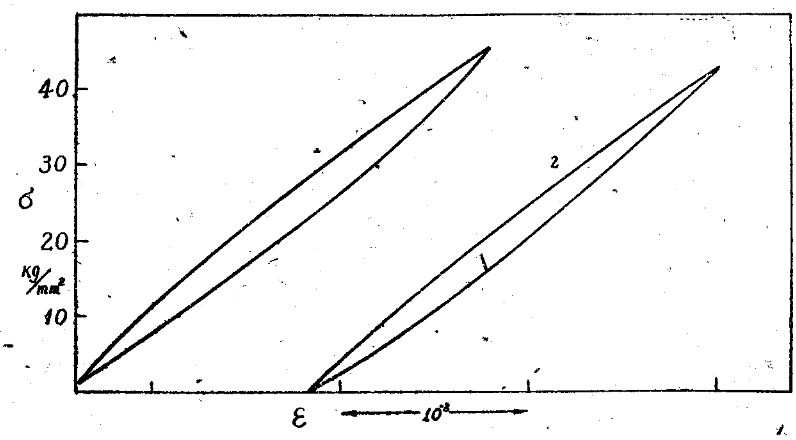


Fig. 15.
0.22% C steel, $\sigma_s=33.6$ kg/mm², $\sigma_B=51.0$ kg/mm².

$$ \Delta W'=\frac{a}{1+m}\sigma^{1+m} \quad (22), $$

Hence work done in the first cycle of stress is

$$ W_1=\frac{ma+a}{1+m}\sigma^{1+m} \quad (23) $$

This indicates that the coefficient $a$ is to be corrected by $a/m$. In a statical tension test of 0.22% steel which differs considerably from fatigue test in its speed of loading and in its range of applied stress, following results as shown in fig. 15 has been obtained. In fig. 16 $\log\sigma$ and $\log\varepsilon_p$ are plotted for loading (o) and for unloading (•), from this we can see that the equation $\varepsilon_p = a\sigma^m$ approximately holds for unloading and also in this case $a \cong \alpha$.

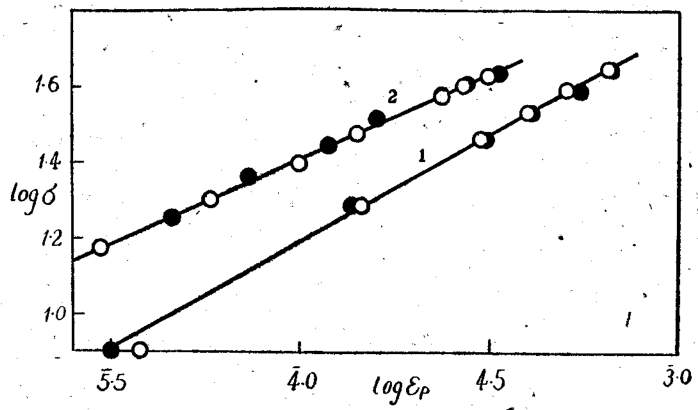


Fig. 16.
○ When the stress increases.
● When the stress decreases.

**ii) Treatment of a Bump in a Stress Wave.** Complex stress wave has, after all, many maxima and minima in one cycle and as the simplest case let us consider a wave where two maxima and one minimum appear, that is, so called a "bump" appears. Firstly, as shown in fig. 17 when a bump appears during loading consider corresponding stress-permanent strain diagram. By ascending to a maximum $\sigma_1$ and then descending to a minimum $\sigma_2$ work is done each time to the weakest part of the material and the geometrical area in the corresponding stress-strain diagram gives the quantity of work. Afterwards by ascending to a maximum $\sigma_3$ and descending to zero the larger part of work is afforded. The curve representing the relation between $\sigma$ and $\varepsilon_p$ in the second ascension is not the same as that in the first ascension. Due to the preceeding loading and unloading certain value of residual force arises and this force, since it is a compressive one after tensile loading, weakens the effect of loading in the second time. The residual stress was roughly assumed to be proportional to $\sigma_2$, that is, equal to $h\sigma_2$ (when $\sigma_2=0$, namely, when the external load was perfectly removed there also exists a residual stress which was neglected in the first approximation). Consequently the work done in the first stress cycle is as follows:

$$W_1 = \frac{ma}{1+m}\left\{\sigma_1^{1+m} + (\sigma_3 - h\sigma_2)^{1+m} - (\sigma_2 - h\sigma_2)^{1+m}\right\} + \frac{a}{1+m}\left\{(\sigma_1-\sigma_2)^m(\sigma_1 + m\sigma_2) + \sigma_3^{1+m}\right\}. \quad (24)$$

Arranging this in the similar form as before as well as possible we get

$$\sigma_r^{1+m} = \frac{ma}{ma+a}\left\{\sigma_1^{1+m} + (\sigma_3 - h\sigma_2)^{1+m} - (\sigma_2 - h\sigma_2)^{1+m}\right\} + \frac{a}{ma+a}\left\{(\sigma_1-\sigma_2)^m(\sigma_1 + m\sigma_2) + \sigma_3^{1+m}\right\}. \quad (25)$$

In other words, the given stress wave can be treated as a simple wave whose maximum value is equal to $\sigma_r$ thus calculated.

A bump in a stress wave during unloading is considered quite similarly as in figure 18 and the same form will be obtained.

As the equation (25) is not symmetrical with $\sigma_1$ and $\sigma_3$, the interchange of them will lead to different results which is seen in figures 17 and 18.

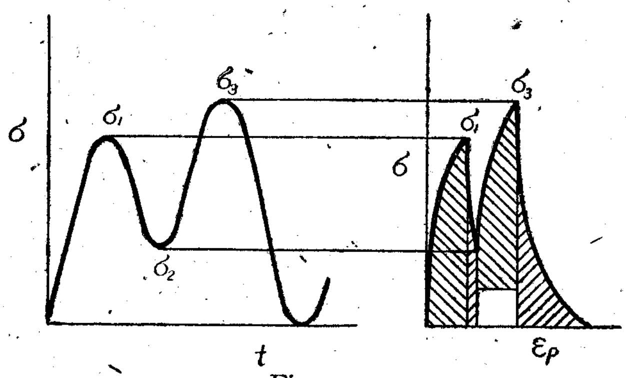


Fig. 17.

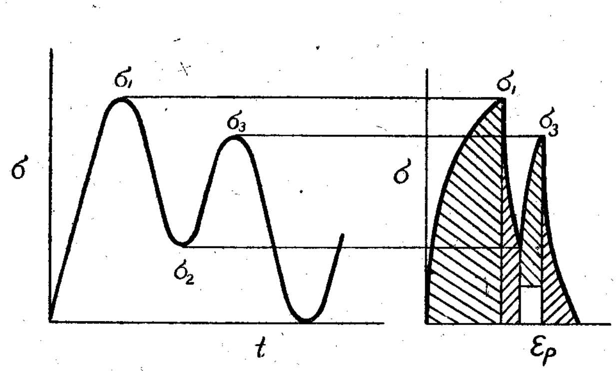


Fig. 18.

Furthermore we can suppose a case where a smaller bump appears in a bump which can be treated similarly as above.

**Example.** A stress wave as shown in fig. 19 was applied to specimens in repeated bending test[6]. Reduced stress is obtained from equation (25) by putting $\sigma_3=\sigma_1$. Now, denote $\sigma_3/\sigma_1=x$ in fig. 19 and we have

$$\sigma_r^{1+m}=\frac{mu}{mu+a}\sigma_1^{1+m}\Big\{1+(1-hx)^{1+m}-x^{1+m}(1-h)^{1+m}\Big\}+\frac{a}{mu+a}\sigma_1^{1+m}\Big\{(1-x)^m(1+mx)+x^{1+m}\Big\} \tag{26}$$

From the experimental results for simple stress wave fatigue curve and consequently $m$ and $\eta$

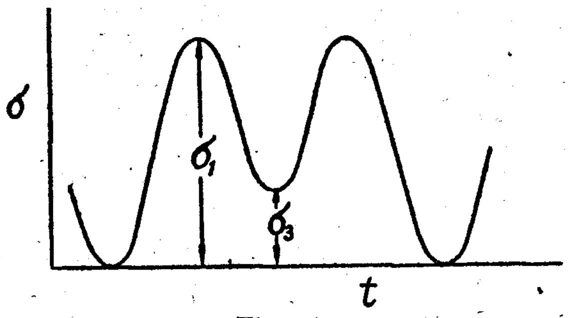


Fig. 19.

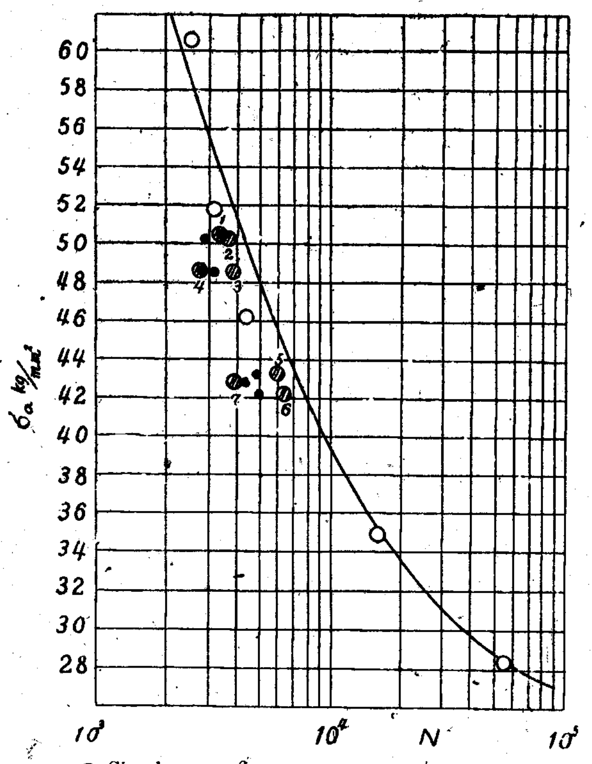


Fig. 20.

were determined (0.65% carbon steel). But the experimental data did not lie completely upon a curve as shown in fig. 20. Putting $m=2.2$ in (26) coefficient of stress reduction $r$ is obtained

$$r=\left[\frac{2.2u}{2.2u+a}\Big\{1+(1-hx)^{3.2}-x^{3.2}(1-h)^{3.2}\Big\}+\frac{a}{2.2u+a}\Big\{(1-x)^{2.2}(1+2.2x)+x^{3.2}\Big\}\right]^{\frac{1}{3.2}} \tag{27}$$

**Table 3.**

| Nr. | $x$ | $r$ | $\sigma_r$ |
|---|---|---|---|
| 1 | 0.558 | 1.016 | 51.40 |
| 2 | 0.385 | 1.053 | 53.03 |
| 3 | 0.314 | 1.084 | 52.77 |
| 4 | 0.230 | 1.112 | 54.16 |
| 5 | 0.325 | 1.081 | 46.82 |
| 6 | 0.263 | 1.104 | 46.58 |
| 7 | 0.212 | 1.118 | 47.85 |

The ratio $u/a$ is yet unknown but using $a=u$ the values of $r$ and $\sigma_r$ for experimented $x$ were summarized in table 3. Lifes are expected corresponding these reduced stresses.

Calculated values were compared with the experimental results in fig. 20. There are cases where both values coincide and also cases where both do not agree well. Principal cause for this fact is supposed due to the scattering of experimental data owing to non-uniformity of the material.

**iii) Case where Mean Stress $\sigma_m$ and Stress Amplitude $\sigma_a$ are Constant and $\sigma_m>\sigma_a$.** Stress wave and corresponding stress-permanent strain diagram are represented in fig. 21. Calculation of the hatched parts in the same figure gives the formula for reduced stress, namely, work done $W_1$ in the first loading is

$$W_1=\frac{mu}{1+m}(\sigma_1-h\sigma_2)^{1+m}-\frac{mu}{1+m}(\sigma_2-h\sigma_2)^{1+m}+\frac{a}{1+m}(\sigma_1-\sigma_2)^m(\sigma_1+m\sigma_2) \tag{28}$$

Comparing with (23) we have the reduced stress,

$$\sigma_r^{1+m}=\frac{mu}{mu+a}(\sigma_1-h\sigma_2)^{1+m}-\frac{mu}{mu+a}(\sigma_2-h\sigma_2)^{1+m}+\frac{a}{mu+a}(\sigma_1-\sigma_2)^m(\sigma_1+m\sigma_2) \tag{29}$$

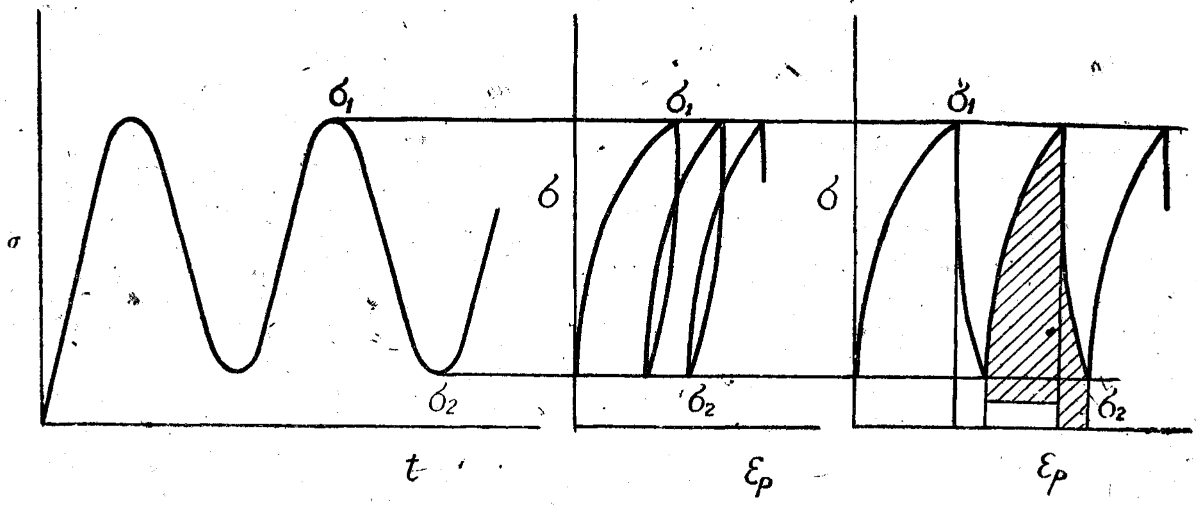


Fig. 21.

or introducing the mean stress $\sigma_m$ and the stress amplitude $\sigma_a$ by

$$\sigma_1=\sigma_m+\sigma_a, \qquad \sigma_2=\sigma_m-\sigma_a$$

we obtain

$$\sigma_r^{1+m}=\frac{ma}{ma+\mathrm{a}}\left\{(1-h)\sigma_m+(1+h)\sigma_a\right\}^{1+m}-\frac{ma}{ma+\mathrm{a}}(1-h)^{1+m}(\sigma_m-\sigma_a)^{1+m}$$
$$+\frac{\mathrm{a}}{ma+\mathrm{a}}(2\sigma_a)^m\left\{(1+m)\sigma_m-(m-1)\sigma_a\right\} \qquad (30)$$

According to the third term correction for unloading strengthens the influence of stress amplitude $\sigma_a$ and agreement with general empirical facts are reached. We often use a diagram representing $\sigma_a-\sigma_m$ relation taking $\sigma_a$ in longitudinal axis and $\sigma_m$ in transversal axis. On the straight line passing through origin and making the angle 45° with both axes $\sigma_a=\sigma_m$ and the inclination of the curve at the point of interception is given by

$$\left(\frac{d\sigma_a}{d\sigma_m}\right)_{\sigma_a=\sigma_m}=\frac{m(1-h)+\frac{\mathrm{a}}{a}}{m(1+h)+\frac{\mathrm{a}}{a}} \qquad (31)$$

**iv) Case where Mean Stress $\sigma_m$ and Stress Amplitude $\sigma_a$ are Constant and $\sigma_a>\sigma_m$.** Stress wave, corresponding stress-permanent strain diagram and the work done in this case are represented in fig. 22. Once ascending to the maximum stress $\sigma_1$ load is decreased along the curve

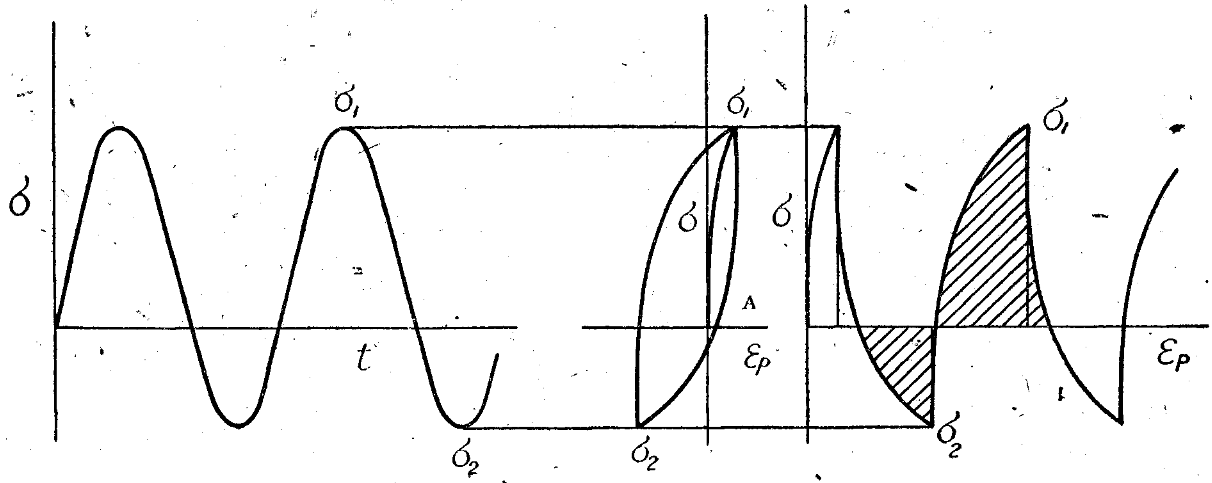


Fig. 22.

$\sigma_1 A$ reaching the value zero at the point $A$ (fig. 22 middle). After the external load changes its sign at $A'$ the $\sigma-\varepsilon_p$ curve extends smoothly along the part $A\sigma_2$ (cf. for instance, the Bauschinger-effect of a brass crystal [7]). Thus the sum of the hatched parts in the same figure, that is, the work done in the first stress cycle is given by

$$W_1=\frac{1}{1+m}(\sigma_1+\sigma_2)^m\left\{(m\alpha-a)\sigma_1+(ma-\alpha)\sigma_2\right\}+\frac{2a}{1+m}\sigma_1^{1+m}+\frac{2\alpha}{1+m}\sigma_2^{1+m}, \qquad (32)$$

and the reduced stress $\sigma_r$ is obtained as

$$\sigma_r^{1+m}=\frac{1}{m\alpha+a}(2\sigma_a)^m\left\{(\alpha+a)(m-1)\sigma_a+(m+1)(\alpha-a)\sigma_m\right\}\frac{2a}{m\alpha+a}(\sigma_m+\sigma_a)^{1+m}+\frac{2\alpha}{m\alpha+a}(\sigma_a-\sigma_m)^{1+m} \quad (33)$$

introducing the mean stress $\sigma_m$ and the stress amplitude $\sigma_a$ by

$$\sigma_1=\sigma_a+\sigma_m, \qquad \sigma_2=\sigma_a-\sigma_m$$

In repeated tension where $\sigma_a=\sigma_m$ we have

$$\sigma_r=2\sigma_a \qquad (34)$$

which agrees with the result obtained from the equation (30). In repeated tension and compression where $\sigma_m=0$

$$\sigma_r^{1+m}=\frac{(\alpha+a)(m-1)}{2(m\alpha+a)}(2\sigma_a)^{1+m}+\frac{2(\alpha+a)}{m\alpha+a}\sigma_a^{1+m} \qquad (35)$$

Using the equations (33) and (30) we can calculate the reduced stress $\sigma_r$ which has the same

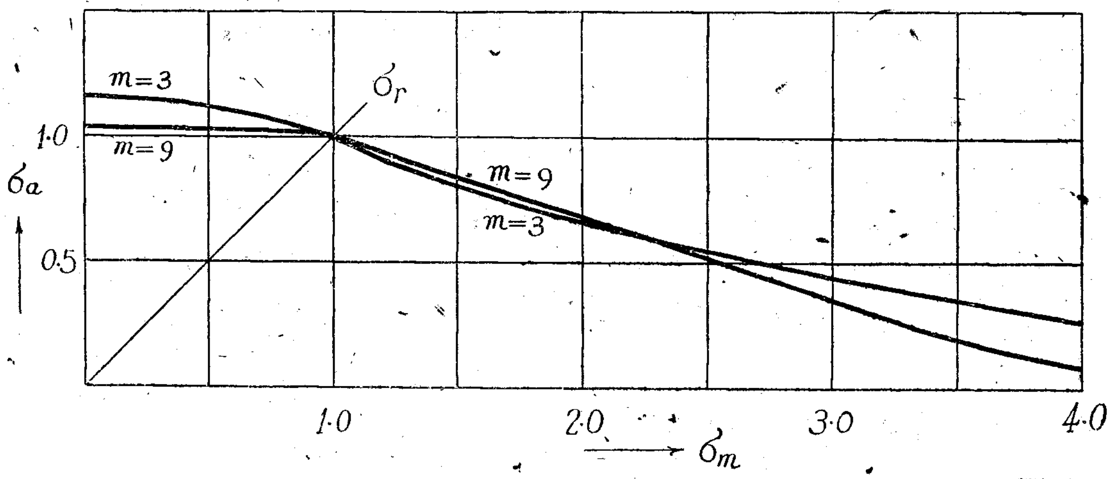


Fig. 23.

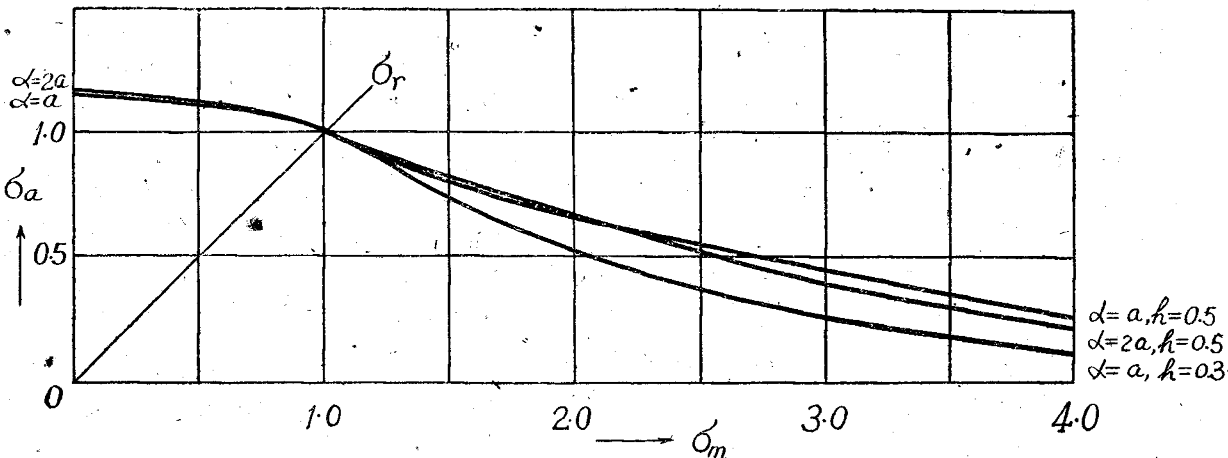


Fig. 24.

effect as the given vibrating stress appointed by $\sigma_a$ and $\sigma_m$. This depends obviously upon the values of $m$, $h$, and $a/\alpha$. Assuming $a=\alpha$ equations (33) and (30) were evaluated for $m=3$ and $m=9$ and are represented in fig. 23. As shown clearly in the same figure inclinations of the two curves differ appreciably according to the value of $m$. In the figure 24 also, the relation between $\sigma_a$ and $\sigma_m$ is represented when $h$ or $a/\alpha$ takes other values. Utilizing these results we can foresee the life for arbitrary set of $\sigma_a$ and $\sigma_m$ when the life for pure repeated tention (for instance) is known. Of course these formula are not the ultimate conclusions and there remain for them strict proofs and refinements and once the perfect theory is attained it will afford considerable conveniences for engineering problem.

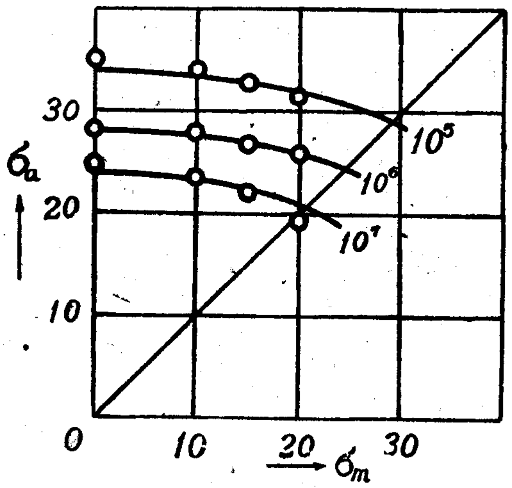

Fig. 25.
Carbon steel, repeated bending.

**Examples.** Repeated bending test was carried out about 0.22% carbon steel applying various values of mean stress and stress amplitude (experimented by Mr. Miyoshi in the Nishihara's laboratory). First fixing mean stress a fatigue curve was obtained from which the stress values corresponding to the fracture at $N=10^5$, $10^6$, $10^7$, were read out. These semi-experimental values are compared with calculated results in fig. 25. In this case $m=3$, $\alpha=a$. Experimental values for notched specimen of the same material were also compared with theoretical ones in fig. 26 for which $m=3$, $\alpha=a$ and $h=0.88$ were taken. In each case the agreement is seen to be tolerably good.

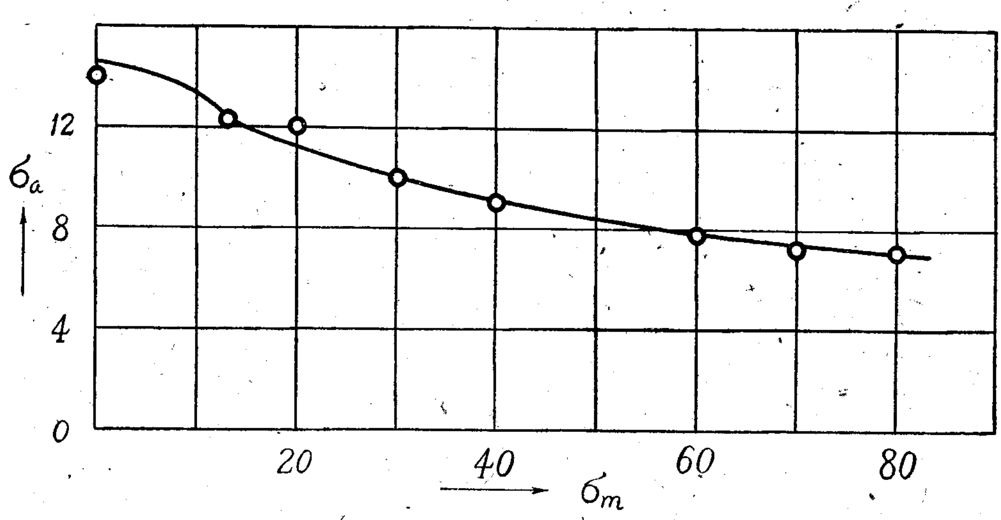

Fig. 26.
Carbon steel, notched, repeated bending.

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## References:

1) T. Nishihara, A. Kobayashi, Mem. Coll. Engg. Kyoto Imp. Univ., XI, 5 (1945), 113.
2) T. Nishihara, M. Kawamoto, Trans. Mech. Soc. Japan, 6 (1940), 1-47.
3) T. Nishihara, T. Kori, Trans. Mech. Soc. Japan, 5 (1939), 1-89.
4) H. W. Russel, W. A. Welcker, A. S. T. M., 36 II (1936), 118.
5) T. Nishihara, Yano Takeda, not yet published.
6) T. Nishihara, T. Yamada, Jour. Mech. Soc. Japan, in press.
7) G. Sachs, H. Shoji, Z. f. physik, 45 (1927), 776.