A New Analytical Method of Steady and Transient Phenomena in Polyphase Transmission Lines

By

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Introduction

Surge propagation phenomena along a single phase transmission line have been studied theoretically as well as experimentally by various authorities for a long time. The present author has also devoted himself to the same investigation for more than 20 years. On the other hand, the studies on travelling waves along polyphase transmission lines are comparatively few, and the analysis for such phenomena is so difficult and complicated that you may fail to give rigorous interpretations for polyphase phenomena by the direct application of the theoretical results already obtained for a single phase line. The actual overhead transmission lines, however, generally consist of 3, 6, 7 or 8 conductors according as they are composed of a single circuit or double circuits, with one ground wire or two. But usually, field engineers seem to discuss polyphase problems by the theoretical results already obtained for travelling wave phenomena in a single phase line. It is verified experimentally that such reasoning is erroneous. In conclusion, the surge propagation phenomena in polyphase lines should be discussed by means of rigorous analytical formulas established for the very polyphase transmission lines themselves, but never conventionally by means of those for a single phase line. The same thing may be said as to the principle of the measurement.

Of course, the polyphase transmission line problems have been discussed theoretically by several authors such as K. W. Wagner, L. A. Pipes, S. Bekku and the present author. But such investigations either contain some unfavorable assumptions or can only discuss the problems in some particular cases, and the courses of calculation become very complicated and tedious.

The present report deals with such polyphase problems by a newly established analytical method quite different from those already published, by means of which we can discuss the phenomena more simply and systematically.

By the application of the so-called Sylvester's theorem in the theory of matrices as well as the new type of operational calculus which was already extended by the present author, we can deduce a new analytical method, by means of which travelling wave phenomena along polyphase lines with concentrated impedances or unifom lines at their terminals can be analysed rigorously and systematically.

In the present report, theorems concerning the new analysis are interpreted with some simple examples, by means of which the readers may aquire the actual knowledge of application.

1. Sylvester's Theorem for Fractional Power Matrix of a given Matrix

We know the so-called Sylvester's theorem in the theory of matrices, by means of which we can reduce any rational function of square matrix $[A]$ to a calculable form by finite times multiplication and addition of square matrices whose degrees are the same with those of $[A]$. By means of this theorem, the present author has already derived a new analytical method for periodically interrupted electric circuits. Moreover, the calculation of circuit constants of cascadely connected recurrent four terminal electric network, numerical solution of rational algebraic integral equation of high degree, and the determination of natural frequencies of oscillating system can also be easily carried out thereby.

But in the above mentioned cases, the functions of $[A]$ under consideration are assumed rational functional matrices, and the verification has been established for such cases.

Now, let us consider another case, where $F([A])$ is a functional matrix of fractional power of $[A]$. Fortunately, such a case has also been discussed by Frazer, Duncan & Collar, and a conclusion has also been got that Sylvester's theorem still holds for fractional powers of a matrix such as $[A]^{\frac{1}{n}}$. We can easily extend this conclusion to the general case, where $F([A])$ represents a functional matrix of fractional powers of $[A]$.

Nevertheless, so far as I know, there is no technical or physical application of Sylvester's theorem in the latter case. The present author has recently found that transient as well as steady currents and voltages of polyphase electric transmission lines can be analysed pure-operationally by Sylvester's theorem for fractional powers of a matrix. Accordingly we can possibly establish a New Operational Calculus for Boundary Problems, which likely corresponds to Heaviside Operational Calculus for Initial Value Problems.

In the first article, a brief interpretation of ordinary Sylvester's theorem and its extended form for the case where $F([A])$ denotes a mere fractional power of $[A]$ such as $[A]^{\frac{1}{n}}$, is given; and a rigorous verification for the general case is left to the readers untouched.

Again, we shall explain the so-called Sylvester's theorem in the ordinary form. Let $a_1, a_2, \ldots a_m$ be distinct characteristic roots of finite square matrix $[A]$; then, by Sylvester's theorem, any rational functional matrix $F([A])$ of a given square matrix $[A]$ is given by

$$\left.\begin{aligned} &F([A])=\sum_{r=1}^{m} F(a_r)[K(a_r)], \\ &\text{where}\quad [K(a_r)]=\frac{\prod\limits_{\substack{s=1\ldots m \\ s\neq r}}(a_s[U]-[A])}{\prod\limits_{\substack{s=1\ldots m \\ s\neq r}}(a_s-a_r)}, \end{aligned}\right\} \tag{1.1}$$

and $[U]$ denotes a unit matrix. At the same time, we must notice the following relations.

$$[K(a_r)][K(a_p)]=[0],\quad r\neq p; \tag{1.2}$$

$$[K(a_r)]^n=[K(a_r)], \tag{1.3}$$

$$\sum_{r=1}^{n}[K(a_r)]=[U], \tag{1.4}$$

where n is a positive integer, and m the number of degrees of $[A]$.

By a slight modification, we can extend the above theorem to the case, where $[A]$ has multiple characteristic roots. Let a_1 be an s-ple characteristic root of $[A]$ of m-th degree, and suppose that the other characteristic roots are distinct; then, any rational functional matrix of $[A]$ can be expressed by the following:

$$\begin{aligned} F([A])=&\frac{F(a_1)}{(a_{s+1}-a_1)(a_{s+2}-a_1)\ldots(a_n-a_1)} \\ &\times(a_{s+1}[U]-[A])\ldots(a_n[U]-[A]) \\ &+\frac{(-)}{1!}\left\{\frac{d}{d a_1}\frac{F(a_1)}{(a_{s+1}-a_1)(a_{s+2}-a_1)\ldots(a_n-a_1)}\right\}\times \\ &\quad(a_1[U]-[A])(a_{s+1}[U]-[A])\ldots(a_n[U]-[A]) \\ &+\ldots\ldots\ldots \\ &+\frac{(-)^{s-1}}{(s-1)!}\left\{\frac{d^{s-1}}{d a_1^{s-1}}\frac{F(a_1)}{(a_{s+1}-a_1)(a_{s+2}-a_1)\ldots(a_n-a_1)}\right\} \\ &\times(a_1[U]-[A])^{s-1}(a_{s+1}[U]-[A])\ldots(a_n[U]-[A]) \end{aligned}$$

$$+\sum_{i=s+1}^{m}\frac{F(\alpha_i)}{(\alpha_1-\alpha_i)^s\prod\limits_{\substack{k=s+1,\dots,m\\ k\neq i}}(\alpha_k-\alpha_i)}$$

$$\times(\alpha_1[U]-([A])^s\prod_{\substack{k=s+1,\dots,m\\ k\neq i}}(\alpha_k[U]-[A]). \qquad (1.5)$$

This formula is valid for the case, where $F([A])$ is a rational functional matrix, as already described. But we have not verified yet, whether this theorem holds in the case, where $F([A])$ is an irrational functional matrix of $[A]$. Again we shall prove this theorem in the latter case.

For simplicity, suppose that $[A]$ is a square matrix of the second degree, and let it be required to find the value of $[A]^{\frac{1}{2}}$. Provided that $[A]$ has distinct characteristic roots α_1 & α_2, and if Sylvester's theorem should still hold in this case, the following relation must hold.

$$\{\alpha_1^{\frac{1}{2}}[K(\alpha_1)]+\alpha_2^{\frac{1}{2}}[K(\alpha_2)]\}^2=[A]. \qquad (1.6)$$

Again, we shall discuss whether this relation should hold or not. Now, the left-hand side of the above formula will be calculated by direct multiplication as follows.

$$\{\alpha_1^{\frac{1}{2}}[K(\alpha_1)]+\alpha_2^{\frac{1}{2}}[(K(\alpha_2)]\}^2=\alpha_1[K(\alpha_1)]^2+\alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}}\{[K(\alpha_1)][K(\alpha_2)]$$
$$+[K(\alpha_2)[K(\alpha_1)]\}+\alpha_2[K(\alpha_2)]^2. \qquad (1.7)$$

Since we have $[K(\alpha_1)][K(\alpha_1)]=[0]$

by (1.2), and $[K(\alpha_1)]^2=[K(\alpha_1)]$ and $[K(\alpha_2)]^2=[K(\alpha_2)]$

by (1.3), the above relation may be reduced to

$$\{\alpha_1^{\frac{1}{2}}[K(\alpha_1)]+\alpha_2^{\frac{1}{2}}[K(\alpha_2)\}^2=\alpha_1[K(\alpha_1)]+\alpha_2[K(\alpha_2)]. \qquad (1.8)$$

The right-hand side of the above formula is nothing but $[A]$ itself. Thus we can verify the relation (1.6).

Evidently by the foregoing method, four possible square roots of $[A]$ can be obtained, namely,

$$[A]^{\frac{1}{2}}=\pm\alpha_1^{\frac{1}{2}}[K(\alpha_1)]\pm\alpha_2^{\frac{1}{2}}[K(\alpha_2)], \qquad (1.9)$$

the signs being here associated in all possible combinations. More generally, if $[A]$ is a square matrix of order m with distinct characteristic roots, then

$$[A]^{\frac{1}{n}}=\sum_{r=1}^{m}\alpha_r^{\frac{1}{2}}\,[K(\alpha_r)]. \qquad (1.10)$$

Now since $\alpha_r^{\frac{1}{n}}$ has n distinct values and there are m characteristic roots, it will be possible to construct n^m such roots. The foregoing arguement, of course, requires some modification by the aid of (1.5), if some of the characteristic roots are repeated.

2. Boundary problems of General Polyphase Transmission Lines

Let the voltages and currents of m-phase transmission lines belong to the elements of the column matrices $[i]$ & $[v]$, and then the differential equations in this case are expressed by the following equations:

$$\left.\begin{aligned} -\frac{\partial[v]}{\partial x} &= \left\{[L]\frac{\partial}{\partial t}+[R]\right\}[i], \\ -\frac{\partial[i]}{\partial x} &= \left\{[C]\frac{\partial}{\partial t}+[G]\right\}[v], \end{aligned}\right\} \tag{2.1}$$

where $[L]$, $[R]$, $[C]$ and $[G]$ denote m-th degree matrices whose elements are composed of inductances, resistances, capacitances and leakances of the transmission lines per unit length respectively. The matrices in the above equations are expressed as follows:

$$\left.\begin{aligned} &[i]=\begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_m \end{pmatrix}, \quad [v]=\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}, \\ &[L]=\begin{pmatrix} L_{11}\ L_{12}\cdots\cdots L_{1m} \\ L_{21}\ L_{22}\cdots\cdots L_{2m} \\ \cdots\cdots\cdots\cdots \\ L_{m1}\ L_{m2}\cdots\cdots L_{mm} \end{pmatrix} \text{ etc..} \end{aligned}\right\} \tag{2.2}$$

Again to solve equations (2.1), we introduce the folllowing Laplace transformations:

$$\left.\begin{aligned} [i] &= \mathfrak{H}[I], \\ [v] &= \mathfrak{H}[V], \end{aligned}\right\} \tag{2.3}$$

where the symbol $\mathfrak{H}$ denotes that

$$\mathfrak{H}[I]=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{[I]}{p}\varepsilon^{pt}dp, \qquad c>0, \tag{2.4}$$

and, if we assume that the initial values of the first kind for currents and voltages are all zero throughout the whole lines, equations (2.1) may be reduced to

$$\left.\begin{aligned} -\frac{d[V]}{dx} &= \left\{[L]p+[R]\right\}[I], \\ -\frac{d[I]}{dx} &= \left\{[C]p+[G]\right\}[V]. \end{aligned}\right\} \tag{2.5}$$

Or for simplicity, if we put

$$\left.\begin{aligned}[Z(p)]&=[L]p+[R],\\ [Y(p)]&=[C]p+[G],\end{aligned}\right\} \tag{2.6}$$

then equations (2.5) are replaced by

$$\left.\begin{aligned}-\frac{d[V]}{dx}&=[Z(p)][I],\\ -\frac{d[I]}{dx}&=[Y(p)][V].\end{aligned}\right\} \tag{2.7}$$

Again let us consider the simplest case, where it is required to solve steady phenomena when the complex alternating electromotive forces proportional to ε^{jwt} are impressed to the lines, and then we get the complex vectors $[V]$ and $[I]$ corresponding to the steady values of $[v]$ and $[i]$ by putting $p=jw$ in $[V]$ and $[I]$ in the above equations. But when the transient phenomena are to be discussed, we must reserve p in the above equations as it is, in order to obtain operational functions. Hence for both cases, it may be enough if we discuss equation (2.7) only.

Now, to solve equations (2.7), eliminate $[I]$ from both equations, and we have

$$\frac{d^2[V]}{dx^2}=[Z(p)][Y(p)][V]. \tag{2.8}$$

Compare this equation with that for a single phase line, which is expressed as follows

$$\frac{d^2V}{dx^2}=Z(p)Y(p)V, \tag{2.9}$$

where $Z(p)=Lp+R$ and $Y(p)=Cp+G$.
The solution of (2.9) is given by the well-known formula, viz.,

$$V=A\cosh\sqrt{Z(p)Y(p)}\ x+B\sinh\sqrt{Z(p)Y(p)}\ x. \tag{2.10}$$

In equations (2.9) and (2.10), V, $Z(p)$ and $Y(p)$ never mean matrices of course, and A and B represent integration constants which are to be determined by the terminal or boundary conditions of the lines.

Now, we can prove by direct substitution that the solution of (2.8) has a form similar to that given by (2.10) and is expressed by

$$[V]=\cosh\sqrt{[Z(p)][Y(p)]}\ x\cdot[A]+\sinh\sqrt{[Z(p)][Y(p)]}\ x\cdot[B], \tag{2.11}$$

where $[A]$ and $[B]$ are column matrices whose elements consist of integration constants and are given by the following:

$$[A]=\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}, \quad [B]=\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}, \tag{2.12}$$

or for simplicity, putting

$$[Q]=\sqrt{[Z(p)][Y(p)]}\ , \tag{2.13}$$

we can reduce equation (2.11) to the form

$$[V]=\cosh[Q]x\cdot[A]+\sinh[Q]x\cdot[B]. \tag{2.14}$$

To obtain $[I]$, substituting (2.14) for $[V]$ in the first equation of (2.7), we have

$$[I]=[Z(p)]^{-1}[Q]\{\cosh[Q]x\cdot[A]+\sinh[Q]x\cdot[B]\}. \tag{2.15}$$

Equations (2.14) and (2.15) possess $2m$ integration constants that are given as the elements of $[A]$ and $[B]$, which can be determined when the currents and voltages are given at the line terminals $x=0$ and $x=l$. For instance, let us consider the case where the voltages and currents at $x=0$ are given. Let

$$\begin{aligned} [V]&=[V_0]\,, \\ [I]&=[I_0] \quad \text{at} \quad x=0\,, \end{aligned} \tag{2.16}$$

then, comparing the first equation of (2.16) with that obtained by putting $x=0$ in (2.14), we have

$$[A]=[V_0]. \tag{2.17}$$

In like manner, from (2.15) and the second of (2.16), we have

$$-[Z(p)]^{-1}[Q][B]=[I_0]\,,$$

or

$$[B]=-[Q]^{-1}[Z(p)][I_0]. \tag{2.18}$$

Substitute (2.17) and (2.18) for $[A]$ and $[B]$ in (2.14) and (2.15), we have finally

$$\left.\begin{aligned} [V]&=\cosh[Q]x\cdot[V_0]-\sinh[Q]x\cdot[Q]^{-1}[Z(p)][I_0]\,, \\ [I]&=[Z(p)]^{-1}[Q]\{\cosh[Q]x\cdot[Q]^{-1}[Z(p)][I_0]-\sinh[Q]x\cdot[V_0]\}. \end{aligned}\right\} \tag{2.19}$$

These are the required solutions ; but as they stand, they contribute to nothing, since they contain hyperbolic-functional matrices of fractional power matrix $[Q]=\sqrt{[Z(p)][Y(p)]}$. Hence it follows that we have merely succeeded in the deduction of formal solutions of (2.7). We must transform these formulas to other different forms whose calculation is practically possible. This question is answered by the application of Sylvester's theorem already explained in the preceding article. It is enough for us to apply this theorem only to the terms containing $[Q]$ in equations (2.19). By such procedure, we have finally

$$\left.\begin{aligned}
[V]&=\sum_{r=1}^{m}[K(q_r^2)]\left\{\cosh q_r x\cdot[V_0]-\sinh q_r x\left[\frac{Z(p)}{q_r}\right][I_0]\right\},\\
[I]&=\sum_{r=1}^{m}[Z(p)]^{-1}q_r[K(q_r^2)]\left\{\cosh q_r x\frac{1}{q_r}[Z(p)][I_0]-\sinh q_r x\right.\\
&\quad\left.\times[V_0]\right\}\\
&=\sum_{r=1}^{m}\left[\frac{Z(p)}{q_r}\right]^{-1}[K(q_r^2)]\left\{\cosh q_r x\cdot\left[\frac{Z(p)}{q_r}\right][I_0]-\sinh q_r x\right.\\
&\quad\left.\times[V_0]\right\},
\end{aligned}\right\}\qquad(2.20)$$

where q_r^2's $(r=1, 2, \ldots\ldots m)$ are characteristic roots of

$$[Q]^2=[Z(p)][Y(p)];$$

or in other words, q_r^2's satisfy the value of q^2 in the following equation:

$$\delta\{q^2[U]-[Q]^2\}=\delta\{q^2[U]-[Z(p)][Y(p)]\}=0, \qquad (2.21)$$

where $[U]$ denotes a unit matrix of m-th degree. And provided that q_1, q_2, ... , q_m are all different, $[K(q_r^2)]$ as interpreted in the preceding Article, is given by

$$[K(q_r^2)]=\frac{\prod\limits_{s\neq r}^{s=1\ldots m}(q_s^2[U]-[Q]^2)}{\prod\limits_{s\neq r}^{s=1\ldots m}(q_s^2-q_r^2)}$$

$$=\frac{\prod\limits_{s\neq r}^{s=1\ldots m}(q_s^2[U]-[Z(p)][Y(p)])}{\prod\limits_{s\neq r}^{s=1\ldots m}(q_s^2-q_r^2)}. \qquad (2.22)$$

The value of $[K(q_r^2)]$ thus derived should be substituted in both formulas of $[V]$ and $[I]$ of equations (2.20) and then the voltages and currents may be determined in calculable forms. The procedure of calculation of $[K(q_r^2)]$ by relation (2.22) is easily carried out practically by $(m-1)$ times multiplication of matrices and then m-times addition of the results. It is necessary for us to notice that m is finite, and is not greater than 8 in our case, since it is the number of the phases of the transmission lines in question. Therefore it must be easy and simple to calculate the matrix $[K(q_r^2)]$ for actual transmission lines. Hence the substitution of this formula for $[K(q_r^2)]$ in (2.20), will make the calculating procedure of $[V]$ and $[I]$ more practical and systematic.

In the above mathematical treatment, the readers may find something akin to Heaviside Operational Calculus, since the calculation is carried out purely symbolically. Hence it comes about that here we have found another type of a symbolic method quite different from, but similar to Heaviside Operational Method, by means of which boundary problems of polyphase transmission lines can be analysed purely operationally and systematically.

As another example, we shall again consider the case, where the values of $[V]$ and $[I]$ are given at $x=l$ as follows:

$$[V]=[V_l],\quad [I]=[I_l]. \tag{2.23}$$

Putting these relations in (2.14) and (2.15), we can derive the equations to determine integration-constants-matrices $[A]$ and $[B]$ as follows:

$$\left.\begin{aligned}&\cosh[Q]l\cdot[A]+\sinh\cdot[Q]l\cdot[B]=[V_l],\\&[Z(p)]^{-1}[Q]\{\sinh[Q]l\cdot[A]+\cosh[Q]l\cdot[B]\}=-[I_l].\end{aligned}\right\} \tag{2.24}$$

To determine $[B]$, transform the above equations to

$$[A]+\{\cosh[Q]l\}^{-1}\sinh[Q]l\cdot[B]=\{\cosh[Q]l\}^{-1}[V_l],$$
$$[A]+\{\sinh[Q]l\}^{-1}\cosh[Q]l\cdot[B]=-\{\sinh[Q]l\}^{-1}[Q]^{-1}[Z(p)][I_l].$$

Subtracting the second equation from the first, we obtain finally

$$\begin{aligned}[B]=&\Big[\{\cosh[Q]l\}^{-1}\sinh[Q]l-\{\sinh[Q]l\}^{-1}\cosh[Q]l\Big]^{-1}\\&\times\Big[\{\cosh[Q]l\}^{-1}[V_l]+\{\sinh[Q]l\}^{-1}[Q]^{-1}[Z(p)][I_l]\Big],\end{aligned} \tag{2.25}$$

and to find $[A]$, transform (2.24) to

$$\{\sinh[Q]l\}^{-1}\{\cosh[Q]l\}[A]+[B]=\{\sinh[Q]l\}^{-1}[V_l],$$
$$\{\cosh[Q]l\}^{-1}\{\sinh[Q]l\}[A]+[B]=-\{\cosh[Q]l\}^{-1}[Z(p)][I_l].$$

Subtracting the second from the first, we obtain finally,

$$\begin{aligned}[A]=&\Big[\{\sinh[Q]l\}^{-1}\cosh[Q]l-\{\cosh[Q]l\}^{-1}\sinh[Q]l\Big]^{-1}\\&\times\Big[\{\sinh[Q]l\}^{-1}[Vl]+\{\cosh[Q]l\}^{-1}[Q]^{-1}[Z(p)][I_l]\Big].\end{aligned} \tag{2.26}$$

We can, finally, deduce the values of $[V]$ and $[I]$ at any given point x, by substituting the values given by equations (2.25) and (2.26) for $[A]$ and $[B]$ in equations (2.13) and (2.14) or into the following relations:

$$\left.\begin{aligned}&[V]=\cosh[Q]x\cdot[A]+\sinh[Q]x\cdot[B],\\&[I]=-[Z(p)]^{-1}[Q]\{\sinh[Q]x\cdot[A]+\cosh[Q]x\cdot[B]\}.\end{aligned}\right\} \tag{2.27}$$

But thus established formulas for $[V]$ and $[I]$, may contribute to nothing as they stand. To ultilize these results for numerical calculation, we must transform them into other practically calculable forms by applying Sylvester's theorem which has been already shown. And accordingly, we have

$$\left.\begin{aligned}[V]&=\sum_{r=1}^{m}[K(q_r^2)]\Big\{\cosh[Q]x\cdot[A]+\sinh[Q]x\cdot[B]\Big\}_{[Q]\to q_r},\\ [I]&=-\sum_{r=1}^{m}[Z(p)]^{-1}[K(q_r^2)]\Big[[Q]\Big\{\sinh[Q]x\cdot[A]+\cosh[Q]x\\ &\quad\times[B]\Big\}\Big]_{[Q]\to q_r},\end{aligned}\right\}\quad(2.28)$$

where the values of $[A]$ and $[B]$ should be replaced by (2.25) and (2.26) respectively.

On the contrary, from (2.25) and (2.26), we have

$$\left.\begin{aligned}[A]_{[Q]\to q_r}&=\Big[\{\sinh q_rl\}^{-1}\cosh q_rl-\{\cosh q_rl\}^{-1}\sinh q_rl\Big]^{-1}\\ &\quad\times\Big[\{\sinh q_rl\}^{-1}[V_l]+\{\cosh q_rl\}^{-1}\frac{1}{q_r}[Z(q)][I_l]\Big]\\ &=\cosh q_rl\cdot[V_l]+\frac{\sinh q_rl}{q_r}[Z(p)][I_l],\\ [B]_{[Q]\to q_r}&=-\Big[\sinh q_rl\cdot[V_l]+\frac{\cosh q_rl}{q_r}[Z(p)][I_l]\Big].\end{aligned}\right\}\quad(2.29)$$

Hence, substitute (2.29) for $[A]$ and $[B]$ in (2.28) and we have finally

$$\left.\begin{aligned}[V]&=\sum_{r=1}^{m}[K(q_r^2)]\Big\{\cosh q_r(l-x)\cdot[V_l]+\sinh q_r(l-x)\cdot\Big[\frac{Z(p)}{q_r}\Big][I_l]\Big\},\\ [I]&=\sum_{r=1}^{m}\Big[\frac{Z(p)}{q_r}\Big]^{-1}[K(q_r^2)]\Big\{\cosh q_r(l-x)\cdot\Big[\frac{Z(p)}{q_r}\Big][I_l]\\ &\quad-\sinh q_r(l-x)\cdot[V_l]\Big\},\end{aligned}\right\}\quad(2.30)$$

where, as has been already shown, in the case where $q_1, q_2, \cdots q_m$ are all different from each other, $[K(q_r^2)]$ is given by

$$\left.\begin{aligned}[K(q_r^2)]&=\frac{\prod\limits_{\substack{s=1\cdots m\\ s\neq r}}(q_s^2[U]-[Q]^2)}{\prod\limits_{\substack{s=1\cdots m\\ s\neq r}}(q_s^2-q_r^2)}\\ &=\frac{\prod\limits_{\substack{s=1\cdots m\\ s\neq r}}(q_s^2[U]-[Z(p)][Y(p)])}{\prod\limits_{\substack{s=1\cdots m\\ s\neq r}}(q_s^2-q_r)^2}.\end{aligned}\right\}\quad(2.31)$$

Comparing the results obtained in equations (2.20) and (2.30) with those for a single phase transmission line, we reach the following very interesting and important conclusion.

In our polyphase lines, each q_r corresponds to propagation constant of a single phase line; but in our case, the number of lines is m, and accordingly there should exist m values of q_r's $(r=1, 2, \cdots m)$. It is worthy of remark that each $[Z(p)/q_r]$ corresponds to characteristic impedance of a single phase line.

But, in this case, we must notice that there exists essential difference between these two quantities. In a single phase line, if $Z(p)$ and $Y(p)$ denote impedance and admittance per unit length of the line, its characteristic impedance z shoud be given by

$$z=\sqrt{\frac{Z(p)}{Y(p)}}\,. \tag{2.32}$$

On the other hand, in case of polyphase lines, as has been already shown, a matrix with elements corresponding to characteristic impedance of a single phase line appears, which is given by

$$\left[\frac{Z(p)}{q_r}\right]. \tag{2.33}$$

Hence mathematical as well as physical meaning of the characteristic impedance matrix in polyphase lines is quite different from that of the characteristic impedance of a single phase line, only their dimensions coinciding with each other. The number of values of $[Z(p)/q_r]$ should be noticed to be m in this case.

3. Travelling Waves in General Polyphase Transmission Lines

In the preceding Article, assuming the lines to be of finite length, we have established general formulas for currents and voltages at any point x, when the terminal conditions at $x=0$ and $x=l$ are given. Making use of these formulas, we can calculate the transient as well as steady values of $[v]$ and $[i]$. But in that Article, we have been restricted to the case, where the voltages or currents at the terminals are known. Such assumption may be well in actual steady conditions; but as for transient phenomena, you had better give another form of terminal conditions. It is surely so, in the case where surge propagation phenomena are taken into consideration. Travelling waves along transmission lines, as has been discussed by various authorities, are composed of two groups of travelling waves, one of which propagates forward, while the other backward, and when the voltages or currents of one group are given, the corresponing currents or voltages belonging to the same group are uniquely determined. At the same time, we must notice that the voltage or current waves belonging to one group starting from the sending terminal are determined solely by concentrated impedance or admittance as well as E. M. F. inserted at the very terminal.

In the present Article, we shall analyse these travelling wave phenomena in polyphase transmission lines by means of Sylvester's theorem.

Consider m-phase transmission lines, and then the voltage and current matrices $[v]$ and $[i]$ at any point x and at any instant t, are given by the following equations.

$$\left.\begin{aligned} -\frac{\partial[v]}{\partial x} &= \left[L\frac{\partial}{\partial t}+R\right][i], \\ -\frac{\partial[i]}{\partial x} &= \left[C\frac{\partial}{\partial t}+G\right][v]. \end{aligned}\right\} \tag{3.1}$$

Eliminating $[i]$ or $[v]$ from the above equations, we have

$$\left.\begin{aligned} \frac{\partial^2[v]}{\partial x^2} &= \left[L\frac{\partial}{\partial t}+R\right]\left[C\frac{\partial}{\partial t}+G\right][v], \\ \frac{\partial^2[i]}{\partial x^2} &= \left[C\frac{\partial}{\partial t}+G\right]\left[L\frac{\partial}{\partial t}+G\right][i]. \end{aligned}\right\} \tag{3.2}$$

By the special analytial procedure which was adopted by the present author, we can establish the formulas of currents and voltages at x and t before reflected waves come about. Assume that the voltage waves which propagate forward (or in the right direction) possess the value $[v]_{x=0}=[v_0]H(t)$ at the sending end, where $H(t)$ denotes Heaviside unit function given by

$$H(t)=\left\{\begin{matrix} 0\ , & t<0\ ; \\ 1\ , & t>0\ ; \end{matrix}\right\} \tag{3.3}$$

and then, provided reflected waves have not reached yet, the voltages at x will be expressed by the following matrix form:

$$\left.\begin{aligned} v &= \mathfrak{H}[V]=\mathfrak{H}\varepsilon^{-[Q]x}[V_0] \\ &= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{\varepsilon^{-[Q]x}\varepsilon^{pt}}{p}[V_0]\,dp\,, \\ &\text{where} \\ [V] &= \varepsilon^{-[Q]x}[V_0]\,, \end{aligned}\right\} \tag{3.4}$$

and $[V_0]$ denotes p-functional matrix corresponding to $[V_0]$, or in other words, the elements of $[V_0]$ are operational forms corresponding to those of $[v_0]$, namely

$$[v_0]H(t)=\mathfrak{H}[V_0]=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{[V_0]}{p}\varepsilon^{pt}\,dp\,. \tag{3.5}$$

And $[Q]$ is given by, the following fractional power matrix,

$$[Q]=\sqrt{[Lp+R][Cp+G]}\,,$$

or for simplicity, if we put

$$\left.\begin{aligned}[Z(p)]&=[Lp+R]\,,\\ [Y(p)]&=[Cp+G]\,,\end{aligned}\right\} \tag{3.6}$$

then the value of $[Q]$ shown above is replaced by

$$[Q]=\sqrt{[Z(p)][Y(p)]}\,. \tag{3.7}$$

To find the currents at x, put

$$[i]=\mathfrak{H}[I]\,, \tag{3.8}$$

where, of course, $[I]$ represents p-functional matrix corresponding to $[i]$. Substituting (3.4) and (3.8) for $[v]$ and $[i]$ in the first equation of (3.1), we can determine $[I]$ as follows,

$$[I]=[Z(p)]^{-1}[Q]\varepsilon^{-[Q]x}[V_0] \tag{3.9}$$

Now, you find functional matrices ,containing the fractional power matrix in both expressions of $[V]$ and $[I]$ given by (3.4) and (3.9). Hence $[V]$ and $[I]$, as they stand, have no practical applicability. To make the results practically calculable, we must, again, apply Sylvester's theorem that was interpreted in the first Article.

Since the number of transmission lines is m, the matrix

$$[Q]^2=[Z(p)][Y(p)] \tag{3.10}$$

is a rectangular matrix of m-th degree; hence it should possess m characteristic roots $q_1^2, q_2^2, \cdots\cdots q_m^2$ respectively, or in other words, $q_1, q_2, \cdots\cdots q_m$ should satisfy the following equation:

$$\delta\left\{q^2[U]-[Q]^2\right\}=0\,. \tag{3.11}$$

And consequently applying Sylvester's theorem to (3.4) and (3.9), we have

$$\left.\begin{aligned}[V]&=\varepsilon^{-[Q]x}[V_0]=\sum_{r=1}^{m}\varepsilon^{-q_r x}[K(q_r^2)][V_0]\,,\\ [I]&=[(Z(p)]^{-1}[Q]\varepsilon^{-[Q]x}[V_0]=\sum_{r=1}^{m}\varepsilon^{-q_r x}\left[\frac{Z(p)}{q_r}\right][K(q_r^2)][V_0]\,,\end{aligned}\right\} \tag{3.12}$$

where

$$[K(q_r^2)]=\frac{\prod\limits_{s\neq r}^{s=1\cdots m}(q_r^2[U]-[Q]^2)}{\prod\limits_{s\neq r}^{s=1\cdots m}(q_s^2-q_r^2)}\,, \tag{3.13}$$

where $q_1, q_2, \cdots q_m$ are assumed all different from each other. Provided some of them are equal, or in other words, if $[Q]^2$ has multiple characteristic roots, another expression should be substituted for $[K(q_r^2)]$, applying equation (1.5).

EXAMPLE 1.

Travelling Waves along Symmetrical Two-phase No-Loss Lines.

For simlicity, assume the resistance and leakance of the lines to be neglected, and suppose that E. M. F. s E_1 and E_2 are impressed at the sending terminals ; then, provided that the lines are same-shaped and symmetrically arranged, we may put

$$\left.\begin{aligned} [Z(p)] &= p\begin{pmatrix} L & M \\ M & L \end{pmatrix}, \\ [Y(p)] &= p\begin{pmatrix} C & C' \\ C' & C \end{pmatrix}. \end{aligned}\right\} \qquad (3.14)$$

Accordingly we have

$$\begin{aligned} [Q]^2 &= [Z(p)][Y(p)] \\ &= p^2\begin{pmatrix} m & m' \\ m' & m \end{pmatrix}, \end{aligned} \qquad (3.15)$$

where

$$\left.\begin{aligned} m &= LC + MC', \\ m' &= LC' + MC. \end{aligned}\right\} \qquad (3.16)$$

Hence the characteristic equation of $[Q]^2$ is

$$\delta(q^2[U] - p^2[m]) = 0,$$

or

$$q^4 - 2mp^2q^2 + (m^2 - m'^2)p^4 = 0. \qquad (3.17)$$

Therefore the characteristic roots of $[Q]^2$ are given by

$$q^2 = \left.\begin{matrix} q_1^2 \\ q_2^2 \end{matrix}\right\} = p^2(m \pm m') = \left\{\begin{matrix} (L+M)(C+C')p^2, \\ (L-M)(C-C')p^2. \end{matrix}\right. \qquad (2.18)$$

Or the above equation may be arranged in the form:

$$\begin{pmatrix} q_1^2 \\ q_2^2 \end{pmatrix} = \begin{pmatrix} p^2/g_1^2 \\ p^2/g_2^2 \end{pmatrix}, \qquad (3.19)$$

where

$$\left.\begin{aligned} g_1 &= \frac{1}{\sqrt{(C+M)(C+C')}}, \\ g_2 &= \frac{1}{\sqrt{(L-M)(C-C')}}, \end{aligned}\right\} \qquad (3.20)$$

Hence potential waves at the point x may be obtained by putting $m=2$ in equation (3.12), and substituting (3.19) in the result thus obtained; viz.,

$$[V]=\left\{\varepsilon^{-q_1x}[K(q_1^2)]+\varepsilon^{-q_2x}[K(q_2^2)]\right\}. \tag{3.21}$$

Putting $m=2$ in (1.3), we have

$$\begin{aligned}[K(q_1^2)]&=\frac{q_2^2[U]-[Q]^2}{q_2^2-q_1^2}=\frac{p^2(m-m')[U]-p^2[m]}{p^2(m-m')-p^2(m+m')}\\&=-\frac{1}{2m'}\begin{pmatrix}m-m'-m & -m'\\ -m' & m-m'-m\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}.\end{aligned} \tag{3.22}$$

Similarly we have

$$[K(q_2^2)]=\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

On the contrary, in our case, we have

$$[V_0]=\begin{pmatrix}E_1\\E_2\end{pmatrix}. \tag{3.23}$$

Hence putting (3.19), (3.22) and (3.23) in (3.21), we have

$$\begin{aligned}[V]&=\frac{\varepsilon^{-px/g_1}}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}E_1\\E_2\end{pmatrix}+\frac{\varepsilon^{-px/g_2}}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}E_1\\E_2\end{pmatrix}\\&=\frac{E_1+E_2}{2}\varepsilon^{-px/g_1}\begin{pmatrix}1\\1\end{pmatrix}+\frac{E_1-E_2}{2}\varepsilon^{-px/g_2}\begin{pmatrix}1\\-1\end{pmatrix}.\end{aligned} \tag{3.24}$$

Similarly we have

$$\begin{aligned}[I]&=\frac{\varepsilon^{-px/g_1}}{2g_1}\begin{pmatrix}L&M\\M&L\end{pmatrix}^{-1}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}E_1\\E_2\end{pmatrix}+\frac{\varepsilon^{-px/g_2}}{2g_2}\begin{pmatrix}L&M\\M&L\end{pmatrix}^{-1}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}E_1\\E_2\end{pmatrix}\\&=\frac{\varepsilon^{-px/g_1}}{2g_1(L^2-M^2)}\begin{pmatrix}L&-M\\-M&L\end{pmatrix}\begin{pmatrix}E_1+E_2\\E_1+E_2\end{pmatrix}+\frac{\varepsilon^{-px/g_2}}{2g_2(L^2-M^2)}\begin{pmatrix}L&-M\\-M&L\end{pmatrix}\begin{pmatrix}E_1-E_2\\-E_1+E_2\end{pmatrix}\\&=\frac{E_1+E_2}{2z_1}\varepsilon^{-px/g_1}\begin{pmatrix}1\\1\end{pmatrix}+\frac{E_1-E_2}{2z_2}\varepsilon^{-px/g_2}\begin{pmatrix}1\\-1\end{pmatrix},\end{aligned} \tag{3.25}$$

where

$$z_1=\sqrt{\frac{L+M}{C+C'}},\quad z_2=\sqrt{\frac{L-M}{C-C'}}. \tag{3.26}$$

Hence, from (3.24) and (3.25), we have finally

$$\left.\begin{aligned}[v]=\mathfrak{H}[V]&=\begin{pmatrix}\frac{E_1+E_2}{2}H(g_1t-x)+\frac{E_1-E_2}{2}H(g_2t-x)\\ \frac{E_1+E_2}{2}H(g_1t-x)-\frac{E_1-E_2}{2}H(g_2t-x)\end{pmatrix},\\ [i]=\mathfrak{H}[I]&=\begin{pmatrix}\frac{E_1+E_2}{2z_1}H(g_1t-x)+\frac{E_1-E_2}{2z_2}H(g_2t-x)\\ \frac{E_1+E_2}{2z_1}H(g_1t-x)-\frac{E_1-E_2}{2z_2}H(g_2t-x)\end{pmatrix},\end{aligned}\right\} \quad (3.27)$$

where

$$H(gt-x)=\begin{cases}0, & gt<x\ ;\\ 1, & gt>x\ .\end{cases} \quad (3.28)$$

EXAMPLE 2

Travelling Waves along Symmetrical Three-Phase No-Loss Lines.

Again, putting

$$[Z(p)]=p\begin{pmatrix}L & M & M\\ M & L & M\\ M & M & L\end{pmatrix},\quad [Y(p)]=p\begin{pmatrix}C & C' & C'\\ C' & C & C'\\ C' & C' & C\end{pmatrix}, \quad (3.29)$$

we have

$$[Q]^2=[Z(p)]\,[Y(p)]=p^2\,[m]$$

$$=p^2\begin{pmatrix}m & m' & m'\\ m' & m & m'\\ m' & m' & m\end{pmatrix}, \quad (3.30)$$

where
$$\left.\begin{aligned}m&=LC+2MC',\\ m'&=LC'+CM+MC'.\end{aligned}\right\} \quad (3.31)$$

The characteristic roots of $[Q]^2$ namely q^2 can be determined from the following equation,

$$\begin{vmatrix}q^2-mp^2 & -m'p^2 & -m'p^2\\ -m'p^2 & q^2-mp^2 & -m'p^2\\ -m'p^2 & -m'p^2 & q^2-mp^2\end{vmatrix}=0,$$

or simplifying this, we have

$$\{q^2-(m-m')\,p^2\}^2\,\{q^2-(m+2m')\,p^2\}=0. \quad (3.32)$$

Hence the characteristic roots of $[Q]^2$ are given by

$$\left.\begin{aligned}q_1^2&=(m-m')\,p^2 \ :\ \text{double root},\\ q_2^2&=(m+2m')\,p^2 \ :\ \text{single root}.\end{aligned}\right\} \quad (3.33)$$

Accordingly, to find $[V]$ and $[I]$, we must use the general formula (1.5). Putting $s=2$ in the same equation, we have

$$F([A])=\left\{F(\alpha_1)[U]-\frac{dF(\alpha_1)}{d\alpha_1}(\alpha_1[U]-[A])-\frac{F(\alpha_1)}{\alpha_3-\alpha_1}\right.$$
$$\left.\times(\alpha_1[U]-[A])\right\}\frac{\alpha_3[U]-[A]}{\alpha_3-\alpha_1}+F(\alpha_3)\frac{(\alpha_1[U]-[A])^2}{(\alpha_1-\alpha_3)^2}, \qquad (3.34)$$

where α_1 represents a double root, and α_3 a single root of third-degree matrix $[A]$.

On the contrary, from (3.4) and (3.9) we have

$$\left.\begin{array}{l}[V]=\varepsilon^{-[Q]x}[V_0],\\ [I]=[Z(p)]^{-1}[Q]\varepsilon^{-[Q]x}[V_0].\end{array}\right\} \qquad (3.35)$$

Hence taking into account the relations given by (3.33) and (3.34), we can transform (3.35) as follows.

$$[V(p)]=\varepsilon^{-[Q]x}[V_0]$$
$$=\left\{\varepsilon^{-q_1x}[U]-\frac{d(\varepsilon^{-q_1x})}{d(q_1^2)}(q_1^2[U]-p^2[m])-\frac{\varepsilon^{-q_1x}}{q_2^2-q_1^2}\right.$$
$$\left.\times(q_1^2[U]-p^2[m])\right\}\cdot\frac{q_2^2[U]-p^2[m]}{q_2^2-q_1^2}[V_0]$$
$$+\frac{\varepsilon^{-q_2x}}{(q_1^2-q_2^2)^2}(q_1^2[U]-p^2[m])^2[V_0]$$
$$=\frac{\varepsilon^{-q_1x}}{q_2^2-q_1^2}(q_1^2[U]-p^2[m])[V_0]$$
$$+\frac{x}{2q_1}\frac{\varepsilon^{-q_1x}}{q_2^2-q_1^2}(q_1^2[U]-p^2[m])(q_2^2[U]-p^2[m])[V_0]$$
$$-\frac{\varepsilon^{-q_1x}}{(q_2^2-q_1^2)^2}(q_1^2[U]-p^2[m])(q_2^2[U]-p^2[m])[V_0]$$
$$+\frac{\varepsilon^{-q_1x}}{(q_2^2-q_1^2)^2}(q_1^2[U]-p^2[m])^2[V_0]. \qquad (3.36)$$

On the contrary, cofactors corresponding to the elements belonging to the first column are respectively calculated as follows.

$$\begin{vmatrix} q^2-mp^2 & -m'p^2 \\ -m'p^2 & q^2-mp^2 \end{vmatrix}=\{q^2-(m-m')p^2\}\ \{q^2-(m+ml)p^2\},$$
$$-\begin{vmatrix} -m'p^2 & -m'p^2 \\ -m'p^2 & q^2-mp^2 \end{vmatrix}=m'p^2\{q^2-(m-m')p^2\},$$
$$\begin{vmatrix} -m'p^2 & q^2-mp^2 \\ -m'p^2 & -m'p^2 \end{vmatrix}=m'p^2\{q^2-(m-m')p^2\}.$$

Hence these cofactors have a common factor.

The same thing may be said about the cofactors corresponding to the second and third column.

Now evidently $q_1^2=(m-m')^2$ is a double root of (3.32), namely $\delta(q^2[U]-p^2[m])=0$, and q_2^2 a single root thereof; and accordingly, by the aid of the extended Cayley-Hamilton's theorem,[1] we have

$$(q_1^2[U]-p^2[m])(q_2^2[U]-p^2[m])=[0]. \qquad (3.37)$$

Hence the second and third terms of the right-hand side of equation (3.36) vanish, and only the first and fourth terms remain, and consequently we have

$$[V]=\frac{\varepsilon^{-q_1x}}{q_2^2-q_1^2}(q_2^2[U]-p^2[m])[V_0]+\frac{\varepsilon^{-q_2x}}{(q_1^2-q_2^2)^2}$$
$$\times(q_1^2[U]-p^2[m])^2[V_0]. \qquad (3.38)$$

Substituting (3.33) in the above equation, we have

$$[V]=\frac{\varepsilon^{-q_1x}}{3m'}\{(m+2m')[U]-[m]\}[V_0]$$
$$+\frac{\varepsilon^{-q_2x}}{9m'^2}\{(m-m')[U]-[m]\}^2[V_0]. \qquad (3.39)$$

On the contrary, since

$$(m-m')[U]-[m]=\begin{pmatrix} m-m' & 0 & 0 \\ 0 & m-m' & 0 \\ 0 & 0 & m-m' \end{pmatrix}-\begin{pmatrix} m & m' & m' \\ m' & m & m' \\ m' & m' & m \end{pmatrix}$$
$$=-m'\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and

$$(m+2m')[U]-[m]=\begin{pmatrix} m+2m' & 0 & 0 \\ 0 & m+2m' & 0 \\ 0 & 0 & m+2m' \end{pmatrix}-\begin{pmatrix} m & m' & m' \\ m' & m & m' \\ m' & m' & m \end{pmatrix}$$
$$=m'\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

equation (3.39) is reduced to

$$[V]=\frac{\varepsilon^{-q_1x}}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} V_{01} \\ V_{02} \\ V_{03} \end{pmatrix}+\frac{\varepsilon^{-q_2x}}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^2\begin{pmatrix} V_{01} \\ V_{02} \\ V_{03} \end{pmatrix}$$

[1]) Cayley-Hamilton's theorem was extended by the present author.

$$=\frac{\varepsilon^{-q_1x}}{3}\begin{pmatrix} 2V_{01}- & V_{02}- & V_{03} \\ -V_{01}+2V_{02}- & V_{03} \\ -V_{01}- & V_{02}+2V_{03} \end{pmatrix}+\frac{\varepsilon^{-q_2x}}{3}\begin{pmatrix} V_{01}+V_{02}+V_{03} \\ V_{01}+V_{02}+V_{03} \\ V_{01}+V_{02}+V_{03} \end{pmatrix}. \tag{3.40}$$

Provided that

$$[v_0]=\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{3.41}$$

is a constant matrix independent of time, and if we put

$q_1=p/g_1, \quad q_2=p/g_2,$

where

$$\left.\begin{aligned} g_1&=(m-m')^{-\frac{1}{2}}=\frac{1}{\sqrt{(L-M)(C-C')}}, \\ g_2&=(m+2m')^{-\frac{1}{2}}=\frac{1}{\sqrt{(L+2M)(C+2C')}}, \end{aligned}\right\} \tag{3.42}$$

the t-matrix corresponding to $[V]$ becomes finally

$$[v]=\frac{H(g_1t-x)}{3}\begin{pmatrix} 2E_1- & E_2- & E_3 \\ -E_1+2E_2- & E_3 \\ -E_1- & E_2+2E_3 \end{pmatrix}+\frac{H(g_2t-x)}{3}\begin{pmatrix} E_1+E_2+E_3 \\ E_1+E_2+E_3 \\ E_1+E_2+E_3 \end{pmatrix}. \tag{3.43}$$

Conclusion

In the present paper, the author has only introduced a new analytical method of steady and transient phenomena in general polyphase transmission lines, and dealt with some ideal simple examples thereof, in order to interprete the new mathematical method, various other actual problems of polyphase transmission lines being abbreviated for the sake of simplicity. Such problems, however, are interesting and really important for electrical engineers, and, at present, assistants and students of my laboratory are studying such problems by means of the above mentioned new method; the results may be published on some occasions in future.